



Signals and Systems

信號與系統

Lecture 1-2



Elementary Signals

- Exponential Signals 指數型訊號
- Sinusoidal Signals 弦波型訊號
- Exponentially Damped Sinusoidal Signals
 指數型弦波型訊號
- Step (Function) Signals 步階訊號
- Impulse (Function) Signals 脈衝訊號
- Ramp Function of Unit Slope 單位斜率斜訊號



Exponential Signals

$$x(t) = A e^{at}$$

$a < 0$: Decaying Exponential Signal

$a > 0$: Growing Exponential Signal

$$e = 2.71828183\dots$$

$$e^0 = 1$$

$$e^1 = 2.71828183\dots$$

...

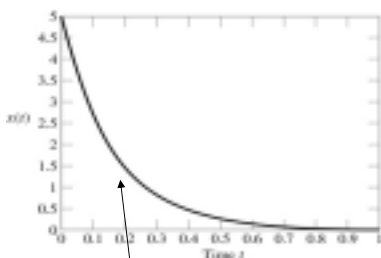
e^t = 以 t 為自變數的指數函數

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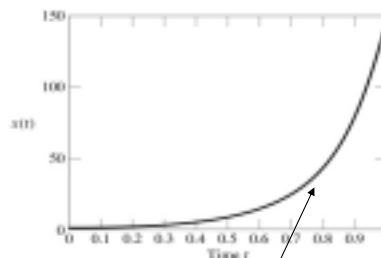


(a) Decaying exponential form of continuous-time signal.

(b) Growing exponential form of continuous-time signal.



衰退指數性訊號



成長指數性訊號

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Example:

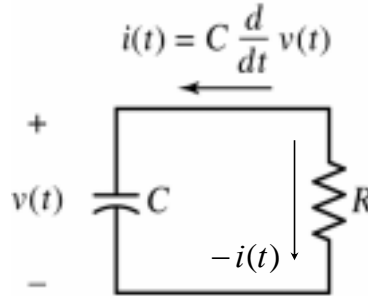
Lossy capacitor, with the loss represented by shunt resistance R .

$$v(t) = \frac{1}{C} \int_{-\infty}^t i(\tau) d\tau$$

$$\therefore v(t) = -i(t) \cdot R = -RC \frac{d}{dt} v(t)$$

$$\therefore RC \frac{d}{dt} v(t) + v(t) = 0$$

$$\therefore v(t) = V_0 e^{-t/(RC)}$$

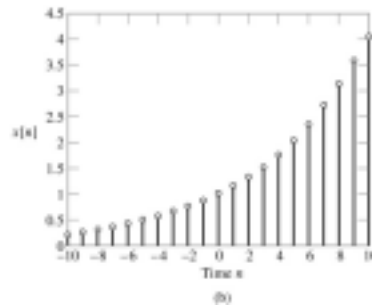
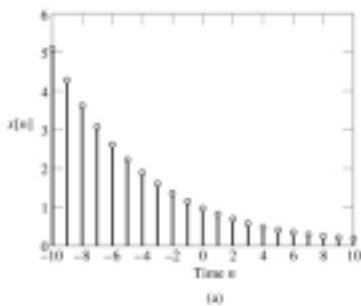


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(a) Decaying exponential form of discrete-time signal.

(b) Growing exponential form of discrete-time signal.



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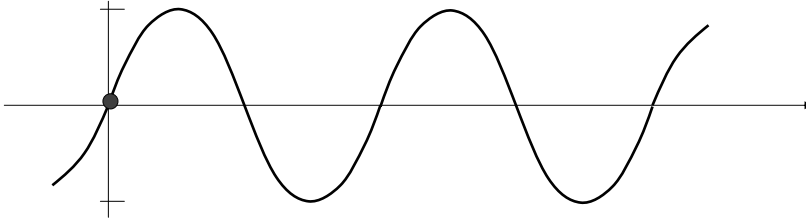
Sinusoidal Signals

$$x(t) = A \sin(\omega t + \phi)$$

A : *amplitude* 訊號振幅

ω : *angular frequency*(radians / sec) 訊號角頻率

ϕ : *phase* 訊號相位移



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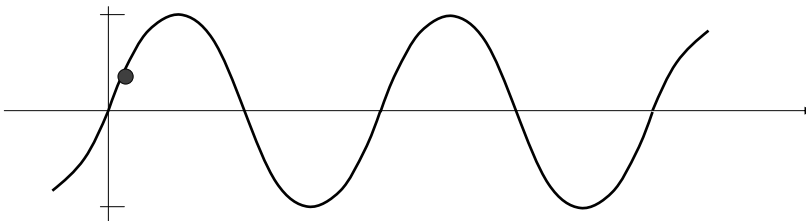
Sinusoidal Signals (cont.)

$$x(t) = A \sin(\omega t + \phi)$$

A : *amplitude* 訊號振幅

ω : *angular frequency*(radians / sec) 訊號角頻率

ϕ : *phase* 訊號相位移



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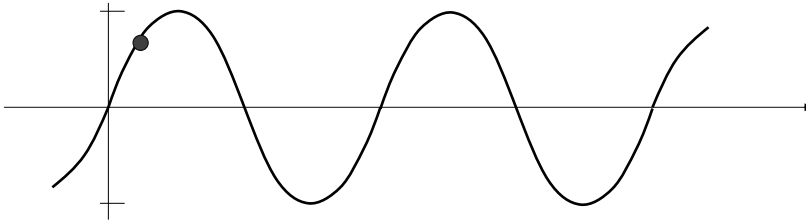
Sinusoidal Signals (cont.)

$$x(t) = A \sin(\omega t + \phi)$$

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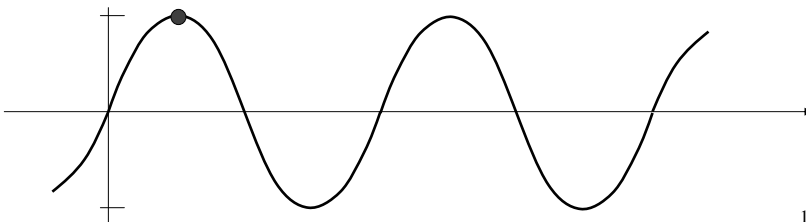
Sinusoidal Signals (cont.)

$$x(t) = A \sin(\omega t + \phi)$$

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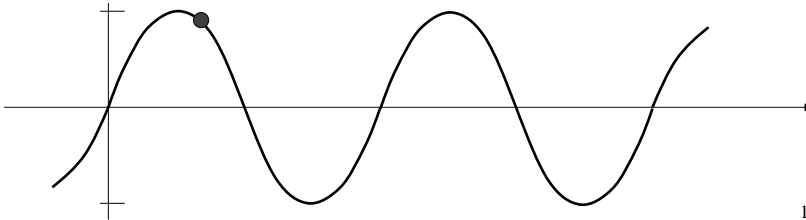
Sinusoidal Signals (cont.)

$$x(t) = A \sin(\omega t + \phi)$$

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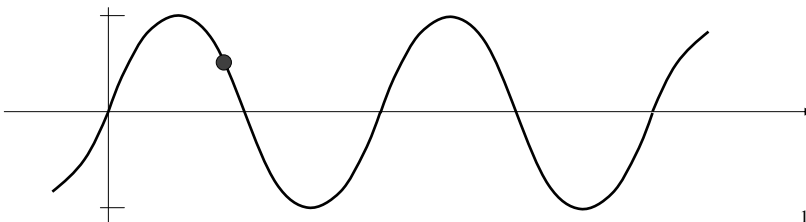
Sinusoidal Signals (cont.)

$$x(t) = A \sin(\omega t + \phi)$$

A : *amplitude* 訊號振幅

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ϕ : *phase* 訊號相位移



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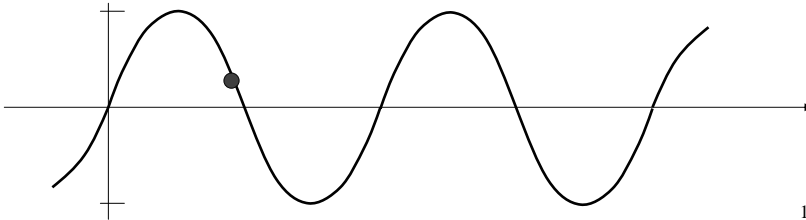
Sinusoidal Signals (cont.)

$$x(t) = A \sin(\omega t + \phi)$$

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ϕ : *phase* 訊號相位移



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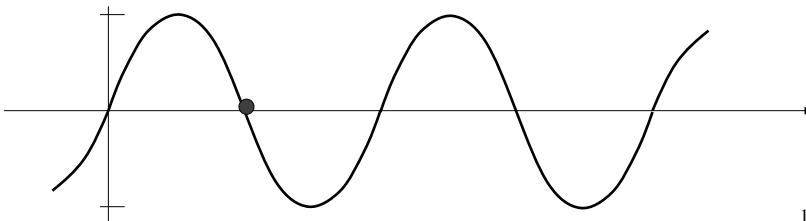
Sinusoidal Signals (cont.)

$$x(t) = A \sin(\omega t + \phi)$$

A : *amplitude* 訊號振幅

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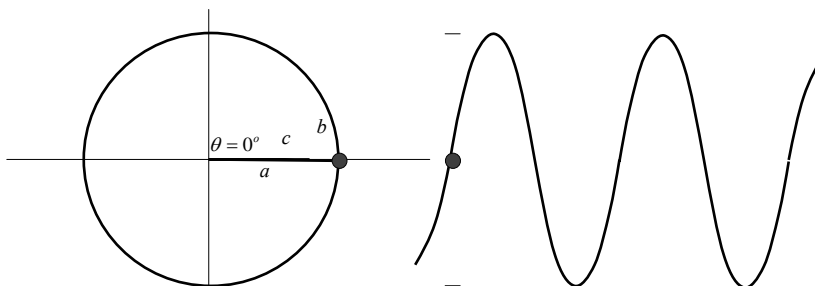
ϕ : *phase* 訊號相位移



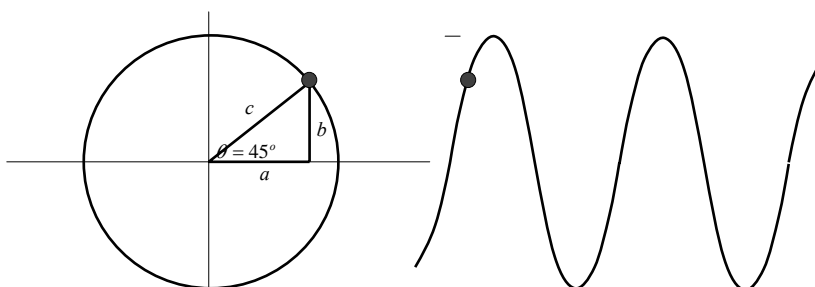
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Phasor and Sinusoids

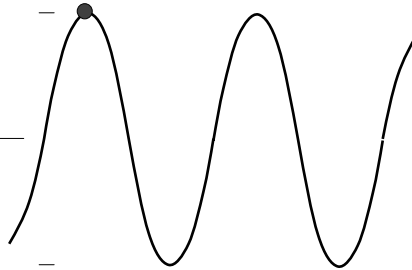
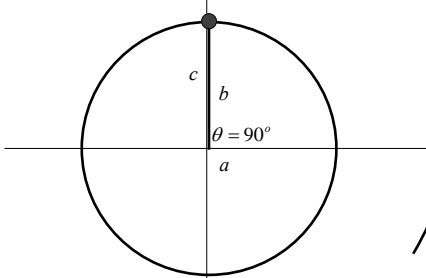


Phasor at 45°





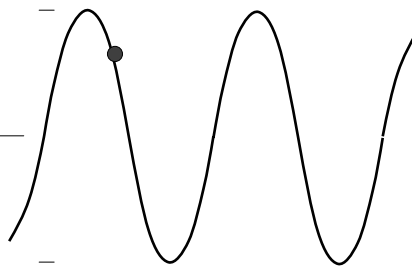
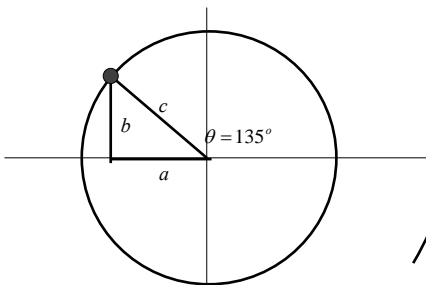
Phasor at 90°



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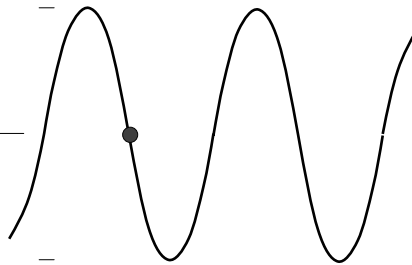
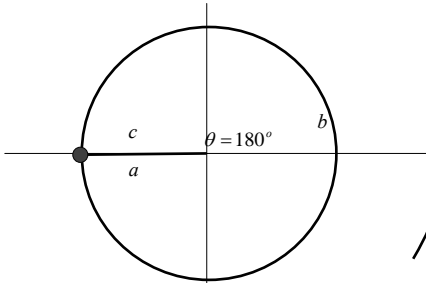
Phasor at 135°



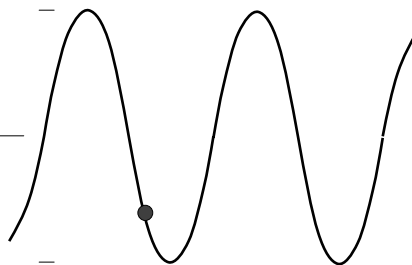
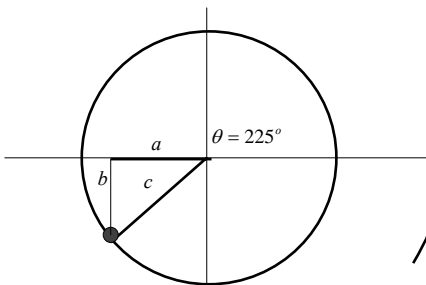
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Phasor at 180°

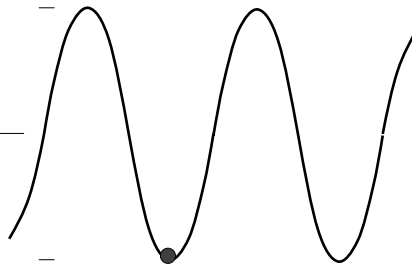
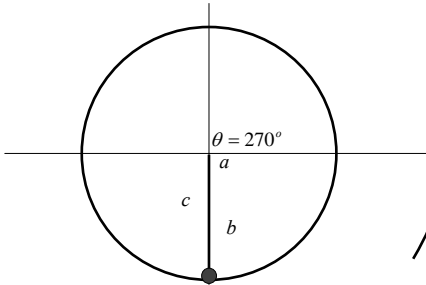


Phasor at 225°

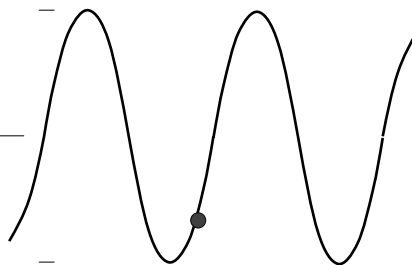
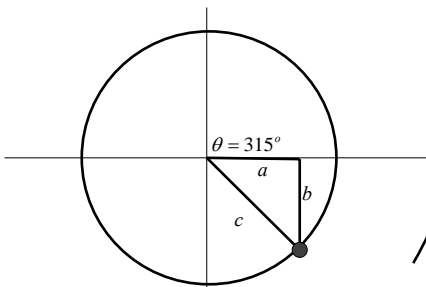




Phasor at 270°

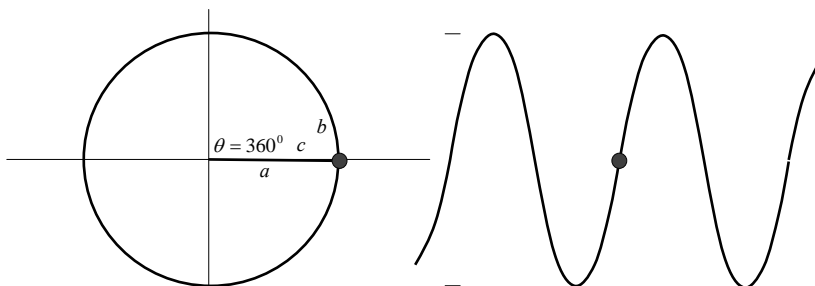


Phasor at 315°





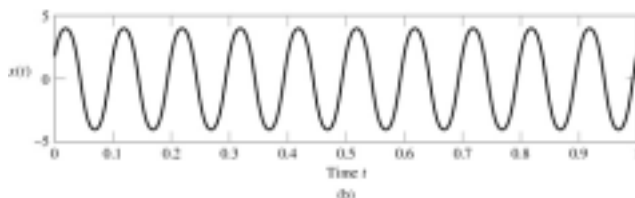
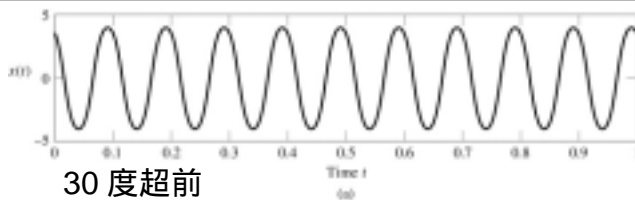
Phasor at 360°



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30 度超前 (T = 0.1 sec., $\omega = 2\pi/T = 20\pi$)



- (a) Sinusoidal signal $A \cos(t +)$ with phase $= +\pi/6$ radians.
 (b) Sinusoidal signal $A \sin(t +)$ with phase $= +\pi/6$ radians.

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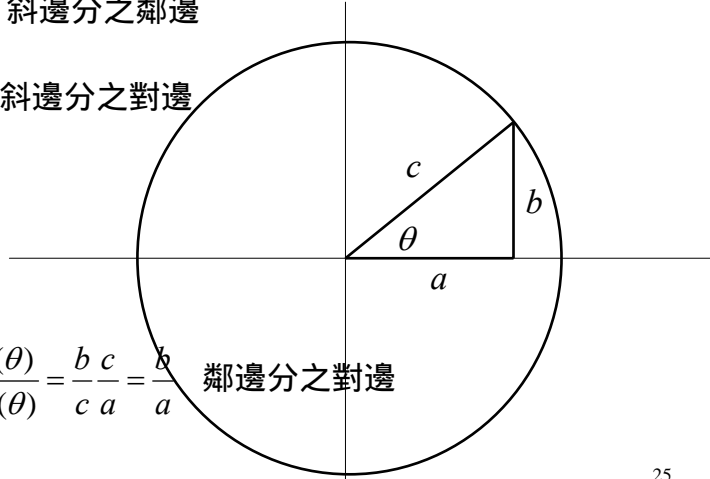


Triangular Function

$$\cos(\theta) = \frac{a}{c} \quad \text{斜邊分之鄰邊}$$

$$\sin(\theta) = \frac{b}{c} \quad \text{斜邊分之對邊}$$

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{b/c}{a/c} = \frac{b}{a} \quad \text{鄰邊分之對邊}$$



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Sinusoidal Function

$$\theta = \omega t \quad (\text{radians} = \frac{\text{radians}}{\text{sec.}} \text{sec.})$$

$$\omega: 2\pi/T$$

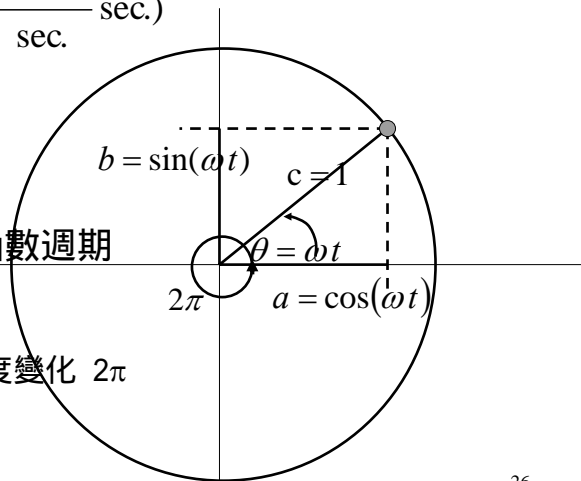
$$\theta: 0 \sim 2\pi$$

$$t: 0 \sim T$$

$$T: \text{Period (訊號函數週期)}$$

逆時針方向繞一圈角度變化 2π

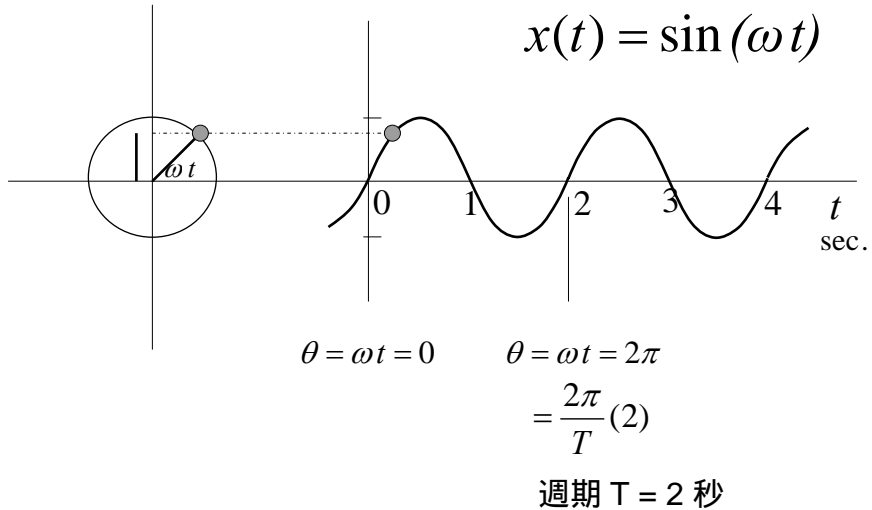
Counter-clockwise



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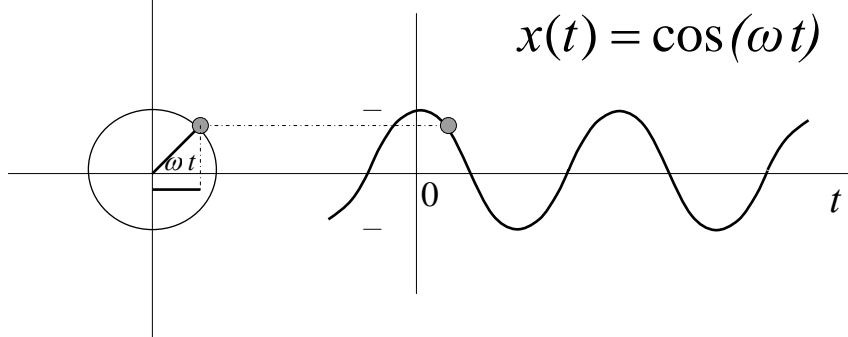
Sine



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Cosine



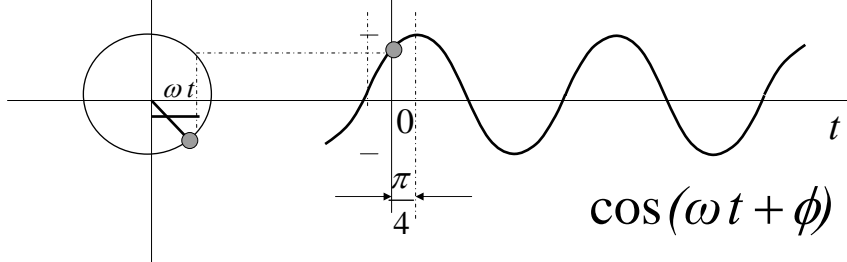
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Cosine with Phase

Time scaling and shifting

複習一下 ?????

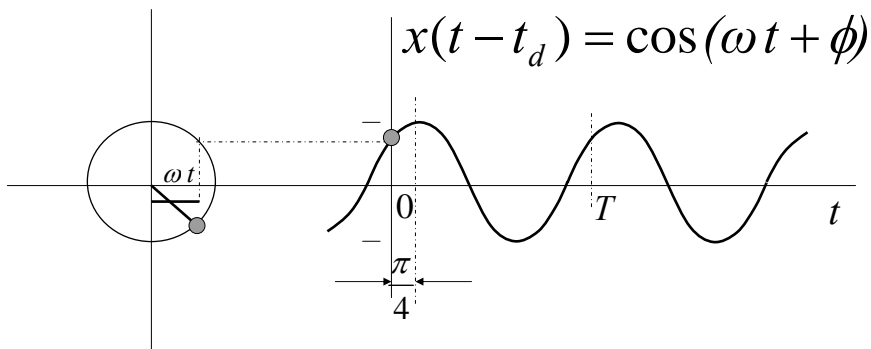


Cosine 訊號延遲了多少時間 ? $t_d = ?$

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Cosine with Time Shift



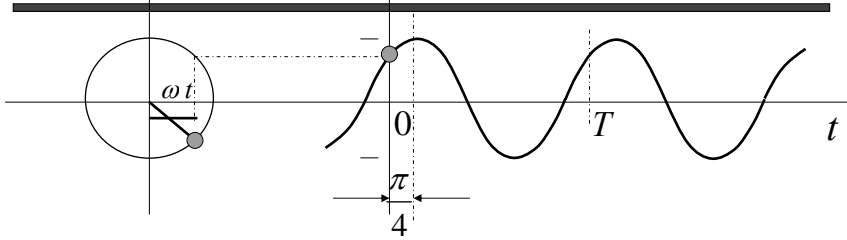
Cosine 訊號延遲了多少時間 ? $t_d = ?$

$$\therefore \omega t_d = \frac{2\pi}{T} t_d = \frac{\pi}{4}, \quad \therefore t_d = \frac{\pi}{4} \frac{T}{2\pi} = \frac{T}{8}$$

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Cosine with Time Shift (Cont.)



$$x(t - t_d) = \cos(\omega(t - t_d)) = \cos(\omega t - \omega t_d)$$

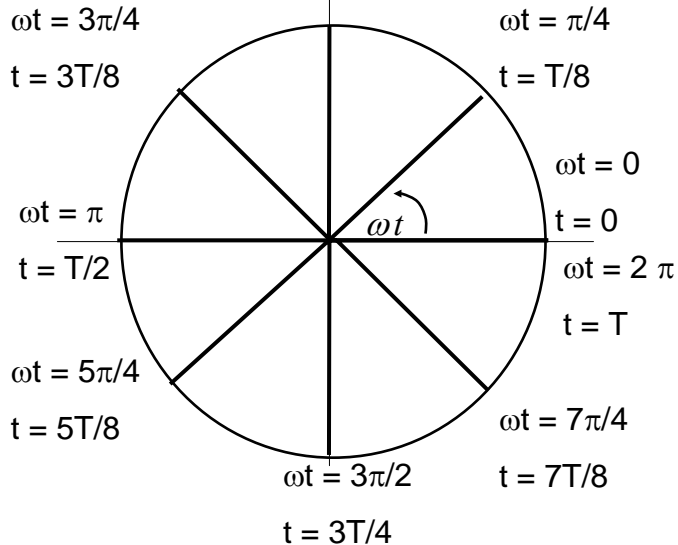
$$= \cos(\omega t + \phi)$$

∴

$$\phi = -\omega t_d = -\frac{\pi}{4}$$



$$\omega t = \pi/2 ; t = T/4$$



**Example:**Parallel LC circuit, assuming ideal inductor L and capacitor C

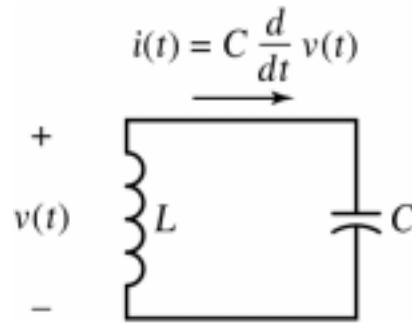
(弦波訊號電路範例)

$$\therefore LC \frac{d^2}{dt^2} v(t) + v(t) = 0$$

solution:

$$v(t) = \cos(\omega_0 t), \quad t \geq 0$$

$$\text{其中 } \omega_0 = \frac{1}{\sqrt{LC}}.$$



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$$\therefore v(t) = \frac{1}{C} \int_{-\infty}^t i(\tau) d\tau \Rightarrow i(t) = C \frac{d}{dt} v(t)$$

$$\therefore v(t) = -L \frac{d}{dt} i(t) = -L \frac{d}{dt} \left(C \frac{d}{dt} v(t) \right) = -LC \frac{d^2}{dt^2} v(t)$$

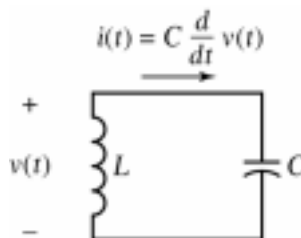
$$\therefore LC \frac{d^2}{dt^2} v(t) + v(t) = 0$$

solution:

$$v(t) = \cos(\omega_0 t), \quad t \geq 0$$

$$\text{其中 } \omega_0 = \frac{1}{\sqrt{LC}}.$$

下一頁說明過程



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$$LCs^2 + 1 = 0,$$

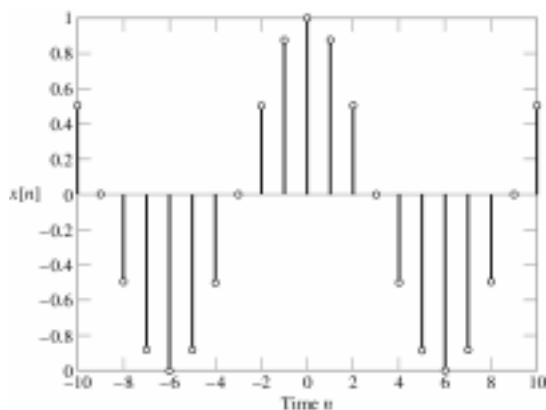
$$\Rightarrow LCs^2 = -1, \Rightarrow s^2 = \frac{1}{-LC}, \Rightarrow s = \pm j \frac{1}{\sqrt{LC}},$$

solution :

$$\begin{aligned} \frac{1}{2}e^{s_1 t} + \frac{1}{2}e^{s_2 t} &= \frac{1}{2} \left(e^{j \frac{t}{\sqrt{LC}}} + e^{-j \frac{t}{\sqrt{LC}}} \right) = \left(\frac{e^{j \frac{t}{\sqrt{LC}}} + e^{-j \frac{t}{\sqrt{LC}}}}{2} \right) \\ &= \cos \left(\frac{t}{\sqrt{LC}} \right) \end{aligned}$$



Discrete-time sinusoidal signal





Sinusoids & Complex Exponential Signals

Euler's Identity:

$$e^{j\theta} = \cos(\theta) + j \sin(\theta)$$

Complex Exponential Signal: 複數指數訊號

$$e^{j(\omega t + \phi)} = \cos(\omega t + \phi) + j \sin(\omega t + \phi)$$

$$\therefore \operatorname{Re}\{e^{j(\omega t + \phi)}\} = \cos(\omega t + \phi) \quad \text{實部}$$

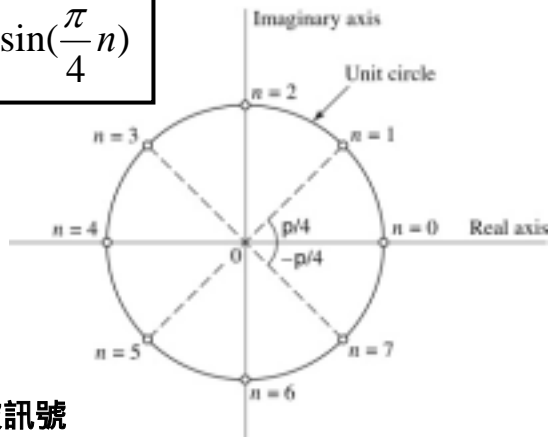
$$\therefore \operatorname{Im}\{e^{j(\omega t + \phi)}\} = \sin(\omega t + \phi) \quad \text{虛部}$$

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Sinusoids on the unit circle

$$e^{j\frac{\pi}{4}n} = \cos\left(\frac{\pi}{4}n\right) + j \sin\left(\frac{\pi}{4}n\right)$$



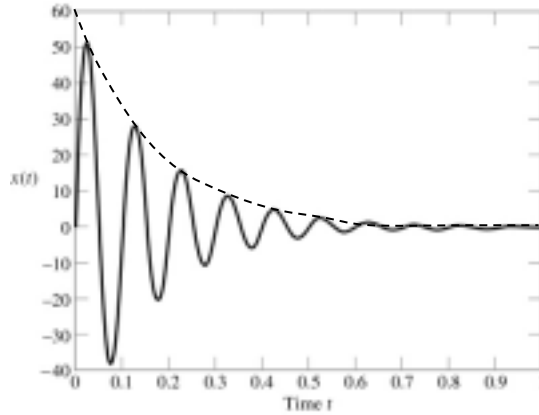
複數平面單位圓與弦波訊號

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Exponentially damped sinusoidal signal

($Ae^{-\alpha t} \sin(\omega t)$, with $A = 60$ and $\alpha = 6$.)

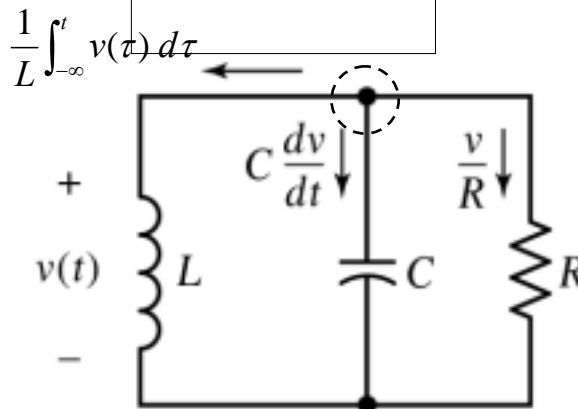


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Parallel LRC circuit, with ideal L, C and R

(衰減型弦波訊號電路範例)



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$$C \frac{dv(t)}{d} + \frac{v(t)}{R} + \frac{1}{L} \int_{-\infty}^t v(\tau) d\tau = 0$$

∴

$$C \frac{dv^2(t)}{dt^2} + \left[\frac{1}{R} \frac{dv(t)}{dt} + \frac{1}{L} v(t) \right] \tau = 0$$

Solution :

$$v(t) = V_0 e^{-\frac{t}{2RC}} \cos(\omega_0 t), \quad t \geq 0,$$

$$\omega_0 = \sqrt{\frac{1}{LC} - \frac{1}{4C^2 R^2}}$$

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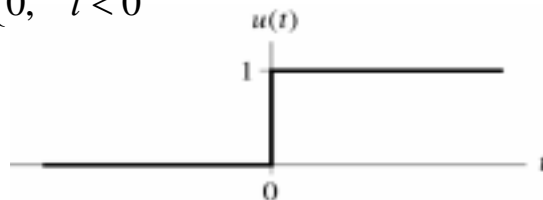
The unit-step function of unit amplitude

Continuous-time Step (function) Signal

(連續性時間步階訊號)

函數定義：

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$



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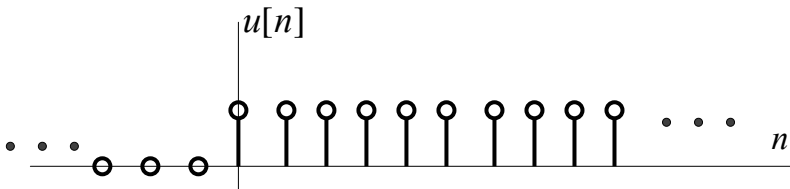
Discrete-Time Step Signal

Discrete-time Step (function) Signal

(離散性時間步階訊號)

函數定義：

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

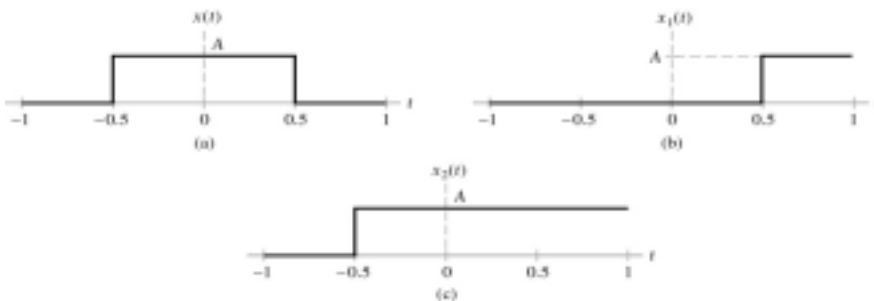


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The difference of two step functions:

Note that $x(t) = x_2(t) - x_1(t)$.

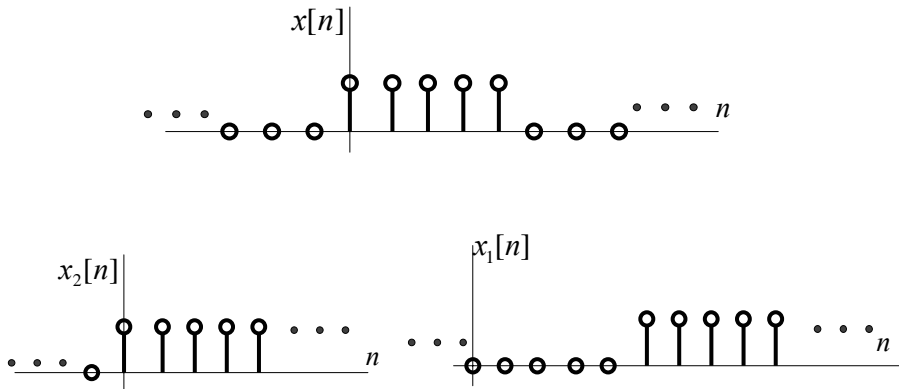


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The difference of two step functions:

Note that $x[n] = x_2[n] - x_1[n]$.



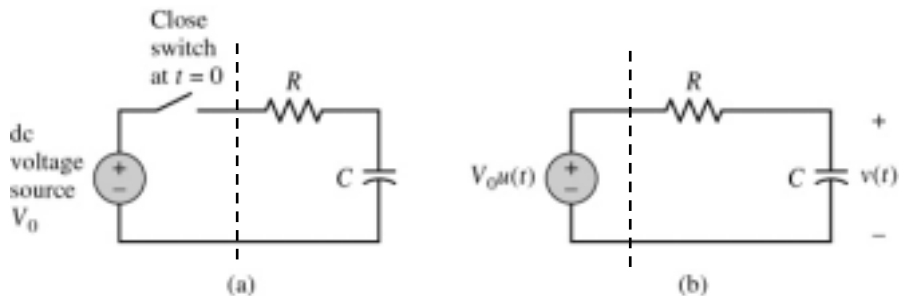
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(連續性時間步階訊號應用範例)

Example:

- Series RC circuit with a switch that is closed at time $t = 0$, thereby energizing the voltage source.
- Equivalent circuit, using a *step function* to replace the action of the switch.

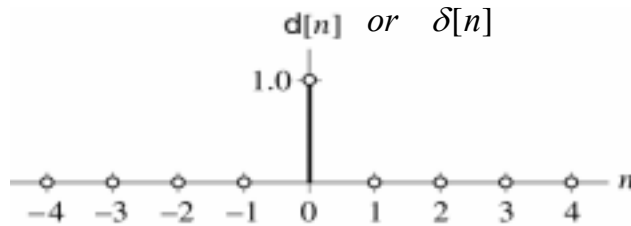


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Discrete-time form of Impulse

離散性時間脈衝訊號定義：
$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$



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Continuous-time form of Impulse

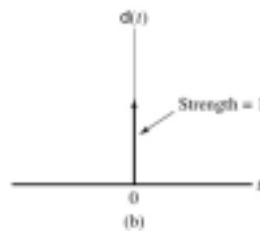
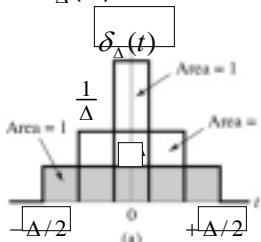
連續性時間脈衝訊號定義：

$$\delta_{\Delta}(t) = \begin{cases} \frac{1}{\Delta}, & -\frac{\Delta}{2} < t < +\frac{\Delta}{2} \\ 0, & \text{others} \end{cases}$$

$$\delta(t) = \lim_{\Delta \rightarrow 0} \delta_{\Delta}(t)$$

$$\delta(t) : \begin{cases} \int_{-\infty}^{+\infty} \delta(t) dt = 1, & t = 0 \\ 0, & t \neq 0 \end{cases}$$

$\delta_{\Delta}(t)$ 是集中於 $t=0$ 的連續時間極窄脈波(面積=1)



(c)

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$\delta(t)$ vs. $u(t)$

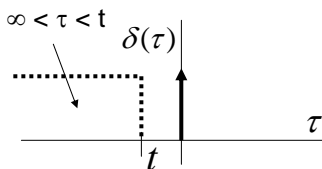
Unit-step function 定義 :

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

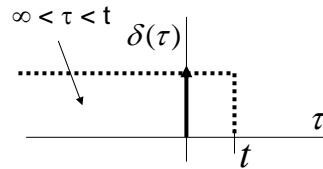
Continuous-time Impulse function 積分 :

$$\int_{-\infty}^t \delta(\tau) d\tau = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

積分區間 $t < 0$



積分區間 $t \geq 0$



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$$\therefore u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases} \quad \& \quad \int_{-\infty}^t \delta(\tau) d\tau = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

\therefore

$$u(t) = \int_{-\infty}^t \delta(\tau) d\tau$$

\therefore

$$\frac{d}{dt}u(t) = \frac{d}{dt} \int_{-\infty}^t \delta(\tau) d\tau = \delta(t)$$

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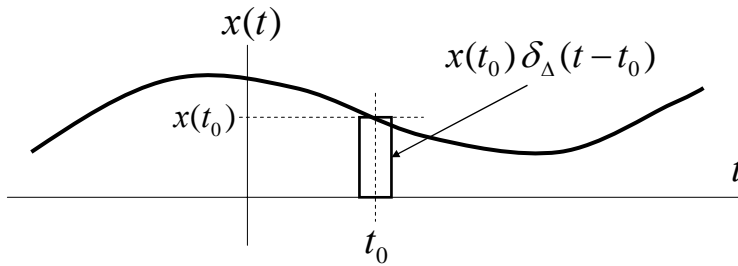


單位脈衝 $\delta(t)$ 時間平移特性

當 $t = t_0$ 時 $x(t)$ 是連續的，也是單位脈衝所在的時間

$$\int_{-\infty}^{\infty} x(t) \delta_{\Delta}(t - t_0) dt = x(t_0)$$

$\delta_{\Delta}(t - t_0)$ 是集中於 $t = t_0$ 的連續時間
極窄脈波 (面積 = 1)

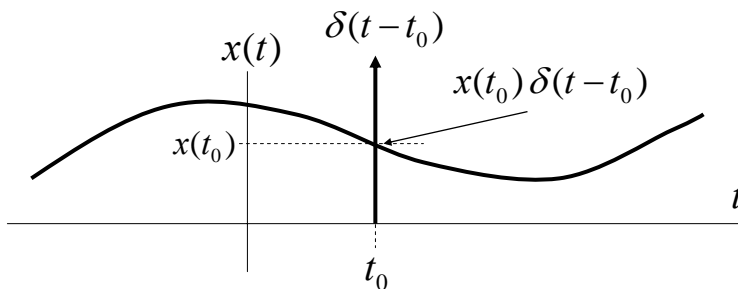


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當 $t = t_0$ 時 $x(t)$ 是連續的，也是單位脈衝所在的時間

$$\int_{-\infty}^{\infty} x(t) \delta(t - t_0) dt = x(t_0)$$



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單位脈衝 $\delta(t)$ 時間比例條調整特性

$$\delta(at) = \frac{1}{a} \delta(t), \quad a > 0$$

$$\therefore \delta_{\Delta}(t) = \begin{cases} \frac{1}{\Delta}, & -\frac{\Delta}{2} < t < +\frac{\Delta}{2} \\ 0, & \end{cases} \quad ; \quad \therefore \delta_{\Delta}(at) = \begin{cases} \frac{1}{\Delta}, & -\frac{\Delta}{2a} < t < +\frac{\Delta}{2a} \\ 0, & \end{cases}$$

壓縮版本

$$\text{area of } \delta_{\Delta}(t) = 1;$$

$$\text{area of } \delta_{\Delta}(at) = \frac{1}{a}$$

$$\therefore \delta(t) = \lim_{\Delta \rightarrow 0} \delta_{\Delta}(t) = \lim_{\Delta \rightarrow 0} a \cdot \delta_{\Delta}(at) = a \cdot \lim_{\Delta \rightarrow 0} \delta_{\Delta}(at) = a \cdot \delta(at)$$

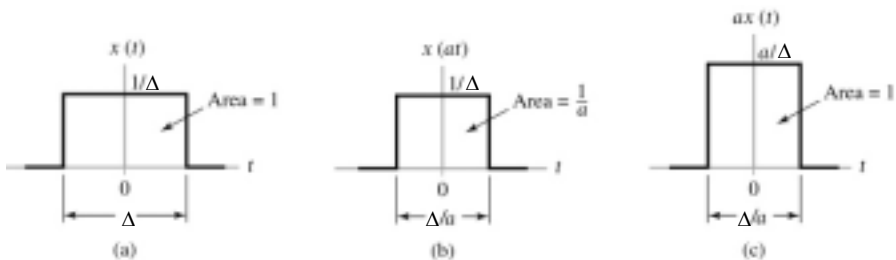
$$\therefore \delta(at) = \frac{1}{a} \delta(t)$$

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Steps involved in proving the time-scaling property of the unit impulse

- Rectangular pulse $x(t)$ of amplitude $1/\Delta$ and duration Δ , symmetric about the origin.
- Pulse $x(t)$ compressed by factor a .
- Amplitude scaling of the compressed pulse, restoring it to unit area.



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單位脈衝 $\delta[n]$ 與 Convolution Sum

任一離散時間系統輸入訊號 $x[n]$ 可以表成離散時間單位脈衝訊號 $\delta[n]$ 的加權疊加:

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k] \delta[n-k]$$

離散時間系統輸出訊號 $y[n]$ 可以表成離散時間脈衝響應 $h[n]$ 的加權疊加:

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k] h[n-k]$$

即為 Convolution Sum



單位脈衝 $\delta(t)$ 與 Convolution Integral

任一連續時間系統輸入訊號 $x(t)$ 可以表成連續時間單位脈衝訊號 $\delta(t)$ 的加權積分:

$$x(t) = \int_{-\infty}^{+\infty} x(\tau) \delta(t-\tau) d\tau$$

連續時間系統輸出訊號 $y(t)$ 可以表成連續時間脈衝響應 $h(t)$ 的加權積分:

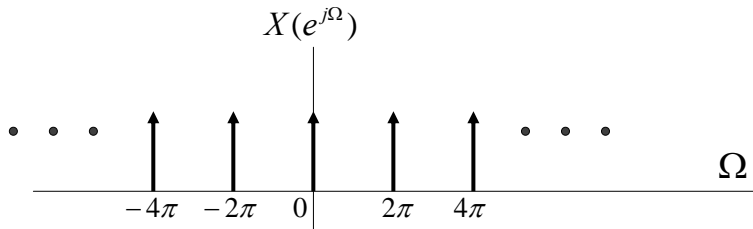
$$y(t) = \int_{-\infty}^{+\infty} x(\tau) h(t-\tau) d\tau$$

即為 Convolution Integral



連續單位脈衝 $\delta(\Omega)$ 序列

$$X(e^{j\Omega}) = \sum_{k=-\infty}^{+\infty} \delta(\Omega - 2\pi k)$$

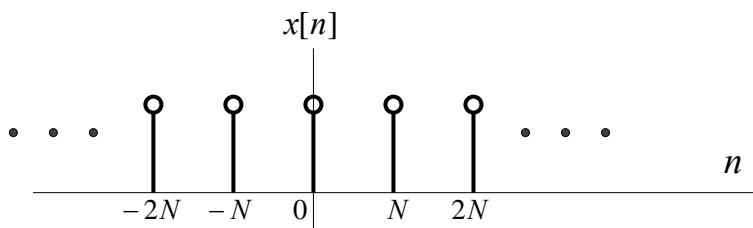


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離散時間單位脈衝 $\delta[n]$ 序列

$$x[n] = \sum_{k=-\infty}^{+\infty} \delta[n - kN]$$



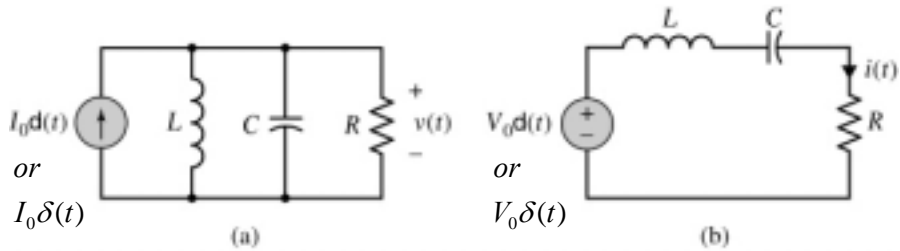
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(連續性時間脈衝訊號應用範例)

Example:

- (a) Parallel *LRC* circuit driven by an impulsive current signal.
- (b) Series *LRC* circuit driven by an impulsive voltage signal.



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離散時間單位脈衝 $\delta[n]$ 乘積與褶積

乘法性質：回憶 only for $n = k$, $\delta[n-k] = 1$.

$$x[n] \cdot \delta[n-k] = x[k] \cdot \delta[n-k] = x[k]$$

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離散時間單位脈衝 $\delta[n]$ 乘積與褶積

褶積性質：

$$x[n] * \delta[n - n_0] = \sum_{k=-\infty}^{+\infty} \delta[k - n_0] x[n - k]$$

$\therefore k = n_0$ only,

\therefore

$$x[n] * \delta[n - n_0] = x[n - n_0]$$

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連續時間單位脈衝 $\delta(t)$ 乘積與褶積

乘法性質：

$$x(t) \cdot \delta(t - \tau) = x(\tau) \cdot \delta(t - \tau)$$

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連續時間單位脈衝 $\delta(t)$ 乘積與褶積

褶積性質：

$$x(t) * \delta(t - t_0) = \int_{-\infty}^{+\infty} \delta(\tau - t_0) x(t - \tau) d\tau$$

$\because \tau = t_0$ only,

\therefore

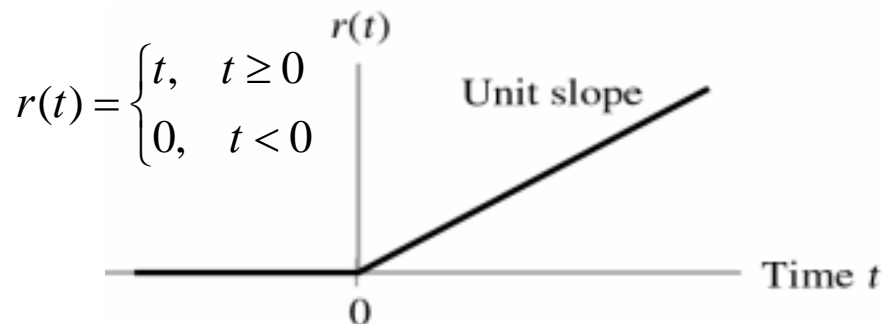
$$x(t) * \delta(t - t_0) = x(t - t_0) \int_{-\infty}^{+\infty} \delta(\tau - t_0) d\tau = x(t - t_0)$$

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Ramp function of unit slope

(連續性時間斜坡訊號)



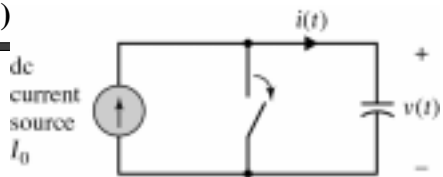
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(連續性時間斜坡訊號應用範例)

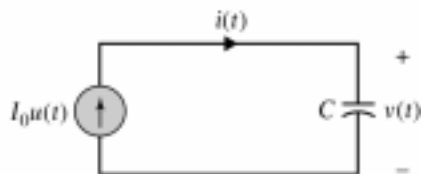
(a) Parallel circuit consisting of a current source, switch, and capacitor, the capacitor is initially assumed to be uncharged, and the switch is opened at time $t = 0$.

(b) Equivalent circuit replacing the action



Switch
is opened
at $t = 0$

(a)



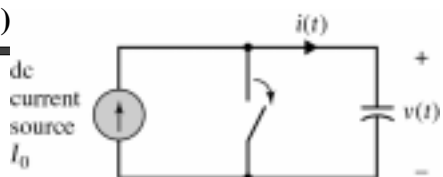
(b)

65



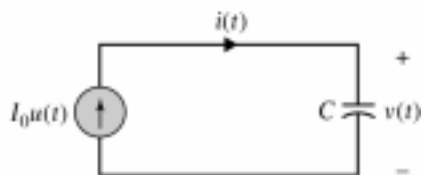
(連續性時間斜坡訊號應用範例)

(b) Equivalent circuit replacing the action
of opening the switch with the step
function $u(t)$.



Switch
is opened
at $t = 0$

(a)



(b)

66



連續性時間斜坡訊號推導

$$\begin{aligned}v(t) &= \frac{1}{C} \int_{-\infty}^t i(\tau) d\tau = \frac{1}{C} \int_{-\infty}^t [I_0 u(\tau)] d\tau \\ &= \begin{cases} \frac{I_0}{C} t, & t \geq 0 \\ 0, & t < 0 \end{cases} \\ &= \frac{I_0}{C} tu(t) = \frac{I_0}{C} r(t)\end{aligned}$$