



Lecture 2-1

Linear Time-Invariant System (LTI System) 線性非時變系統

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Review

Linear Property:

- Scaling
 - $x(t) \rightarrow y(t)$
 - $c x(t) \rightarrow c y(t)$
- Superposition
 - $x_1(t) \rightarrow y_1(t)$
 - $x_2(t) \rightarrow y_2(t)$
 - $x_1(t)+x_2(t) \rightarrow y_1(t)+y_2(t)$

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Review

Time-Invariant Property:

- $x(t) \rightarrow y(t)$
- $x(t - t_0) \rightarrow y(t - t_0)$



Overview

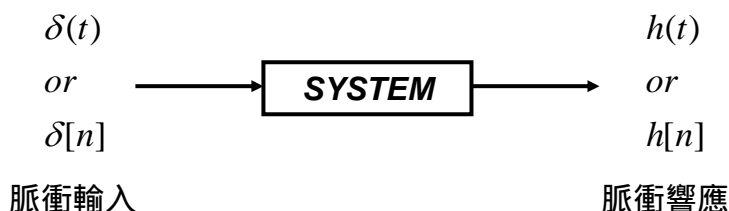
- 褶積和 (Convolution Sum)
- 褶積和計算程序 (~ Evaluation Procedure)
- 褶積積分 (Convolution Integral)
- 褶積積分計算程序 (~ Evaluation Procedure)
- LTI 串、並聯連結
- LTI 系統特性
- 步階響應(Step Response)
- 微分、差分方程式表示LTI系統
- MATLAB 實作



• **脈衝響應 (Impulse Response):**

–系統對於一個脈衝函數 (訊號) δ 輸入所做出的輸出反應。

–實物上，為系統對於一個持續時間非常短的高振幅輸入所做出的反應。



•任何系統輸入可表成時間平移脈衝訊號的加權疊加，系統輸出必為時間平移脈衝響應的加權疊加。

•離散時間系統的加權疊加以來“摺積和” (convolution sum)表示。

•連續時間系統的加權疊加以來“摺積積分” (convolution Integral)表示。



脈衝訊號時間平移的加權疊加

$$\therefore \delta[n] = \begin{cases} 1, & n = 0 \\ 0, & \textit{otherwise} \end{cases}, \quad \therefore x[n] \delta[n] = x[0] \delta[n]$$

注意：x[n] 為因應不同自變數 n 而獲得的應變數值

注意：x[0] 為自變數 n=0 而獲得的應變數值 = constant

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脈衝訊號時間平移的加權疊加

$$\therefore \delta[n] = \begin{cases} 1, & n = 0 \\ 0, & \textit{otherwise} \end{cases}, \quad \therefore x[n] \delta[n] = x[0] \delta[n]$$

$$\therefore x[n] \delta[n-k] = x[k] \delta[n-k]$$

注意：x[k] 為自變數 n=k

而獲得的應變數值 = constant

時間平移 k 的脈衝訊號

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脈衝訊號時間平移的加權疊加

$$\therefore \delta[n] = \begin{cases} 1, & n = 0 \\ 0, & \text{otherwise} \end{cases}, \quad \therefore x[n] \delta[n] = x[0] \delta[n]$$

$$\therefore x[n] \delta[n - k] = x[k] \delta[n - k]$$

$$x[n] = \dots + x[-1] \delta[n + 1] + x[0] \delta[n] + x[1] \delta[n - 1] + \dots$$

$$= \sum_{k=-\infty}^{+\infty} x[k] \delta[n - k]$$

$x[n]$ 可寫成時間平移脈衝的加權疊加

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If the system input is $x[n]$ and the $H\{\}$ represents the system, the system output $y[n]$ will be,

$$y[n] = H\{x[n]\}$$

$$= H\left\{\sum_{k=-\infty}^{+\infty} x[k] \delta[n - k]\right\}, \quad \text{考慮線性特性: 疊加性}$$



If the system input is $x[n]$ and the $H\{\}$ represents the system, the system output $y[n]$ will be,

$$y[n] = H\{x[n]\}$$

$$= H\left\{\sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]\right\}, \text{ 考慮線性特性: 疊加性}$$

$$= \sum_{k=-\infty}^{+\infty} H\{x[k]\delta[n-k]\}, \text{ 考慮線性特性: Scaling}$$



If the system input is $x[n]$ and the $H\{\}$ represents the system, the system output $y[n]$ will be,

$$y[n] = H\{x[n]\}$$

$$= H\left\{\sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]\right\}, \text{ 考慮線性特性: 疊加性}$$

$$= \sum_{k=-\infty}^{+\infty} H\{x[k]\delta[n-k]\}, \text{ 考慮線性特性: Scaling}$$

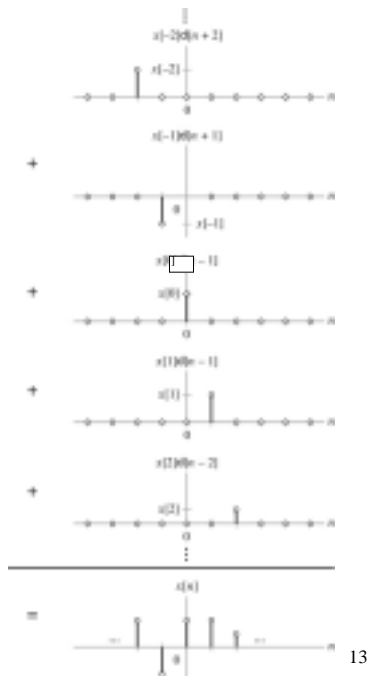
$$= \sum_{k=-\infty}^{+\infty} x[k]H\{\delta[n-k]\}, \quad \because \text{let } h[n] \text{ be the impulse response of } \delta[n].$$

$$= \sum_{k=-\infty}^{+\infty} x[k]h[n-k].$$



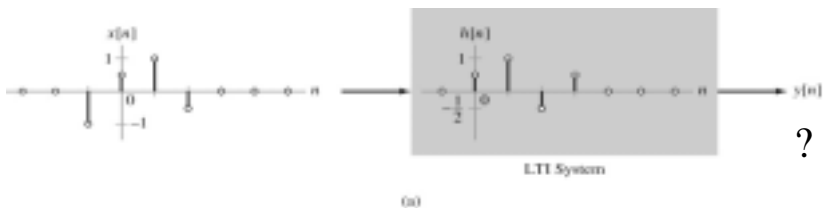
輸入訊號時間平移的加權疊加圖解說明

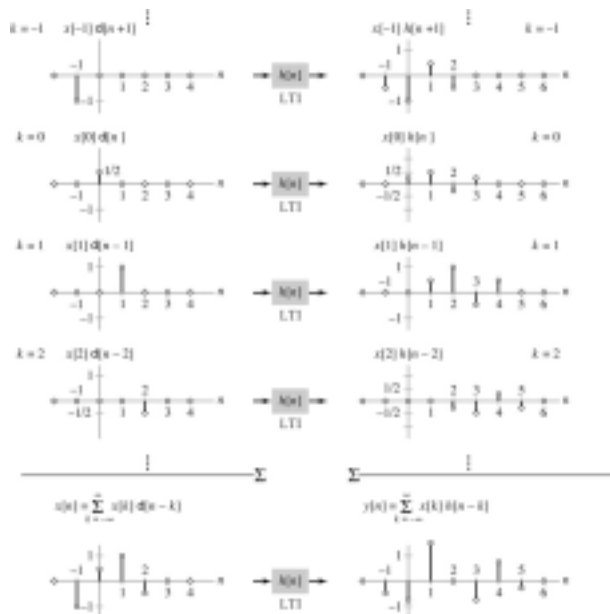
$$x[n] = \sum_{k=-\infty}^{+\infty} x[k] \delta[n-k]$$



Notation of the Convolution Sum

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = x[n] * h[n]$$





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$$x[n] = x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2]$$

$$h[n] = h[0]\delta[n] + h[1]\delta[n-1] + h[2]\delta[n-2] + h[3]\delta[n-3]$$

$$x[-1]\delta[n+1] \rightarrow x[-1]\delta[n+1] * h[n] = x[-1]h[n+1]$$

$$x[0]\delta[n] \rightarrow x[0]\delta[n] * h[n] = x[0]h[n]$$

$$x[1]\delta[n-1] \rightarrow x[1]\delta[n-1] * h[n] = x[1]h[n-1]$$

$$x[2]\delta[n-2] \rightarrow x[2]\delta[n-2] * h[n] = x[2]h[n-2]$$

$$y[n] = x[-1]h[n+1] + x[0]h[n] + x[1]h[n-1] + x[2]h[n-2]$$

$$= \sum_{k=-1}^2 x[k]h[n-k]$$

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Example 2.2 摺積和 計算練習

系統脈衝響應，輸入如下：

$$h[n] = \left(\frac{3}{4}\right)^n u[n], \quad x[n] = u[n],$$

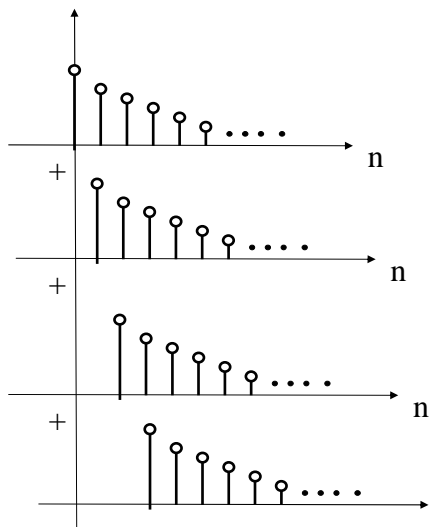
$$y[n] = ? \quad y[-5] = ?, \quad y[5] = ?, \quad y[10] = ?$$

系統輸出為何？當 $n = -5, +5, +10$ 時

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對應輸入，系統脈衝響應： $y[n] = ?$



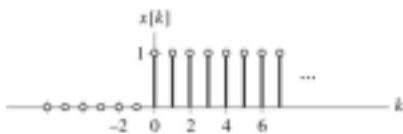
$$x[0]\delta[n] * h[n] = x[0]h[n]$$

$$x[1]\delta[n-1] * h[n] = x[1]h[n-1]$$

$$x[2]\delta[n-2] * h[n] = x[2]h[n-2]$$

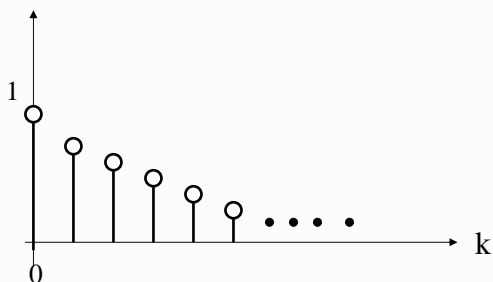
⋮

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$$x[k] = u[k]$$

$$h[k] = \left(\frac{3}{4}\right)^k u[k],$$

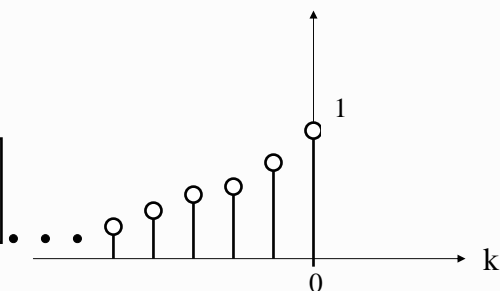


先找出 $x[k]$ 和 $h[k] = ?$

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$$h[-k] = \left(\frac{3}{4}\right)^{-k} u[-k]$$

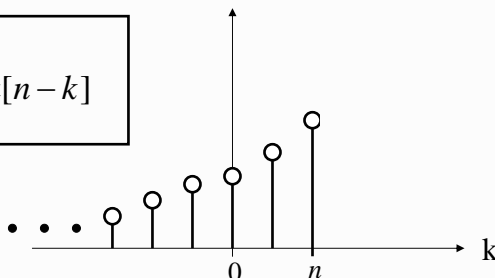


再找出反射運算後 $h[-k] = ?$

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$$h[n-k] = \left(\frac{3}{4}\right)^{n-k} u[n-k]$$



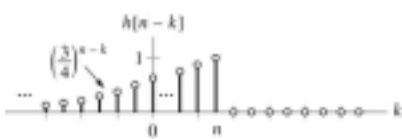
再找出反射運算以及時間平移後 $h[-k+n] = ?$



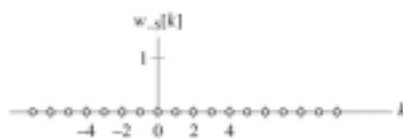
$$x[k] = u[k]$$

$$h[n-k] = \left(\frac{3}{4}\right)^{n-k} u[n-k]$$

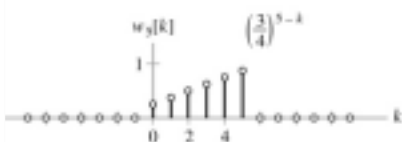
$$w_n[k] = u[k]h[n-k]$$



(a)



(b) $n = -5$



(c) $n = 5$



(d) $n = 10$



$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = \sum_{k=-\infty}^{+\infty} u[k] \left(\frac{3}{4}\right)^{n-k} u[n-k]$$

$$= \begin{cases} \sum_{k=0}^n \left(\frac{3}{4}\right)^{n-k}, & n \geq 0 \\ 0, & \textit{otherwise} \end{cases}$$



$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = \sum_{k=-\infty}^{+\infty} u[k] \left(\frac{3}{4}\right)^{n-k} u[n-k]$$

$$= \begin{cases} \sum_{k=0}^n \left(\frac{3}{4}\right)^{n-k}, & n \geq 0 \\ 0, & \textit{otherwise} \end{cases}$$

∴

(1) $y[-5] = 0$

(2) $y[5] = \sum_{k=0}^5 \left(\frac{3}{4}\right)^{5-k} = \left(\frac{3}{4}\right)^5 \sum_{k=0}^5 \left(\frac{3}{4}\right)^{-k} = \left(\frac{3}{4}\right)^5 \sum_{k=0}^5 \left(\frac{4}{3}\right)^k$

$$= \left(\frac{3}{4}\right)^5 \frac{1 - \left(\frac{4}{3}\right)^6}{1 - \frac{4}{3}} = 3.288 \quad \text{有限幾何級數}$$

(3) $y[10] = ?$



$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = \sum_{k=-\infty}^{+\infty} u[k] \left(\frac{3}{4}\right)^{n-k} u[n-k]$$

$$= \begin{cases} \sum_{k=0}^n \left(\frac{3}{4}\right)^{n-k}, & n \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

∴

(1) $y[-5] = 0$

(2) $y[5] = \sum_{k=0}^5 \left(\frac{3}{4}\right)^{5-k} = \left(\frac{3}{4}\right)^5 \sum_{k=0}^5 \left(\frac{3}{4}\right)^{-k} = \left(\frac{3}{4}\right)^5 \sum_{k=0}^5 \left(\frac{4}{3}\right)^k$

$$= \left(\frac{3}{4}\right)^5 \frac{1 - \left(\frac{4}{3}\right)^6}{1 - \frac{4}{3}} = 3.288 \quad \text{有限幾何級數}$$

(3) $y[10] = ?$ 學生試一試

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Example 2.3 系統 脈衝響應 = ?

系統輸出入關係如右: $y[n] = \frac{1}{4} \sum_{k=0}^3 x[n-k]$

令輸入 $x[n] = \delta[n]$, 輸出即為系統 脈衝響應.

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Example 2.3 (cont.)

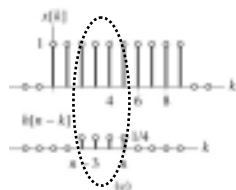
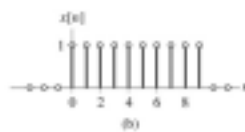
系統輸出入關係如右:
$$y[n] = \frac{1}{4} \sum_{k=0}^3 x[n-k]$$

令輸入 $x[n] = \delta[n]$, 輸出即為系統脈衝響應:

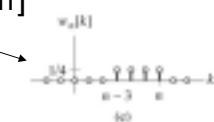
$$\begin{aligned} h[n] &= \frac{1}{4} \sum_{k=0}^3 \delta[n-k] = \frac{1}{4} (\delta[n] + \delta[n-1] + \dots + \delta[n-3]) \\ &= \frac{1}{4} (u[n] - u[n-4]) \end{aligned}$$

今若輸入: $x[n] = u[n] - u[n-10]$, 輸出 = ?

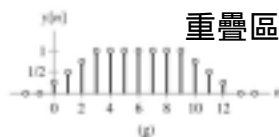
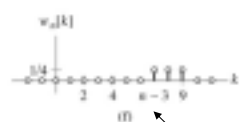
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重疊區間 $k: [0 \sim n]$



重疊區間 $k: [n-3 \sim n]$



重疊區間 $k: [n-3 \sim 9]$

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Solution:
$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

(1) $n < 0$, $y[n] = 0$, no overlay

(2) $0 \leq n \leq 3$, $y[n] = \sum_{k=0}^n \frac{1}{4}$ overlay $k : 0 \sim n$

(3) $4 \leq n \leq 9$, $y[n] = \sum_{k=n-3}^n \frac{1}{4}$ overlay $k : (n-3) \sim n$

(4) $10 \leq n \leq 12$, $y[n] = \sum_{k=n-3}^9 \frac{1}{4}$ overlay $k : (n-3) \sim 9$

(5) $n > 12$, $y[n] = 0$, no overlay

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Example 2.4 迴歸性系統

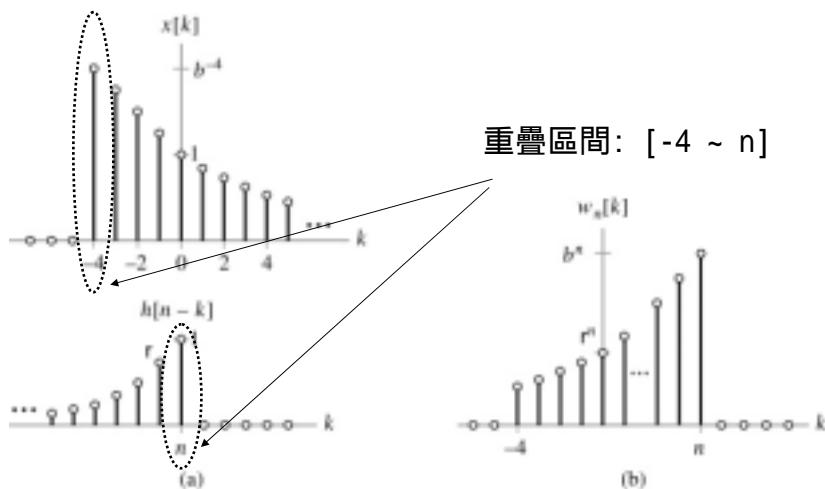
系統輸出入關係如右: $y[n] - \rho y[n-1] = x[n]$

若輸入 $x[n] = b^n u[n+4]$, 輸出 = ?

(系統為已知 因果性系統 , 且 $b \neq \rho$)

Hint: for a causal system, $h[n] = 0$ for $n < 0$.

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Solution: impulse response $h[n]$: $h[n] = \rho h[n-1] + \delta[n]$

\therefore causal system: $h[n] = 0$, for $n < 0$

$$h[0] = \delta[0] = 1$$

$$h[1] = \rho h[0] + \delta[1] = \rho$$

$$h[2] = \rho h[1] + \delta[2] = \rho^2$$

\vdots

$$h[n] = \rho^n u[n]$$

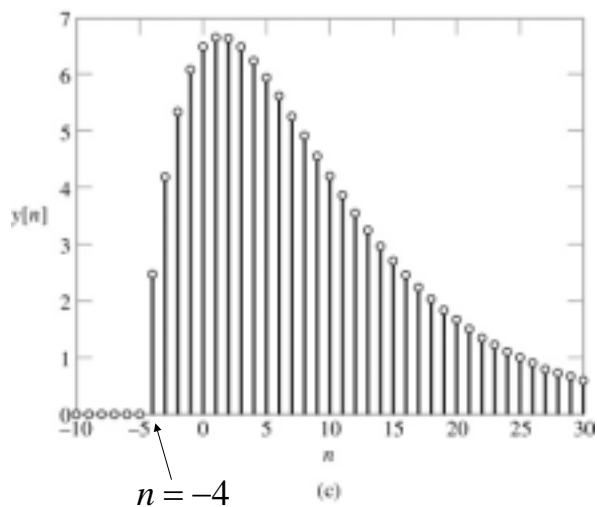
若輸入 $x[n] = b^n u[n+4]$ 重疊區間: $[-4, n]$

$$y[n] = x[n] * h[n] = \{\rho^n u[n]\} * \{b^n u[n+4]\} = \sum_{k=-4}^n b^k \rho^{n-k}$$

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Plot $y[n]$, $\rho = 0.9$ and $b = 0.8$.



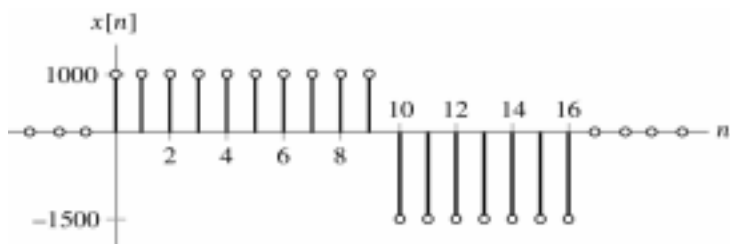
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Example 2.5 (Applies Ex2.4 solution)

在一投資帳戶中前10年皆存入1000元，後7年則皆提出1500元，若年利率是8%，請問餘額 = ？

帳戶存取可用下圖表示：



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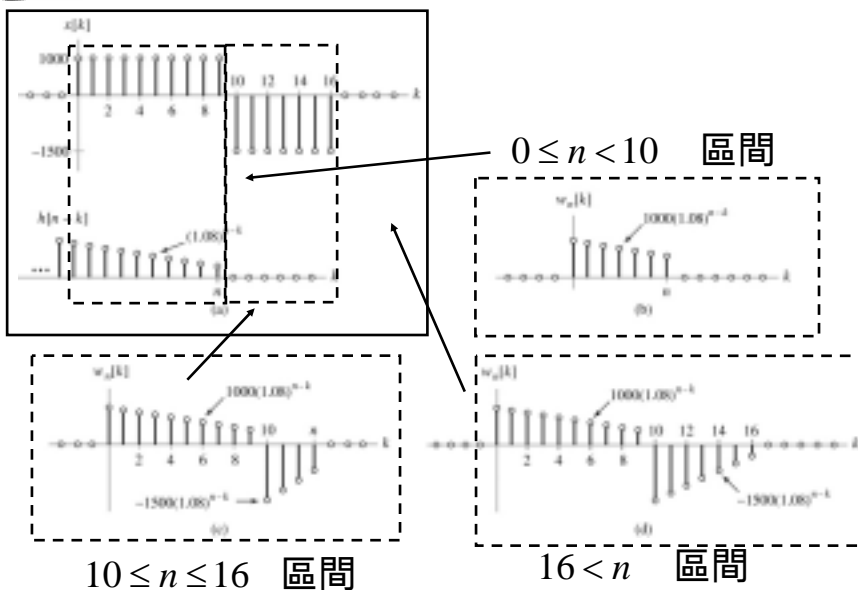
在投資帳戶中，餘額計算可用一階遞迴方程式表示：

$$y[n] - \rho y[n-1] = x[n]$$

若存款利率：8%， ρ 為 1.08

在 Ex2.4 中得知系統脈衝響應： $h[n] = \rho^n u[n] = (1.08)^n u[n]$

餘額可由 convolution sum 求得 $y[n] = x[n] * h[n] = ?$





(1) $n < 0$, $y[n] = 0$, no overlay

$$(2) 0 \leq n \leq 9, \quad y[n] = \sum_{k=0}^n 1000(1.08)^{n-k} = 12500\left((1.08)^{n+1} - 1\right)$$

$$(3) 10 \leq n \leq 16, \quad y[n] = \sum_{k=0}^9 1000(1.08)^{n-k} - \sum_{k=10}^n 1500(1.08)^{n-k}$$

$$= 7246.89(1.08)^n - 18750\left((1.08)^{n-9} - 1\right)$$

$$(4) n \geq 17 \quad y[n] = \sum_{k=0}^9 1000(1.08)^{n-k} - \sum_{k=10}^{16} 1500(1.08)^{n-k}$$

$$= 3340.17(1.08)^n$$



計算式參考：

$$\sum_{k=0}^9 1000(1.08)^{n-k}$$

$$= 1000(1.08)^n \sum_{k=0}^9 (1.08)^{-k} = 1000(1.08)^n \sum_{k=0}^9 (1.08^{-1})^k$$

$$= 1000(1.08)^n \frac{1 - (1.08^{-1})^{10}}{1 - 1.08^{-1}}; \quad \text{finite geometry series}$$

$$= 1000(1.08)^n \frac{0.5368065}{1 - 0.9259259} = 1000(1.08)^n (7.2468852)$$

$$\approx 7246.89(1.08)^n$$



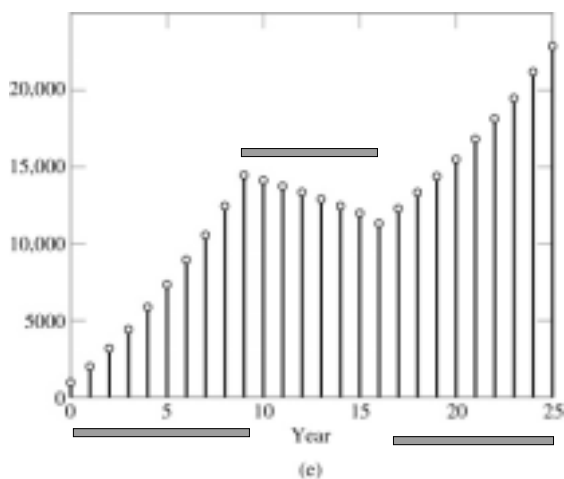
計算式參考：

$$\begin{aligned}
 & \sum_{k=10}^n 1500(1.08)^{n-k}, \quad \text{let } m = k - 10, \\
 &= 1500(1.08)^n \sum_{k=10}^n (1.08)^{-k} = 1500(1.08)^n \sum_{m=0}^{n-10} (1.08^{-1})^{m+10}, \\
 &= 1500(1.08)^{n-10} \sum_{m=0}^{n-10} (1.08^{-1})^m = 1500(1.08)^{n-10} \frac{1 - (1.08^{-1})^{n-9}}{1 - 1.08^{-1}}, \\
 &= 1500(1.08)^{n-9} (1.08^{-1}) \frac{1 - (1.08^{-1})^{n-9}}{1 - 1.08^{-1}}, \\
 &= \frac{1500(1.08)^{n-9} (1.08^{-1})}{1 - 1.08^{-1}} - \frac{1500(1.08^{-1})}{1 - 1.08^{-1}}, \\
 &= \frac{1500(1.08^{-1})}{1 - 1.08^{-1}} \left\{ (1.08)^{n-9} - 1 \right\} = 18750 \left\{ (1.08)^{n-9} - 1 \right\}
 \end{aligned}$$

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The $y[n]$ represents the value of the investment immediately after the deposit or withdrawal at the start of year n .



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EX: $Y[n] = x[n]*h[n] = ?$

