



## Lecture 2-2

### Linear Time-Invariant System (LTI System) 線性非時變系統

1



## Convolution Integral

連續時間訊號也可表示為時間平移脈衝的加權疊加 (積分) :

$$x(t) = \int_{-\infty}^{+\infty} x(\tau) \delta(t - \tau) d\tau$$

連續時間訊號的系統輸出 :

$$y(t) = H\{x(t)\} = H\left\{\int_{-\infty}^{+\infty} x(\tau) \delta(t - \tau) d\tau\right\}$$

2



由於線性特性，連續時間訊號的系統輸出：

$$y(t) = H\{x(t)\} = \int_{-\infty}^{+\infty} x(\tau) H\{\delta(t-\tau)\} d\tau$$

定義系統對單一脈衝輸入的響應： $H\{\delta(t)\} = h(t)$

定義系統滿足非時變性： $H\{\delta(t-\tau)\} = h(t-\tau)$

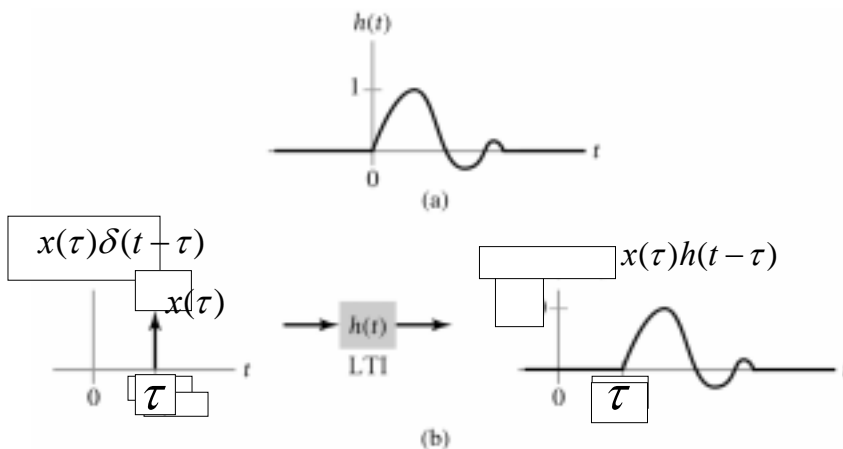
因此在 LTI 系統，連續時間訊號的系統輸出：

$$y(t) = \int_{-\infty}^{+\infty} x(\tau) h(t-\tau) d\tau = x(t) * h(t)$$

3



Convolution Integral 圖解說明：



4

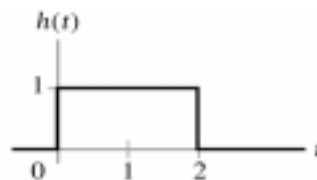
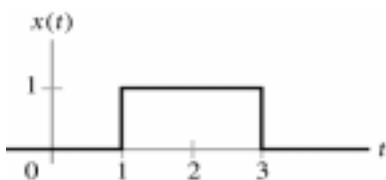


Convolution Integral 計算程序： $y(t) = \int_{-\infty}^{+\infty} x(\tau) h(t-\tau) d\tau$

- (1) 畫出  $x(\tau)$  和  $h(\tau)$  圖形
- (2) 應用反射特性獲得  $h(-\tau)$  圖形
- (3) 將  $h(-\tau)$  圖形向  $t$  平移獲得  $h(t-\tau)$  圖形
- (4) 若  $t > 0$ ，向右平移，若  $t < 0$ ，向左平移
- (5) 寫出中繼式： $w_i(\tau) = x(\tau)h(t-\tau)$
- (6) 將對每個平移  $t$  區間，做積分運算： $\int_{-\infty}^{+\infty} w_i(\tau) d\tau$
- (7) 積分區域： $\tau = -\infty \sim +\infty$

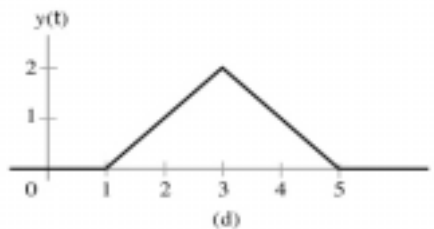
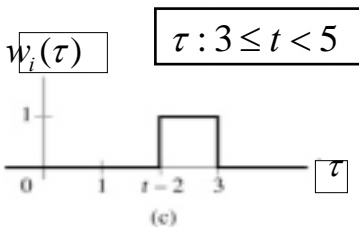
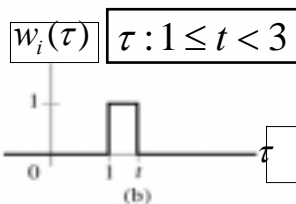
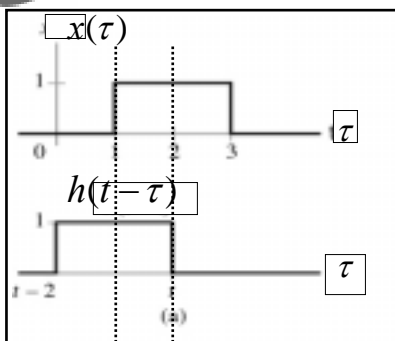
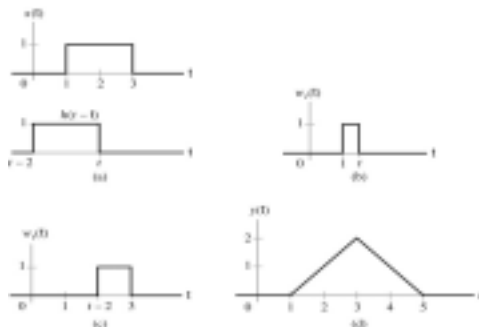


EX2.6:  $y(t) = x(t) * h(t) = ?$



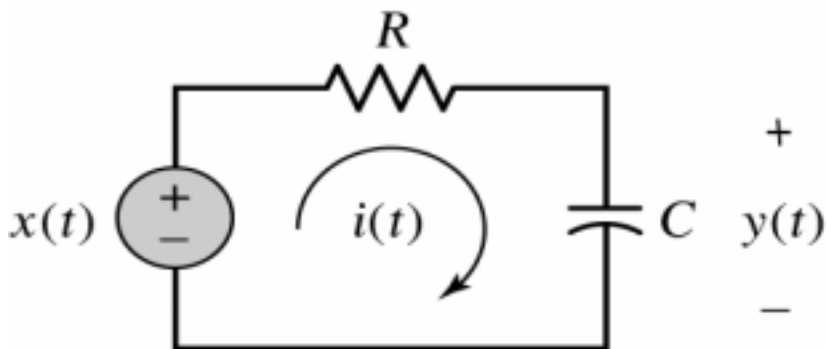


Find  $x(\tau)$  and the reflected, time-shifted impulse,  $h(t-\tau)$ .  
 Find the product signal  $w_i(\tau)$  for  $1 \leq t < 3$ .  
 Find the product signal  $w_i(\tau)$  for  $3 \leq t < 5$ .  
 Find the system output  $y(t)$ .





EX: A RC circuit system with the voltage source  $x(t)$  as input and the voltage measured across the capacitor  $y(t)$ , as output.

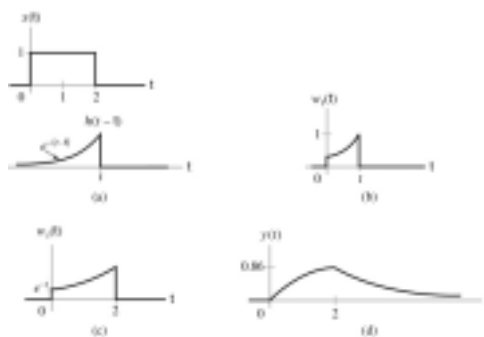


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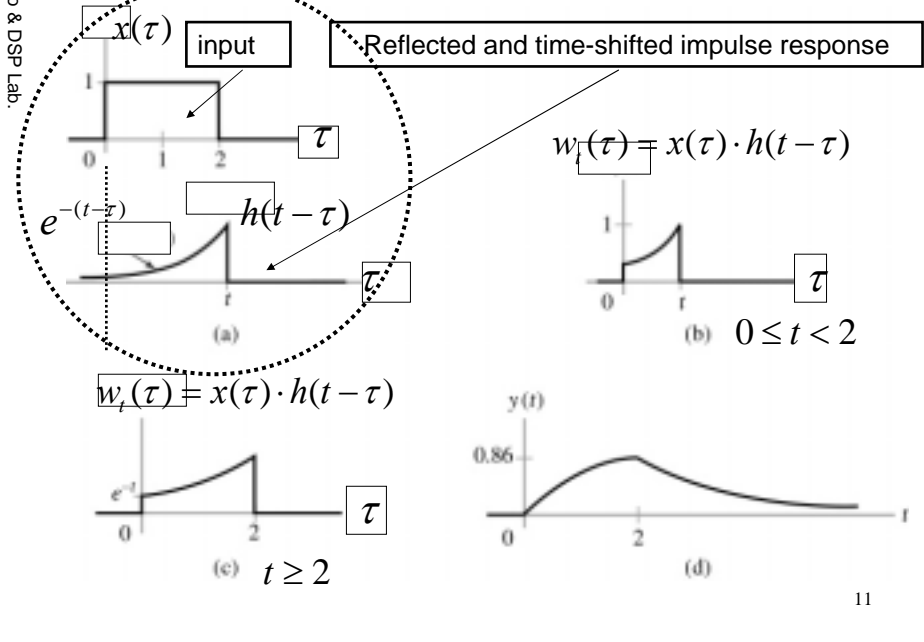


Ex 2.7:

- The input  $x(\tau)$  superimposed over the reflected and time-shifted impulse response  $h(t - \tau)$ , depicted as a function of  $\tau$ .
- The product signal  $w_1(\tau)$  for  $0 \leq t < 2$ .
- The product signal  $w_2(\tau)$  for  $t \geq 2$ .
- The system output  $y(t)$ .

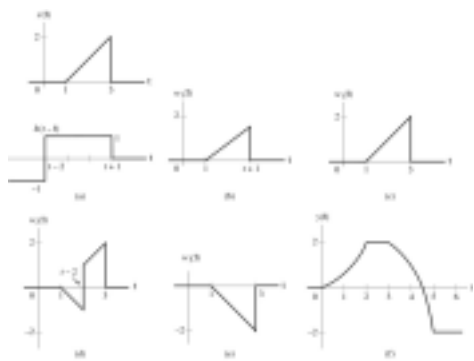


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Ex 2.8:

- (a) The input  $x(\tau)$  superimposed on the reflected and time-shifted impulse response  $h(t-\tau)$ , depicted as a function of  $\tau$ .
- (b) The product signal  $w_A(\tau)$  for  $0 \leq t < 2$ .
- (c) The product signal  $w_A(\tau)$  for  $2 \leq t < 3$ .
- (d) The product signal  $w_A(\tau)$  for  $3 \leq t < 5$ .
- (e) The product signal  $w_A(\tau)$  for  $t \geq 5$ . The system output  $y(t)$ .





$x(\tau)$

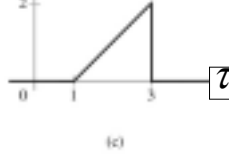


- (b)  $w_t(\tau)$  for  $0 \leq t < 2$ .
- (c)  $w_t(\tau)$  for  $2 \leq t < 3$ .
- (d)  $w_t(\tau)$  for  $3 \leq t < 5$ .
- (e)  $w_t(\tau)$  for  $t \geq 5$ .

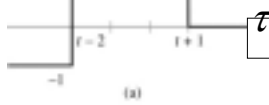
$w_t(\tau)$



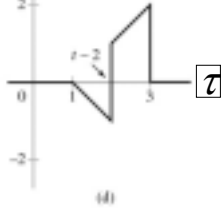
$w_t(\tau)$



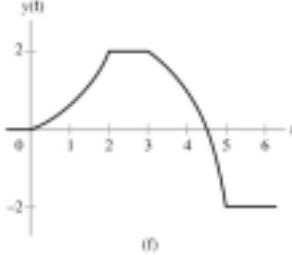
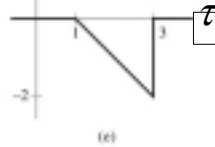
$h(t-\tau)$



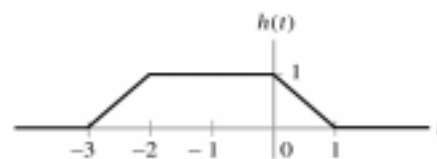
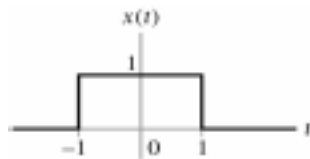
$w_t(\tau)$



$w_t(\tau)$



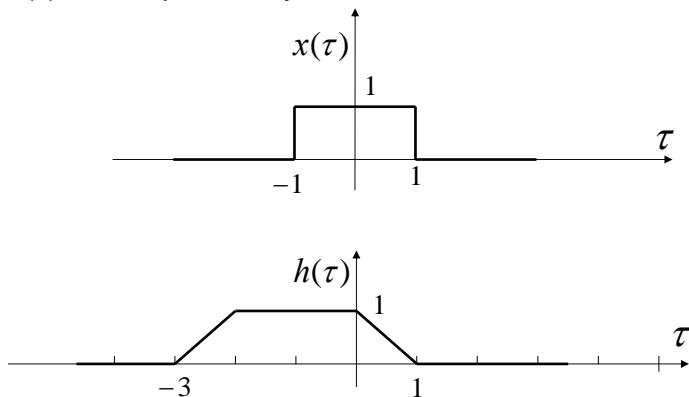
P2.5: 試求系統輸出： $y(t) = x(t) * h(t)$





### Step 1:

Produce new set of dependent variables,  $x(\tau)$  and  $h(\tau)$  with replace  $t$  by  $\tau$ .

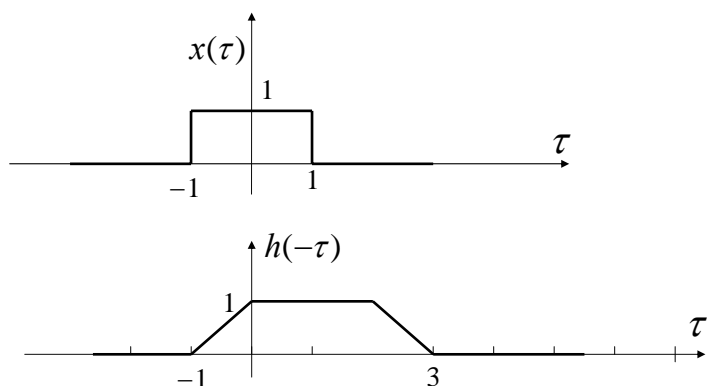


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### Step 2:

Obtain the  $h(-\tau)$  from  $h(\tau)$  by reflection operation.



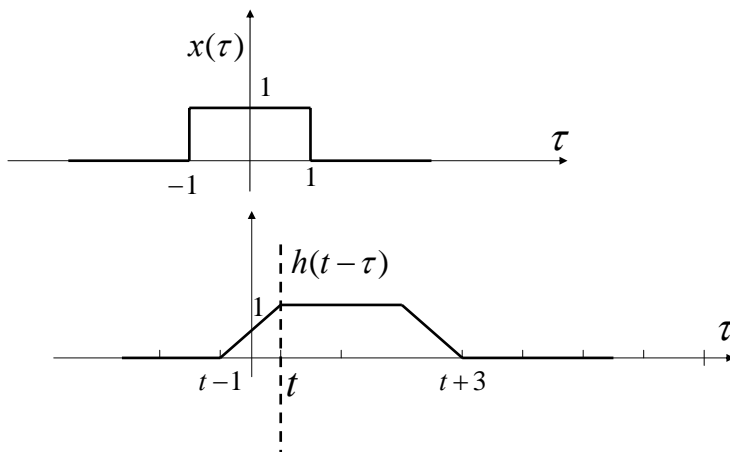
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## Step 3:

Obtain the  $h(t-\tau)$  from  $h(-\tau)$  by time-shifting operation.



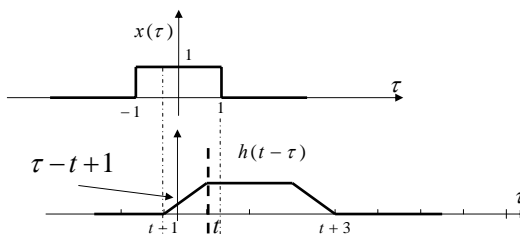
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## 詳細推導

e.g.,

if  $0 \leq t < 1$ ,



$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau) d\tau = \int_{t-1}^t (\tau-t+1) d\tau + \int_t^1 1 d\tau$$

$$= \left( \frac{\tau^2}{2} - \tau(t-1) \right) \Big|_{t-1}^t + \left( \tau \right) \Big|_t^1$$

$$= \frac{t^2}{2} - t(t-1) - \frac{(t-1)^2}{2} + (t-1)^2 + (1-t)$$

$$= \frac{t^2}{2} - t^2 + t - \frac{t^2}{2} + t - \frac{1}{2} + t^2 - 2t + 1 + 1 - t$$

$$= \frac{3}{2} - t$$

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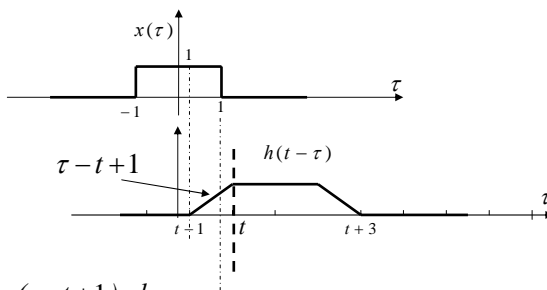
### 詳細推導

e.g.,

if  $1 \leq t < 2$ ,

$$\begin{aligned}
 y(t) &= \int_{-\infty}^{+\infty} x(\tau)h(t-\tau) d\tau = \int_{t-1}^1 (\tau-t+1) d\tau \\
 &= \left( \frac{\tau^2}{2} - \tau(t-1) \right) \Big|_{t-1}^1 \\
 &= \frac{1}{2} - (t-1) - \frac{(t-1)^2}{2} + (t-1)^2 \\
 &= \frac{1}{2} - t + 1 - \frac{t^2}{2} + t - \frac{1}{2} + t^2 - 2t + 1 \\
 &= \frac{t^2}{2} - 2t + 2
 \end{aligned}$$

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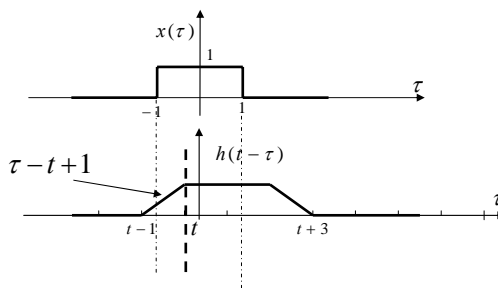
### 詳細推導

e.g.,

if  $-1 \leq t < 0$ ,

$$\begin{aligned}
 y(t) &= \int_{-\infty}^{+\infty} x(\tau)h(t-\tau) d\tau = \int_{-1}^t (\tau-t+1) d\tau + \int_t^1 1 d\tau \\
 &= \left( \frac{\tau^2}{2} - \tau(t-1) \right) \Big|_{-1}^t + \left( \tau \right) \Big|_t^1 \\
 &= \frac{t^2}{2} - t(t-1) - \frac{(-1)^2}{2} - (t-1) + (1-t) \\
 &= \frac{t^2}{2} - t^2 + t - \frac{1}{2} - t + 1 + 1 - t \\
 &= -\frac{t^2}{2} - t + \frac{3}{2}
 \end{aligned}$$

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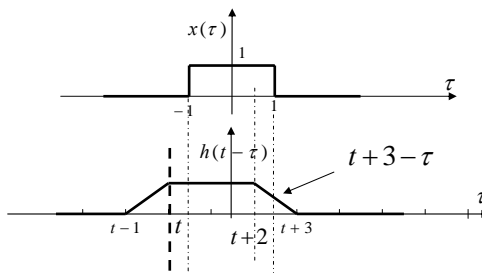




### 詳細推導

e.g.,

if  $-2 \leq t < -1$ ,



$$\begin{aligned}
 y(t) &= \int_{-\infty}^{+\infty} x(\tau)h(t-\tau) d\tau = \int_{t+2}^1 (t+3-\tau) d\tau + \int_{-1}^{t+2} 1 d\tau \\
 &= \left(\tau(3+t) - \frac{\tau^2}{2}\right) \Big|_{t+2}^1 + \left(\tau\right) \Big|_{-1}^{t+2} \\
 &= (3+t) - \frac{1}{2} - (t+2)(3+t) + \frac{(t+2)^2}{2} + (t+2+1) \\
 &= 3+t - \frac{1}{2} - t^2 - 5t - 6 + \frac{t^2}{2} + 2t + 2 + t + 3 \\
 &= -\frac{t^2}{2} - t + \frac{3}{2}
 \end{aligned}$$

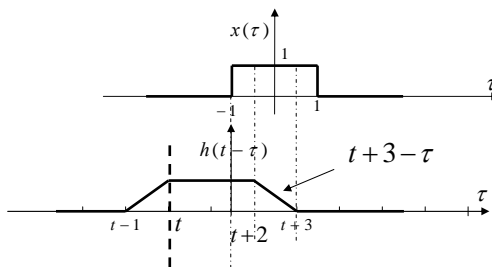
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### 詳細推導

e.g.,

if  $-3 \leq t < -2$ ,



$$\begin{aligned}
 y(t) &= \int_{-\infty}^{+\infty} x(\tau)h(t-\tau) d\tau = \int_{t+2}^{t+3} (t+3-\tau) d\tau + \int_{-1}^{t+2} 1 d\tau \\
 &= \left(\tau(3+t) - \frac{\tau^2}{2}\right) \Big|_{t+2}^{t+3} + \left(\tau\right) \Big|_{-1}^{t+2} \\
 &= (3+t)^2 - \frac{(t+3)^2}{2} - (t+2)(3+t) + \frac{(t+2)^2}{2} + (t+2+1) \\
 &= 9 + 6t + t^2 - \frac{t^2}{2} - 3t - \frac{9}{2} - t^2 - 5t - 6 + \frac{t^2}{2} + 2t + 2 + t + 3 \\
 &= t + \frac{7}{2}
 \end{aligned}$$

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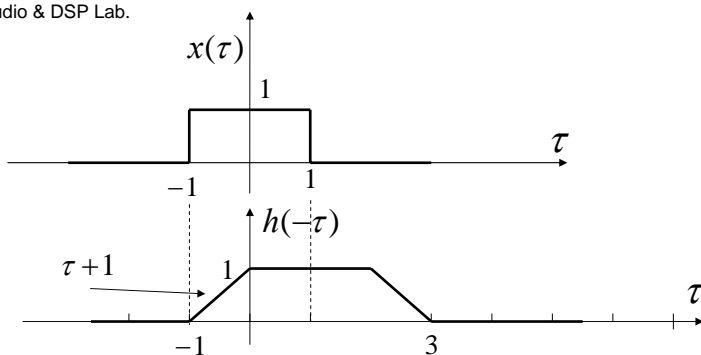
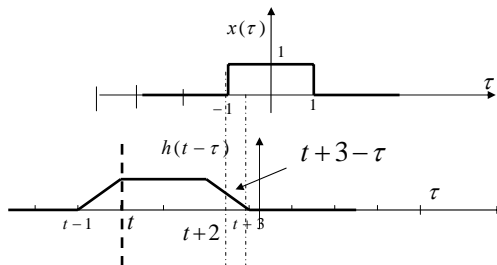
### 詳細推導

e.g.,

if  $-4 \leq t < -3$ ,

$$\begin{aligned}
 y(t) &= \int_{-\infty}^{+\infty} x(\tau)h(t-\tau) d\tau = \int_{-1}^{t+3} (t+3-\tau) d\tau \\
 &= \left( \tau(3+t) - \frac{\tau^2}{2} \right) \Big|_{-1}^{t+3} \\
 &= (3+t)^2 - \frac{(t+3)^2}{2} + (3+t) + \frac{1}{2} \\
 &= 9 + 6t + t^2 - \frac{t^2}{2} - 3t - \frac{9}{2} + 3 + t + \frac{1}{2} \\
 &= \frac{t^2}{2} + 4t + 8
 \end{aligned}$$

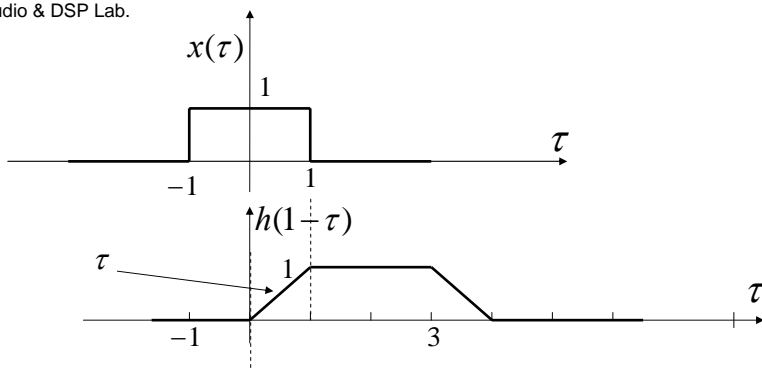
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簡易代入：

$$\begin{aligned}
 y(0) &= \int_{-\infty}^{+\infty} x(\tau)h(0-\tau) d\tau = \int_{-1}^0 (\tau+1) d\tau + \int_0^1 1 d\tau \\
 &= \left( \frac{\tau^2}{2} + \tau \right) \Big|_{-1}^0 + \tau \Big|_0^1 = \frac{1}{2} + 1 = \frac{3}{2}
 \end{aligned}$$

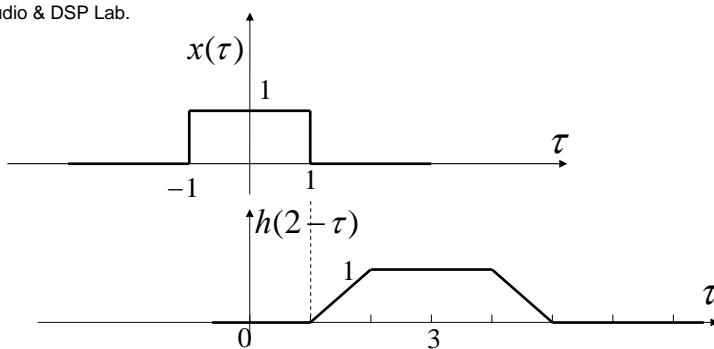
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簡易代入

$$\begin{aligned}
 y(1) &= \int_{-\infty}^{+\infty} x(\tau) h(1-\tau) d\tau = \int_0^1 \tau d\tau \\
 &= \frac{\tau^2}{2} \Big|_0^1 = \frac{1}{2}
 \end{aligned}$$

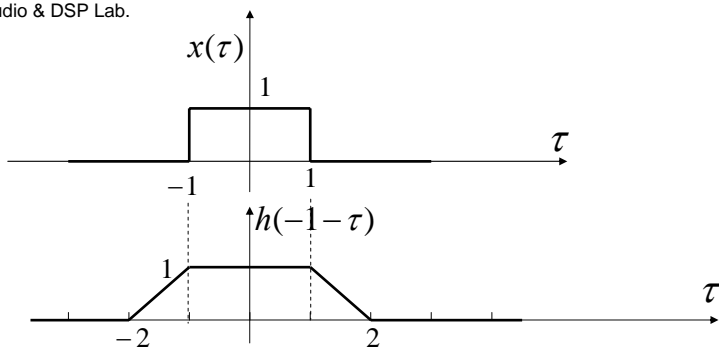
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簡易代入

$$y(2) = \int_{-\infty}^{+\infty} x(\tau) h(2-\tau) d\tau = 0$$

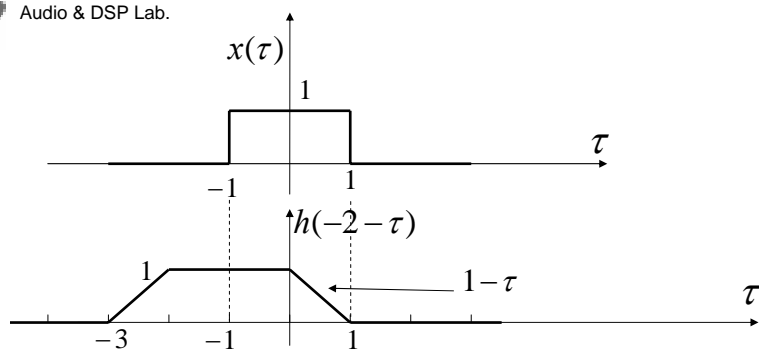
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簡易代入

$$y(-1) = \int_{-\infty}^{+\infty} x(\tau) h(-1-\tau) d\tau = \int_{-1}^{+1} 1 d\tau = \tau \Big|_{-1}^{+1} = 2$$

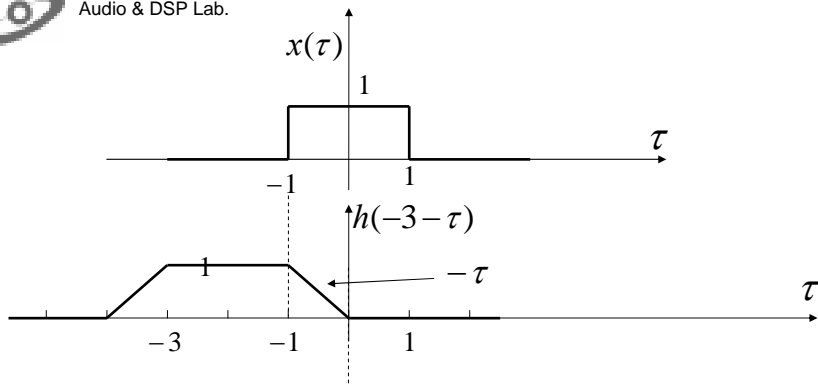
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簡易代入

$$\begin{aligned} y(-2) &= \int_{-\infty}^{+\infty} x(\tau) h(-2-\tau) d\tau = \int_{-1}^0 1 d\tau + \int_0^1 (1-\tau) d\tau \\ &= \tau \Big|_{-1}^0 + \left( \tau - \frac{\tau^2}{2} \right) \Big|_0^1 = 1 + \frac{1}{2} = \frac{3}{2} \end{aligned}$$

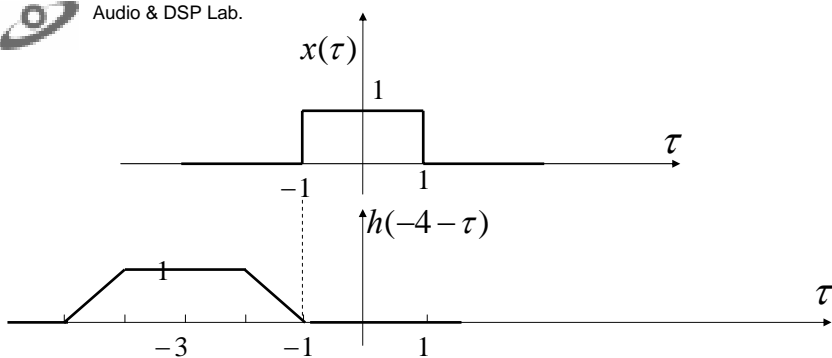
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簡易代入

$$\begin{aligned}
 y(-3) &= \int_{-\infty}^{+\infty} x(\tau) h(-3-\tau) d\tau = -\int_{-1}^0 \tau d\tau \\
 &= -\frac{\tau^2}{2} \Big|_{-1}^0 = \frac{1}{2}
 \end{aligned}$$

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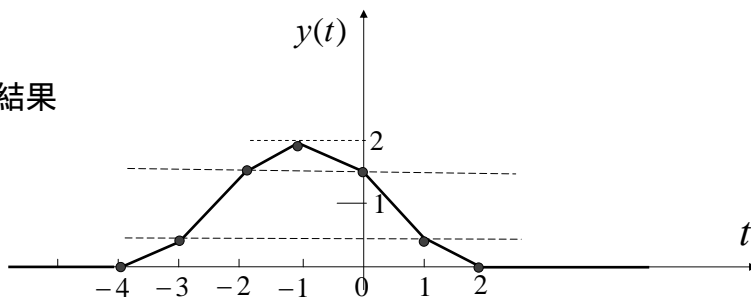
簡易代入

$$y(-4) = \int_{-\infty}^{+\infty} x(\tau) h(-4-\tau) d\tau = 0$$

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結果

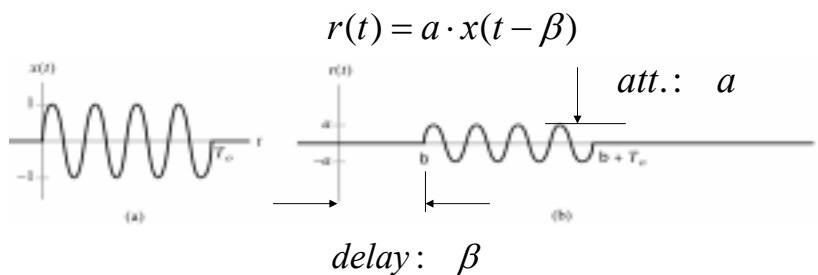


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## Radar range measurement

- (a) Transmitted RF pulse.  
 (b) The received echo is an attenuated and delayed version of the transmitted pulse.



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Radar sends an impulse “ $\delta(t)$ ” to target, and the impulse response is an attenuated “ $a$ ” and delayed “ $\beta$ ”:

$$h(t) = a\delta(t - \beta)$$

$$\therefore h(t) = a\delta(t - \beta), \quad \therefore h(\tau) = a\delta(\tau - \beta)$$

$$h(-\tau) = a\delta(-\tau - \beta) = a\delta(-(\tau + \beta));$$

$$r(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t - \tau)d\tau$$

$$= \int_{-\infty}^{+\infty} x(\tau) (a\delta(t - (\tau + \beta)))d\tau$$

$$= a \int_{-\infty}^{+\infty} x(\tau) [\delta(-\tau + (t - \beta))]d\tau = a x(t - \beta)$$

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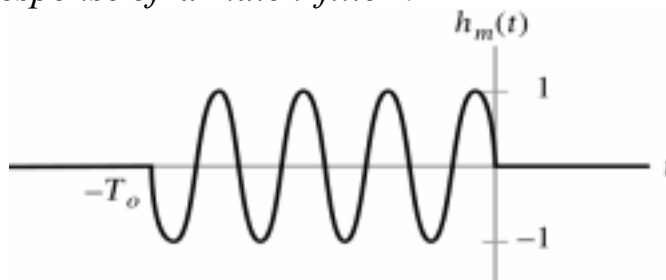


## Match Filter for Radar Range Measurement

(a) Impulse response of the matched filter for processing the received signal.

*The impulse response of a match filter :*

$$h_m(t) = x(-t)$$



(a)

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- (b) The received signal  $r(\tau)$  time-shifted matched filter impulse response  $h_m(t - \tau)$
- (c) Matched filter output  $y(t)$ .

$$y(t) = r(t) * h_m(t)$$

$$= \int_{-\infty}^{+\infty} r(\tau) h_m(t - \tau) d\tau$$

