Lecture 2-3

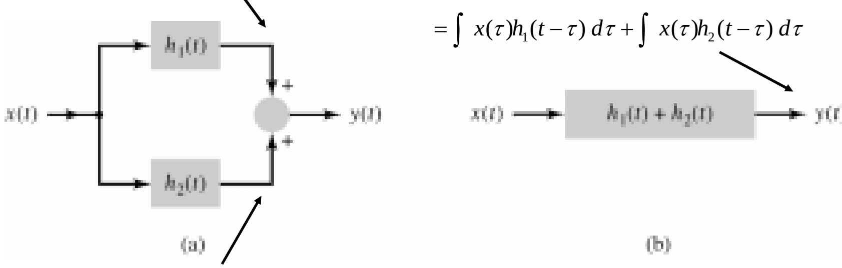
Linear Time-Invariant System
(LTI System)
線性非時變系統

Interconnections of LTI Systems

Interconnection of two LTI systems. (a) Parallel connection of two systems. (b) Equivalent system.

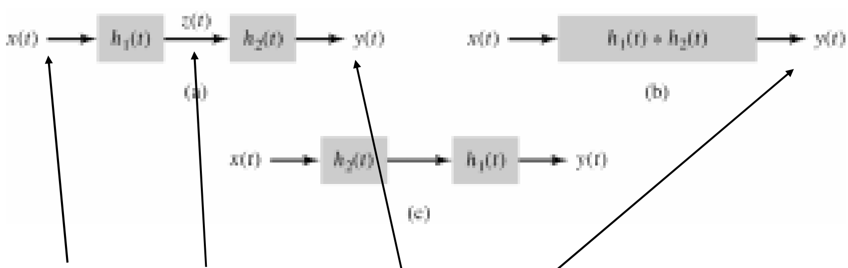
$$x(t)*h_1(t)=\int x(\tau)h_1(t-\tau)\,d\tau$$

$$x(t)*(h_1(t)+h_2(t))=\int x(\tau)(h_1(t-\tau)+h_2(t-\tau))\,d\tau$$
$$=\int (x(\tau)h_1(t-\tau)+x(\tau)h_2(t-\tau))\,d\tau$$
$$=\int x(\tau)h_1(t-\tau)\,d\tau+\int x(\tau)h_2(t-\tau)\,d\tau$$



$$x(t)*h_2(t)=\int x(\tau)h_2(t-\tau)\,d\tau$$

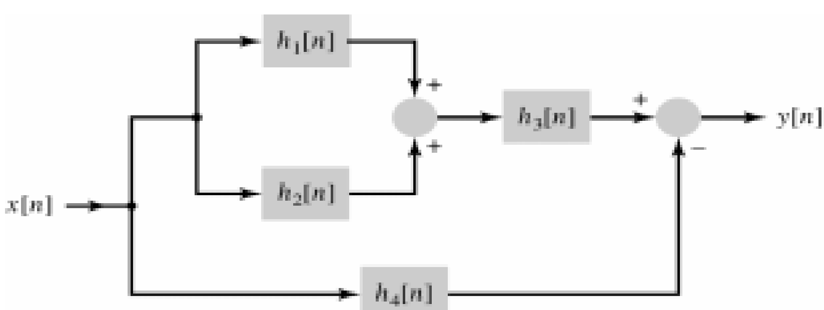
Interconnection of two LTI systems. (a) Cascade connection of two systems. (b) Equivalent system. (c) Equivalent system: Interchange system order.



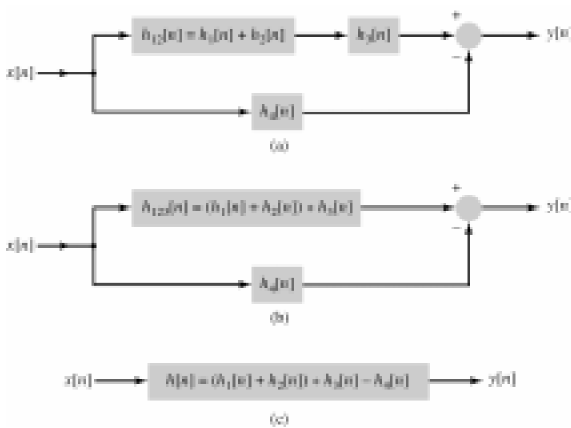
$$if\ \ x(t) = \delta(t), \quad z(t) = h_1(t)$$

$$\therefore y(t) = h_{all}(t) = z(t) * h_2(t) = h_1(t) * h_2(t)$$

Given impulse responses of each subsystem for the interconnection of systems for Example 2.11, find the total system output =?

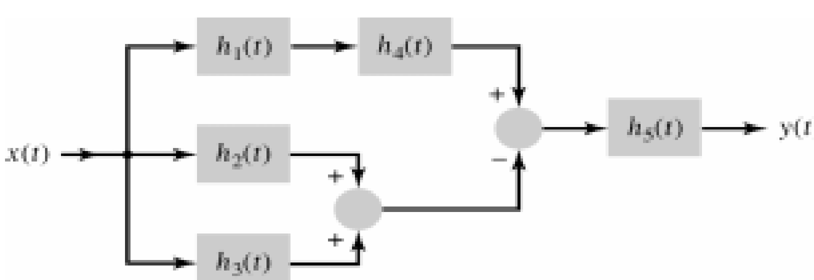


(a) Reduction of parallel combination of LTI systems in upper branch of Fig. 2.20. (b) Reduction of cascade of systems in upper branch of Fig. 2.21(a).
(c) Reduction of parallel combination of systems in Fig. 2.21(b) to obtain an equivalent system for Fig. 2.20.

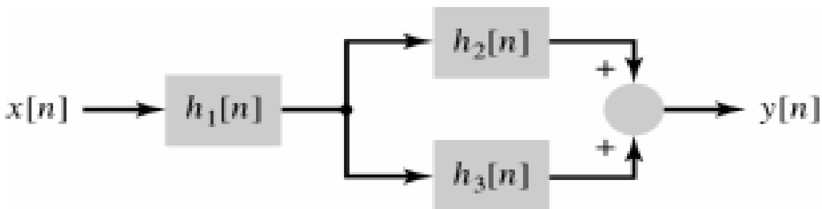


Interconnection of LTI systems for Problem 2.8.

$$h_{all}(t) = h_5(t) * \left(h_1(t) * h_4(t) - \left(h_2(t) + h_3(t)\right)\right)$$



Interconnection of LTI systems for Problem 2.9.



LTI System Properties

- Memory-less LTI System
- Causal LTI System
- Stable LTI System
- Invertible LTI System and De-convolution

Memory-less LTI System: 輸出只與現在輸入有關， 輸出為輸入乘以一個純量。

$$y[n] = x[n] * h[n] = h[n] * x[n]$$
$$= \sum_{k=-\infty}^{+\infty} h[k]x[n-k] = h[0]x[n],$$
$$if \ \ and\ only\ if \quad h[k] = c\delta[k].$$

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau = h(0)x(t),$$
$$if \ \ and\ only\ if \quad h(\tau) = c\delta(\tau).$$

Discrete-Time Causal LTI System:

輸出只與現在與過去的輸入有關，系統在脈衝輸入之前不能有輸出。

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k] \quad or$$

$$y[n] = \cdots + h[-2]x[n+2] + h[-1]x[n+1] + h[0]x[n]$$
$$+ h[1]x[n-1] + h[2]x[n-2] + \cdots$$

因為輸出只與現在與過去的輸入有關，令

$$h[k] = 0, \quad for\ k < 0$$

Discrete-Time Causal LTI System:

$$y[n] = h[n] * x[n] = \sum_{k=0}^{+\infty} h[k]x[n-k]$$

Continuous-Time Causal LTI System:

$$h(\tau) = 0, \quad for\ \tau < 0$$

$$y(t) = h(t) * x(t) = \int_0^{\infty} h(\tau)x(t-\tau)\, d\tau$$

Discrete-Time Stable LTI System:

$$\because \quad |y[n]| = \left|\sum_{k=-\infty}^{+\infty} h[k]x[n-k]\right| \le \sum_{k=-\infty}^{+\infty} |h[k]||x[n-k]|$$

If the input is bounded (BI), $|x[n]| \le M_x < \infty$,

it implies that $|x[n-k]| \le M_x < \infty$.

Hence, $|y[n]| \le M_x \sum_{k=-\infty}^{+\infty} |h[k]|$

If the impulse response is absolutely summable,

$$\sum_{k=-\infty}^{+\infty} |h[k]| < \infty$$

Hence, $|y[n]| \le M_x \sum_{k=-\infty}^{+\infty} |h[k]| < \infty$

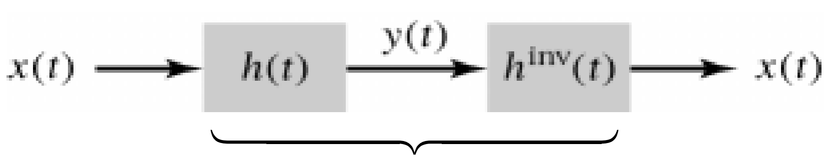
Continuous-Time Stable LTI System:

連續時間 LTI 系統是穩定(BIB0)，若且唯若其脈衝響應是絕對可積分的。 (if and only if)

Iff:

$$\int_{-\infty}^{+\infty} \left| h(\tau) \right| d\tau < \infty$$

Invertible System and Deconvolution



$$h(t) * h^{inv}(t) = \delta(t)$$

回復為 x(t) 的過程稱為：Deconvolution

Invertible System and Deconvolution

Continuous-Time:

$$x(t) = x(t) * \{h(t) * h^{inv}(t)\},$$

$$implies, \quad h(t) * h^{inv}(t) = \delta(t)$$

Discrete-Time:

$$x[n] = x[n] * \{h[n] * h^{inv}[n]\},$$

$$implies, \quad h[n] * h^{inv}[n] = \delta[n]$$

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EX 2.13 $y[n] = x[n] + a\ x[n-1]$

Impulse Response h[n] = ?

$$h[n] = \delta[n] + a\ \delta[n-1]$$

How to find the impulse response from the inverse system, $h^{inv}[n]$ = ?

$$h[n] * h^{inv}[n] = \delta[n]$$

$$\because h[n] * h^{inv}[n] = \delta[n]$$

$$\therefore h[n] * h^{inv}[n] = \{\delta[n] + a\,\delta[n-1]\} * h^{inv}[n]$$

$$= h^{inv}[n] + a\,h^{inv}[n-1] = \delta[n]$$

$$when\ n = 0, \quad h^{inv}[0] + a\,h^{inv}[-1] = 1$$

If the $h^{inv}[n]$ is causal, $h^{inv}[-1] = 0,$

$$\therefore h^{inv}[0] = 1$$

$$when\ n > 0, \quad h^{inv}[n] + a\,h^{inv}[n-1] = 0,$$

$$or \quad rewrite: \quad h^{inv}[n] = -a\,h^{inv}[n-1]$$

$$h^{inv}[0] = 1$$

$$h^{inv}[1] = -ah^{inv}[0] = -a$$

$$h^{inv}[2] = -ah^{inv}[1] = a^2$$

$$h^{inv}[3] = -ah^{inv}[2] = -a^3$$

$$\vdots$$

$$h^{inv}[n] = -ah^{inv}[n-1] = (-a)^n$$

The impulse response from the inverse system,

$$h^{inv}[n] = (-a)^n u[n]$$

Is the inverse system stable?

$$\sum_{k=0}^{\infty} \left| h^{inv}[n] \right| = \sum_{k=0}^{\infty} |a|^k$$

$$if \ |a| < 1, \quad system \ \ stable; \ otherwise \ \ not \ stable.$$

Discrete-Time Step Response

當以單位步階訊號 Unit-Step 輸入時，系統輸出以 s[n] 表示 ***步階響應***：

$$s[n] = u[n] * h[n] = \sum_{k=-\infty}^{+\infty} h[k]u[n-k]$$

$$u[n-k] = \begin{cases} 1, & n-k \geq 0, \quad or \quad n \geq k \\ 0, & n-k < 0, \quad or \quad n < k \end{cases}$$

$$\because u[n-k]=\begin{cases}1, & n\geq k\\ 0, & n<k\end{cases}$$

步階響應是脈衝響應的總和：

$$s[n]=\sum_{k=-\infty}^{n} h[k]$$

脈衝響應亦可用步階響應相減獲得：

$$h[n]=s[n]-s[n-1]$$

Continuous-Time Step Response

步階響應是脈衝響應的積分：

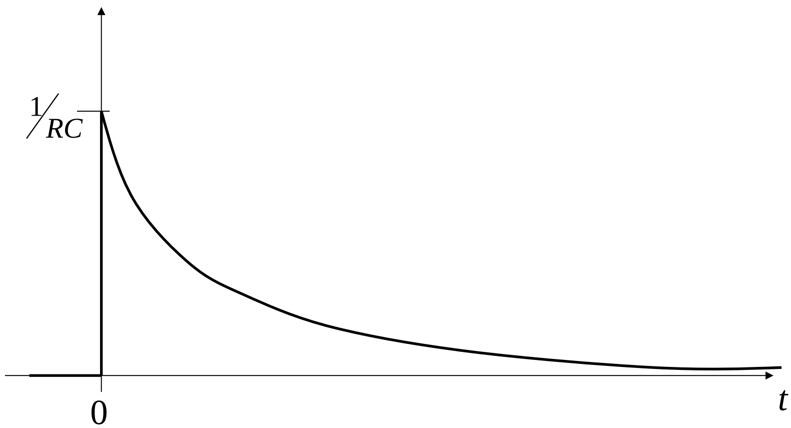
$$s(t)=\int_{-\infty}^{t} h(\tau)\,d\tau$$

脈衝響應亦可用步階響應微分獲得：

$$\frac{d}{dt}s(t)=\frac{d}{dt}\int_{-\infty}^{t} h(\tau)\,d\tau=h(t)$$

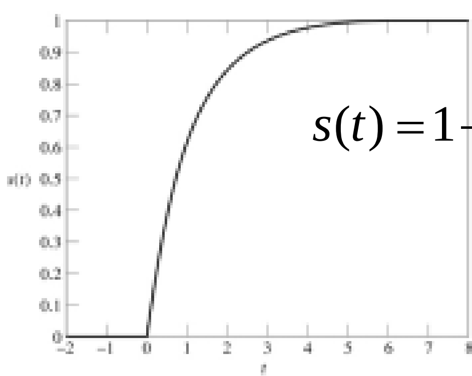
EX 2.14 已知脈衝響應， $h(t)=\dfrac{1}{RC}e^{-t/(RC)}u(t)$

找出步階響應 s(t) = ?



找出步階響應 = ?

Solution:



$$s(t)=1-e^{-\frac{t}{RC}}$$

RC circuit step response for *RC* = 1 s

$$\because \quad s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau$$

$$s(t) = \int_{-\infty}^{t} \left[\frac{1}{RC} e^{-\frac{\tau}{RC}} u(\tau) \right] d\tau$$

$$= \frac{1}{RC} \int_{0}^{t} e^{-\frac{\tau}{RC}} d\tau = \left[-e^{-\frac{\tau}{RC}} \right]\Bigg|_{0}^{t} = -e^{-\frac{t}{RC}} + 1$$

$$= 1 - e^{-\frac{t}{RC}}$$

Differential and Difference Equation

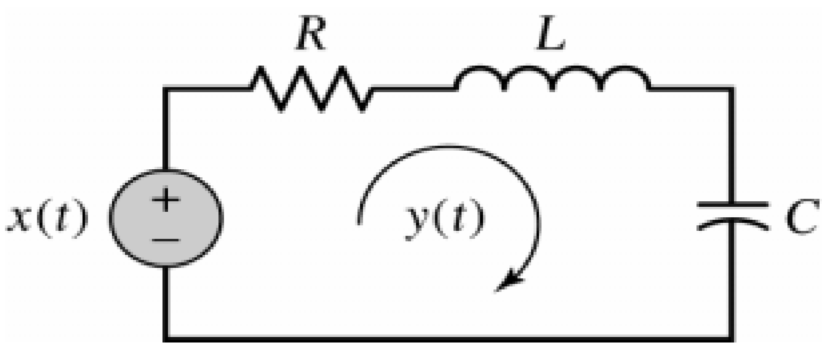
<u>微分方程式</u>：　描述連續時間系統

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

<u>差分方程式</u>：　描述離散時間系統

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Example of an *RLC* circuit described by a differential equation.



微分方程式：

輸入為電壓源: $x(t)$ 輸出為電流: $y(t)$

$$L\frac{d}{dt}y(t)+Ry(t)+\frac{1}{C}\int_{-\infty}^{t}y(\tau)\,d\tau=x(t)$$

對上式等號兩邊對t微分:

$$\boxed{L\frac{d^2}{dt^2}y(t)+R\frac{d}{dt}y(t)+\frac{1}{C}y(t)=\frac{d}{dt}x(t)}$$

差分方程式：

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$y[n] = \frac{1}{a_0}\sum_{k=0}^{M} b_k x[n-k] - \frac{1}{a_0}\sum_{k=1}^{N} a_k y[n-k]$$

EX: 2.16 Difference Equation:

$$y[n] - 1.143y[n-1] + 0.4128y[n-2] =$$
$$0.0675x[n] + 0.1349x[n-1] + 0.675x[n-2]$$

Initial Condition: x[n] = 0, y[-1] = 1, and y[-2] = 2,
Please write a recursive rule to find y[n] = ?

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$$y[n] = 1.143y[n-1] - 0.4128y[n-2] + 0.0675x[n] + 0.1349x[n-1] + 0.675x[n-2]$$

Solution:

$$\because x[n] = 0, \quad \therefore y[n] = 1.143y[n-1] - 0.4128y[n-2]$$

$y[-2] = 2$ ← 已知

$y[-1] = 1$

$y[0] = 1.143(1) - 0.4128(2) = 0.3174$

$y[1] = 1.143(0.3174) - 0.4128(1) = 0.3628 - 0.4128 = -0.05$

$y[2] = 1.143(-0.05) - 0.4128(0.3174) = -0.0572 - 0.1311 = -0.1883$

$y[3] = 1.143(-0.1883) - 0.4128(-0.05) = -0.0406 + 0.0207 = -0.0199$

$\vdots$

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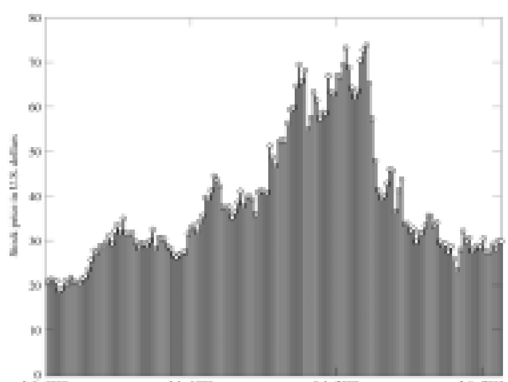
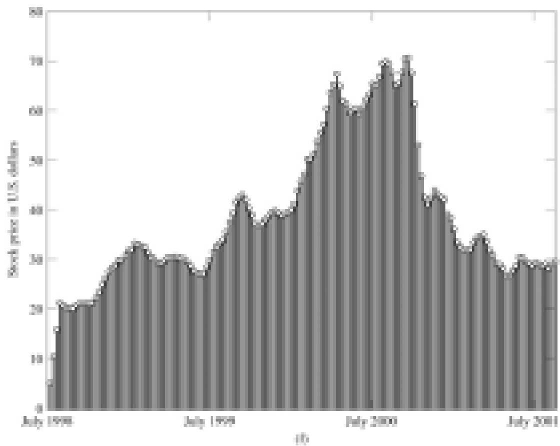
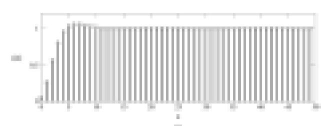
EX: 2.16 : If the input x[n] is described by the following closing price of INTEL stock, please find the output y[n] = ?



Weekly closing price of Intel stock

Output associated with the weekly closing price of Intel stock. (討論輸出、入差異)





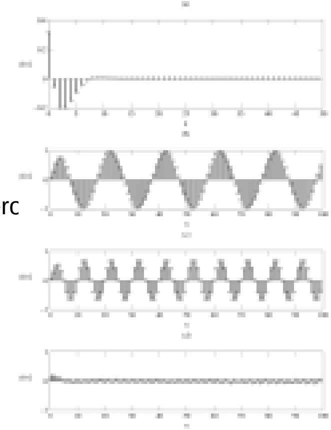


Illustration of the solution to Example 2.16.
(a) Step response of system.
(b) Output due to nonzero initial conditions with zero input.
(c) Output due to $x_1[n] = \cos(1/10\pi n)$.
(d) Output due to $x_2[n] = \cos(1/5\pi n)$.
(e) Output due to $x_3[n] = \cos(7/10\pi n)$.

EX2.14 請用*微分方程式*描述下述 RL circuit，其x(t)表輸入電壓、y(t)表輸出電流。

學生試一試？