Audio & DSP Lab.

Fourier Transform of Signals

訊號的傅立葉轉換

Lecture 3-1

1

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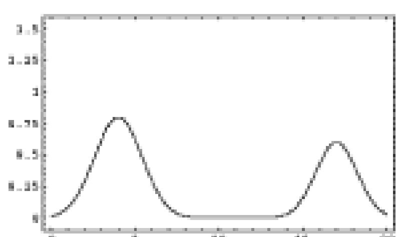
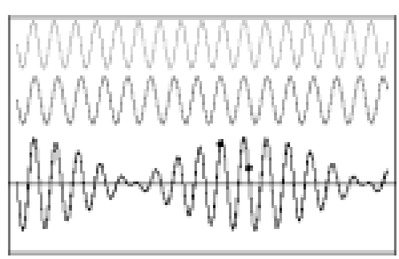
Examples of Waveforms





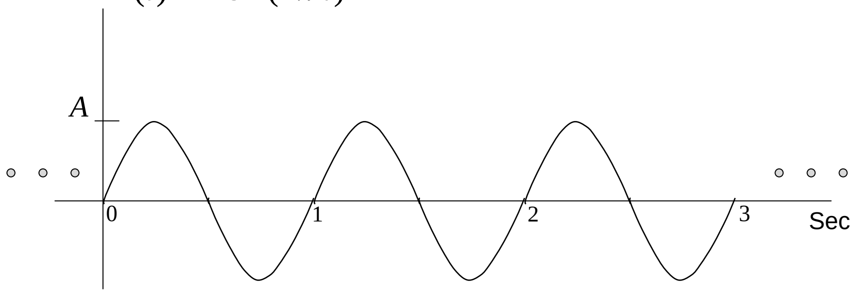
Animation courtesy of Dr. Dan Russell, Kettering University

2

Sinusoidal Signal Review

Continuous Time-domain Waveform:

$$x(t) = A\sin(2\pi t)$$



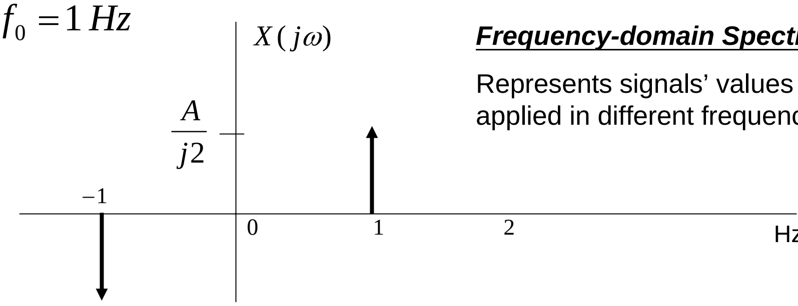
Signal Spectrum

$$x(t) = A\sin(\omega_0 t) = A\sin(2\pi t)$$

$$\omega_0 = 2\pi f$$

$$f_0 = 1\,Hz$$

$$x(t) = \frac{A}{j2} e^{j2\pi t} - \frac{A}{j2} e^{-j2\pi t}$$



Frequency-domain Spectrum:

Represents signals' values applied in different frequencies.

Continuous vs. Discrete-Time Waveforms

Example: $x(t) = A\sin(\omega_0 t) = A\sin(2\pi t)$

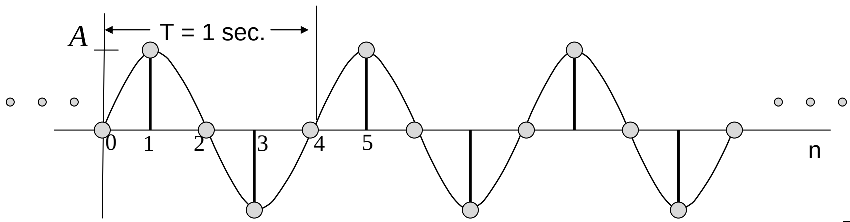
sampling period: $T_s = 0.25\,\text{sec}.$ (抽樣週期)

sampling frequency: $f_s = 4\ Hz$ (抽樣頻率)

$x[n] = x(nT_s) = A\sin(\omega_0 T_s n) = A\sin(\Omega_0 n)$

$\Omega_0 = \omega_0 T_s = 2\pi \times 0.25 = \pi/2$

$\therefore x[n] = A\sin(\Omega_0 n) = A\sin(\pi n/2) = \dfrac{A}{j2} e^{j\pi n/2} - \dfrac{A}{j2} e^{-j\pi n/2}$



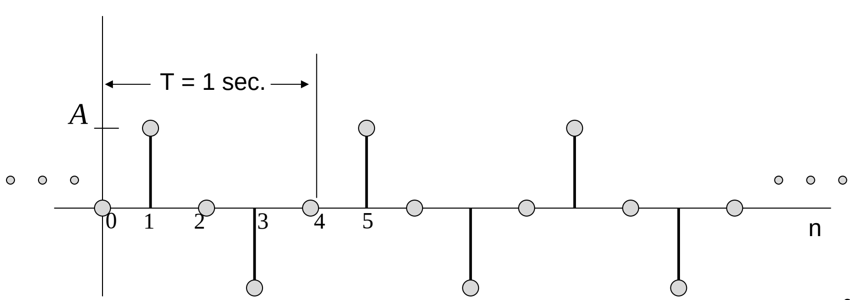
discrete period: $N = 4$

Discrete-Time Waveform

Example:

$\Omega_0 = \omega_0 T_s = 2\pi \times 0.25 = \pi/2$

$x[n] = A\sin(\Omega_0 n) = A\sin(\pi n/2) = \dfrac{A}{j2} e^{j\pi n/2} - \dfrac{A}{j2} e^{-j\pi n/2}$



discrete period: $N = 4$

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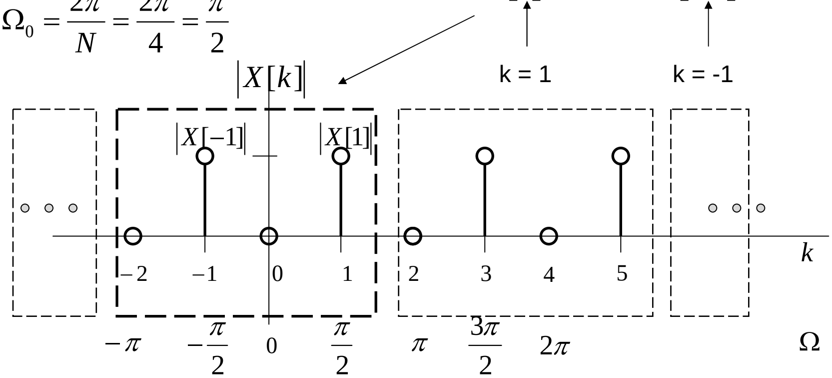
Discrete & Periodic Spectrum

$$x[n] = A\sin(\Omega_0 n) = A\sin(\pi n/2) = \frac{A}{j2}e^{j\pi n/2} - \frac{A}{j2}e^{-j\pi n/2}$$

$N = 4$

$$= X[1]e^{j\frac{2\pi}{4}n} + X[-1]e^{-j\frac{2\pi}{4}n}$$

$$\Omega_0 = \frac{2\pi}{N} = \frac{2\pi}{4} = \frac{\pi}{2}$$



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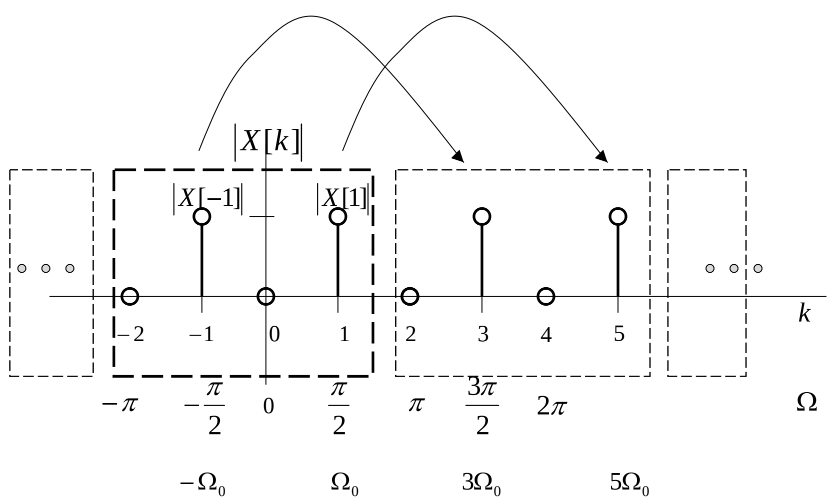
Periodic of 2π



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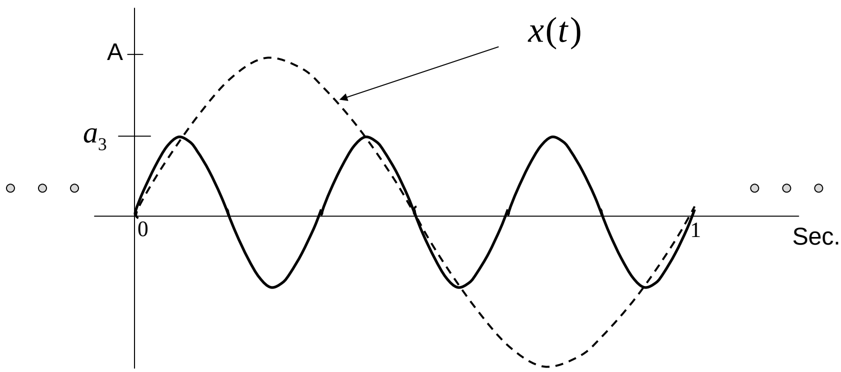
Fundamental & 3rd Harmonics



$x_3(t)$ 為 x(t) 的 3 次 諧波　　$x_3(t) = a_3 \sin(6\pi t)$

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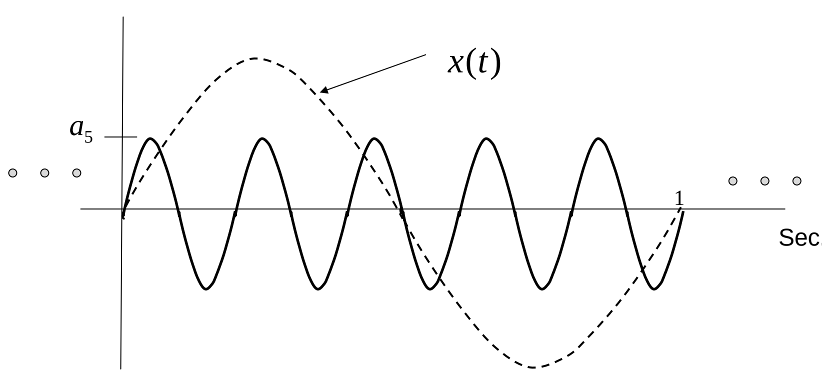
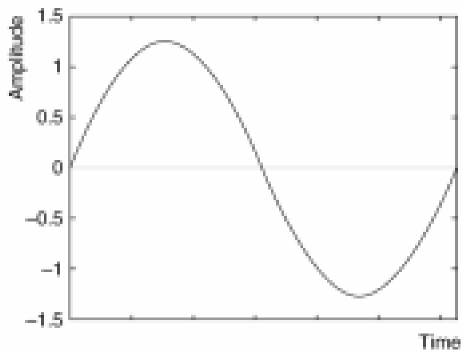
Fundamental & 5th Harmonics



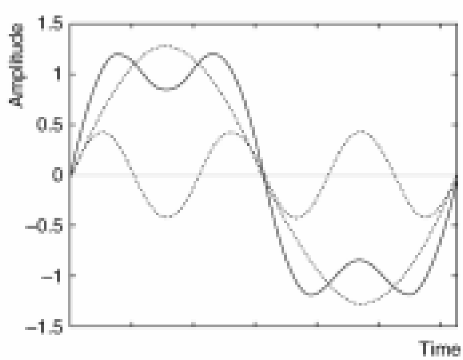
$x_5(t)$ 為 x(t) 的 5 次 諧波　　$x_5(t) = a_5 \sin(10\pi t)$

Fundamental or 1st harmonic



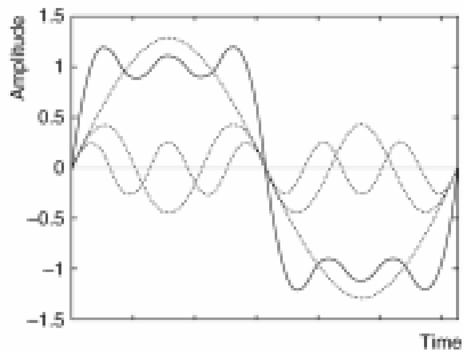
(a) Lowest Frequency Sine Wave

1st + 3rd Harmonics



(b) Sum of Two Sine Waves

$1^{st} + 3^{rd} + 5^{th}$ Harmonics



(c) Sum of Three Sine Waves

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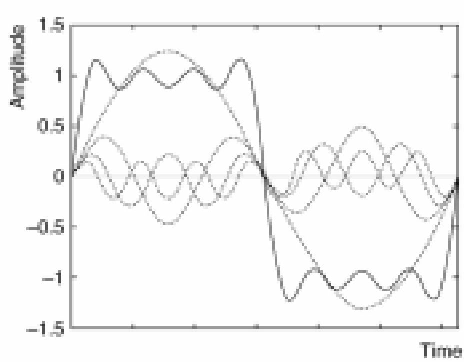
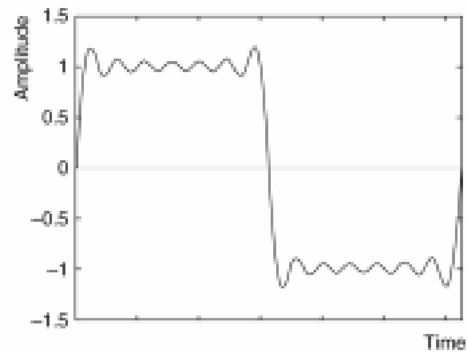
$1^{st} + 3^{rd} + 5^{th} + 7^{th}$ Harmonics



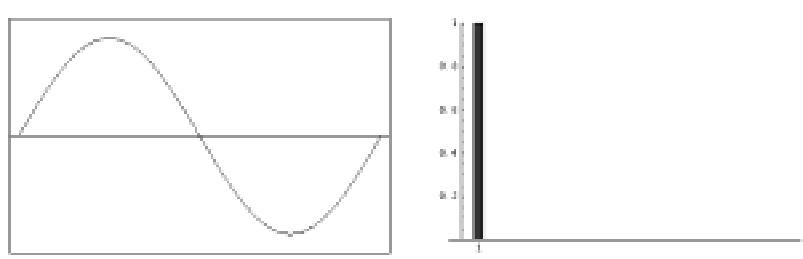
(d) Sum of Four Sine Waves

$1^{st} + 3^{rd} + 5^{th} + 7^{th} + 9^{th} + 11^{th} + 13^{th}$ Harmonics



(e) Sum of Seven Sine Waves

Fourier Series and Waves



Animation courtesy of Dr. Dan Russell, Kettering University

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Complex Sinusoids and Frequency Response of LTI Systems

- LTI 系統對弦波輸入響應稱為 “頻率響應” (Frequency Response)
- 系統輸入若為單位振幅的複數弦波，系統輸出則為與輸入頻率相同的複數弦波乘上系統頻率響應
- 頻率響應分成 “振幅響應” (Magnitude Response) 與 “相位響應” (Phase Response)

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Discrete-Time LTI System

$$\text{if system input, } x[n] = e^{j\Omega n}, \quad \text{複數弦波}$$

$$y[n] = \sum_{k=-\infty}^{+\infty} h[k]\, x[n-k] = \sum_{k=-\infty}^{+\infty} h[k]\, e^{j\Omega(n-k)}$$

$$= e^{j\Omega n} \sum_{k=-\infty}^{+\infty} h[k]\, e^{-j\Omega k}$$

$$= e^{j\Omega n}\, H\left(e^{j\Omega}\right), \quad \text{where } H\left(e^{j\Omega}\right) = \sum_{k=-\infty}^{+\infty} h[k]\, e^{-j\Omega k}$$

複數弦波 × 頻率響應

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Continuous –Time System

$$if\ \ system\ input,\ x(t)=e^{j\omega t},$$ 複數弦波

$$y(t)=\int_{-\infty}^{+\infty}h(\tau)\ x(t-\tau)\ d\tau=\int_{-\infty}^{+\infty}h(\tau)\ e^{j\omega(t-\tau)}\ d\tau$$

$$=e^{j\omega t}\int_{-\infty}^{+\infty}h(\tau)\ e^{-j\omega\tau}\ d\tau$$

$$=e^{j\omega t}\ H(j\omega),\quad where\ \ H(j\omega)=\int_{-\infty}^{+\infty}h(\tau)\ e^{-j\omega\tau}\ d\tau$$

複數弦波 × 頻率響應

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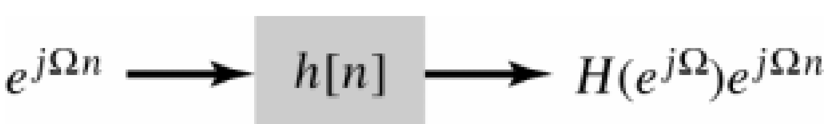
Block Diagram of the LTI System



The output of a complex sinusoidal input to an LTI system is a complex sinusoid of the same frequency as the input, multiplied by the frequency response of the system.

Ex: 3.1 Continuous-Time RC Circuit

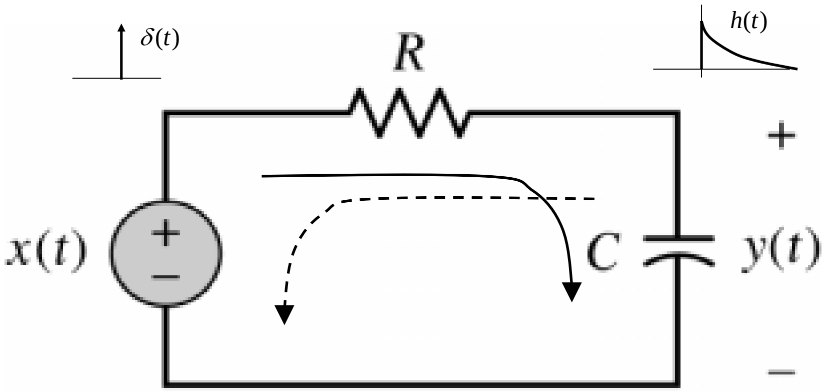


Figure 3.2 *RC* circuit for Example 3.1.

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Frequency Response

$$\because h(t) = \frac{1}{RC} e^{-\frac{t}{RC}} u(t),$$

$$H(j\omega) = \int_{-\infty}^{+\infty} h(\tau)\, e^{-j\omega\tau} d\tau$$ ：頻率響應

$$= \int_{-\infty}^{+\infty} \left[\frac{1}{RC} e^{-\frac{\tau}{RC}} u(\tau) \right] e^{-j\omega\tau} d\tau = \frac{1}{RC} \int_{0}^{+\infty} \left[e^{-\frac{\tau}{RC}} \right] e^{-j\omega\tau} d\tau$$

$$= \frac{1}{RC} \int_{0}^{+\infty} e^{-\left(j\omega + \frac{1}{RC}\right)\tau} d\tau = -\frac{1}{RC} \frac{1}{j\omega + \frac{1}{RC}} e^{-\left(j\omega + \frac{1}{RC}\right)\tau} \Bigg|_{0}^{+\infty}$$

$$= -\frac{1}{1 + j\omega RC} [0 - 1] = \frac{1}{1 + j\omega RC} = \frac{\frac{1}{RC}}{\frac{1}{RC} + j\omega}$$ ：複數函式

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Magnitude Response

$$|H(j\omega)| = \left| \frac{\frac{1}{RC}}{\frac{1}{RC} + j\omega} \right| = \frac{\frac{1}{RC}}{\left| \frac{1}{RC} + j\omega \right|}$$

: 振幅響應

$$= \frac{\frac{1}{RC}}{\sqrt{\omega^2 + \left(\frac{1}{RC} \right)^2}}$$

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Phase Response

argument: 幅角

$$\arg\{H(j\omega)\} = \arg\left\{ \frac{1}{RC} \right\} - \arg\left\{ \frac{1}{RC} + j\omega \right\}$$

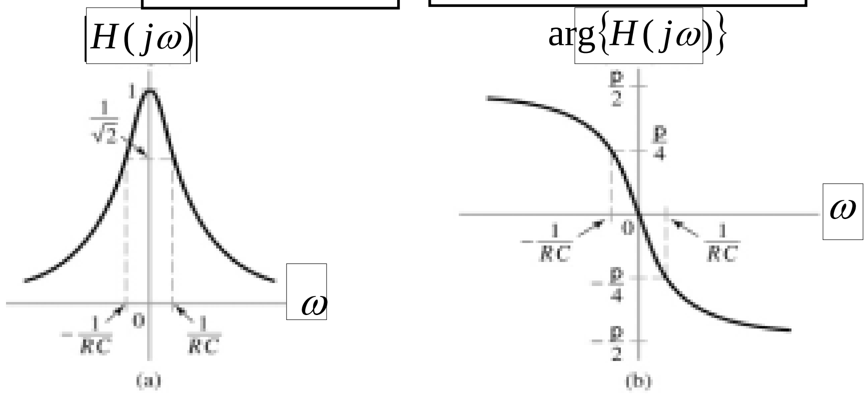
$$= \tan^{-1}[\frac{0}{1/RC}] - \tan^{-1}[\omega RC] = 0 - \tan^{-1}[\omega RC]$$

$$= -\arctan[\omega RC]$$

: 相位響應

$$|H(j\omega)| = \frac{\frac{1}{RC}}{\sqrt{\omega^2 + \left(\frac{1}{RC}\right)^2}}$$

$$\arg\{H(j\omega)\} = -\tan^{-1}[\omega RC]$$



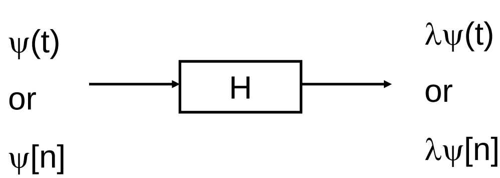
Frequency response of the *RC* circuit :
(a) Magnitude response (b) Phase response

Eigenfunction and Eigenvalue

特徵函數(eigenfunction) 與 特徵值(eigenvalue)

The action of the system on an eigenfunction input is multiplication by the corresponding eigenvalue.

General eigenfunction $\Psi(t)$ or $\Psi[n]$ and eigenvalue λ:



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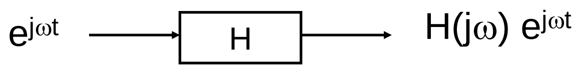
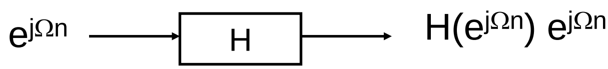
Complex sinusoidal eigenfunction $e^{j\omega t}$ and eigenvalue $H(j\omega)$:



Complex sinusoidal eigenfunction $e^{j\Omega n}$ and eigenvalue $H(e^{j\Omega})$:



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Convolution vs. Multiplication

將任意輸入訊號表示為特徵函數的加權疊加，求取輸出的摺積運算可轉換成乘法運算。

$$let \quad x(t)=\sum_{k=1}^{M} a_k e^{j\omega_k t},$$

若輸入訊號 $e^{j\omega_k t}$ 為特徵值 $H(j\omega_k)$ 的特徵函數，則輸入每一項 $a_k e^{j\omega_k t}$ 皆會產生一個輸出 $a_k H(j\omega_k)e^{j\omega_k t}$ 項。

$$\therefore \quad y(t)=\sum_{k=1}^{M} a_k H(j\omega_k) e^{j\omega_k t},$$