



Fourier Transform of Signals

訊號的傅立葉轉換 Lecture 3-2

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Introduction

- 訊號表示為一組複數弦波的加權疊加
- 把複雜的訊號看成頻率的函數
- 傅立葉表示法
 - 連續時間傅立葉級數 (週期性) : FS
 - 連續時間傅立葉轉換 (非週期性) : FT
 - 離散時間傅立葉級數 (週期性) : DTFS
 - 離散時間傅立葉轉換 (非週期性) : DTFT

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Discrete-Time Periodic Signals (DTPS): The Discrete-Time Fourier Series (DTFS)

基本週期 N 基本頻率 $\Omega_0 = 2\pi/N$ 的週期訊號 $x[n]$:

$X[k]$ is discrete spectrum & $X[k]$ is periodic function.

$$x[n] = \sum_{k=0}^{N-1} X[k] e^{jk\Omega_0 n} \quad (1)$$

$x[n]$ is discrete-time signal and $x[n]$ is periodic signal.

$$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk\Omega_0 n} \quad (2)$$

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Orthogonal Property

$$\begin{aligned} \sum_{n=0}^{N-1} e^{-jk\Omega_0 n} \cdot \sum_{m=0}^{N-1} e^{jm\Omega_0 n} &= \sum_{n=0}^{N-1} \cdot \sum_{m=0}^{N-1} e^{-jk\Omega_0 n} \cdot e^{jm\Omega_0 n} \\ &= \sum_{n=0}^{N-1} \cdot \sum_{m=0}^{N-1} e^{j(m-k)\Omega_0 n} \end{aligned}$$

if $m = k$,

$$= \sum_{n=0}^{N-1} \cdot \sum_{m=k}^k e^{j(0)\Omega_0 n} = \sum_{n=0}^{N-1} 1 = N$$

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$$\begin{aligned} & \sum_{n=0}^{N-1} e^{-jk\frac{2\pi}{N}n} \cdot \sum_{m=0}^{N-1} e^{jm\frac{2\pi}{N}n} = \sum_{n=0}^{N-1} \cdot \sum_{m=0}^{N-1} e^{-jk\frac{2\pi}{N}n} \cdot e^{jm\frac{2\pi}{N}n} \\ & = \sum_{n=0}^{N-1} \cdot \sum_{m=0}^{N-1} e^{j(m-k)\frac{2\pi}{N}n} \end{aligned}$$

if $m \neq k$, let $l = m - k$; The l is an integer.

$$\begin{aligned} & = \sum_{n=0}^{N-1} \cdot \sum_{m=0}^{N-1} e^{j\frac{2\pi l}{N}n} = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \left(e^{j\frac{2\pi l}{N}} \right)^n = \sum_{m=0}^{N-1} \frac{1 - e^{j\frac{2\pi l}{N}N}}{1 - e^{j\frac{2\pi l}{N}}} \\ & = \sum_{m=0}^{N-1} \frac{1 - e^{j2\pi l}}{1 - e^{j\frac{2\pi l}{N}}} = \sum_{m=0}^{N-1} \frac{1 - 1}{1 - e^{j\frac{2\pi l}{N}}} = 0 \end{aligned}$$

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$$\therefore \text{ given } x[n] = \sum_{m=0}^{N-1} X[m] e^{jm\Omega_0 n}$$

$$\begin{aligned} & \frac{1}{N} \sum_{n=0}^{N-1} \underline{x[n]} e^{-jk\Omega_0 n} = \frac{1}{N} \sum_{n=0}^{N-1} \left\{ \underline{\sum_{m=0}^{N-1} X[m] e^{jm\Omega_0 n}} \right\} e^{-jk\Omega_0 n} \\ & = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} X[m] e^{j(m-k)\Omega_0 n} \\ & = \begin{cases} \frac{1}{N} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} X[m] e^{j(m-k)\Omega_0 n} = 0, & \forall k \neq m \\ \frac{1}{N} X[k] \sum_{n=0}^{N-1} 1 = X[k], & \forall k = m \end{cases} \end{aligned}$$

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$$x[n] \xleftrightarrow{DTFS; \Omega_0} X[k]$$

- $X[k]$ 被稱為 $x[n]$ 的頻域表示法，
- $X[k]$ 和 $x[n]$ 皆以有限 N 個數表示週期，
- DTFS 是 唯一可用電腦運算的傅立葉表示法，
- 複數弦波 $e^{jk\Omega_0 n}$ 只需有 N 個。 (N 週期函數)

$$\begin{aligned} \text{i.e., } e^{j(k+N)\Omega_0 n} &= e^{jN\Omega_0 n} e^{jk\Omega_0 n} = e^{jN\frac{2\pi}{N}n} e^{jk\Omega_0 n} \\ &= e^{j2\pi n} e^{jk\Omega_0 n} = e^{jk\Omega_0 n} \end{aligned}$$

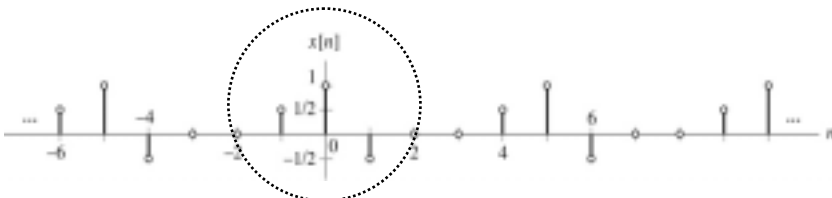
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Example 3.2

試以下圖求出訊號 DTFS 係數 $X[k] = ?$

週期 $N = ?$ 如何選取 那一個週期區間？優、缺點？



Time-domain signal for Example 3.2.

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solution :

$$\therefore N = 5, \quad \therefore \Omega_0 = 2\pi/5,$$

$\therefore x[n]$ is odd symmetric signal, select $n = -2$ to $+2$:

$$\begin{aligned} X[k] &= \frac{1}{5} \sum_{n=-2}^{+2} x[n] e^{-jk \frac{2\pi}{5} n} \\ &= \frac{1}{5} \left[0 \cdot e^{-jk \frac{2\pi}{5} (-2)} + \frac{1}{2} \cdot e^{-jk \frac{2\pi}{5} (-1)} + 1 \cdot e^{-jk \frac{2\pi}{5} (0)} - \frac{1}{2} \cdot e^{-jk \frac{2\pi}{5} (1)} + 0 \cdot e^{-jk \frac{2\pi}{5} (2)} \right] \\ &= \frac{1}{5} \left[\frac{1}{2} \cdot e^{jk \frac{2\pi}{5}} + 1 - \frac{1}{2} \cdot e^{-jk \frac{2\pi}{5}} \right] = \frac{1}{5} \left[1 + \frac{1}{2} \cdot e^{jk \frac{2\pi}{5}} - \frac{1}{2} \cdot e^{-jk \frac{2\pi}{5}} \right] \\ &= \frac{1}{5} \left[1 + j \frac{e^{jk \frac{2\pi}{5}} - e^{-jk \frac{2\pi}{5}}}{j2} \right] = \frac{1}{5} \left[1 + j \sin\left(2\pi k/5\right) \right] \end{aligned}$$

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$$\therefore X[k] = \frac{1}{5} \left[1 + j \sin\left(2\pi k/5\right) \right]$$

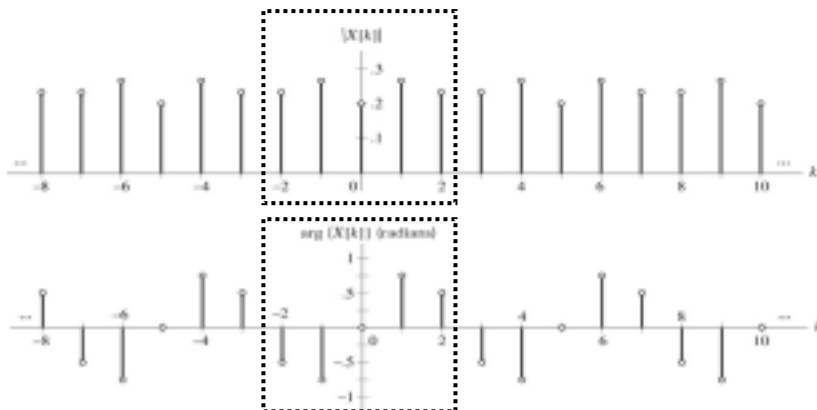
\therefore *Magnitude Spectrum :*

$$|X(k)| = \frac{1}{5} \sqrt{1 + \sin^2\left(2\pi k/5\right)}$$

\therefore *Phase Spectrum :*

$$\arg\{X(k)\} = \tan^{-1} \left\{ \sin\left(2\pi k/5\right) \right\}$$

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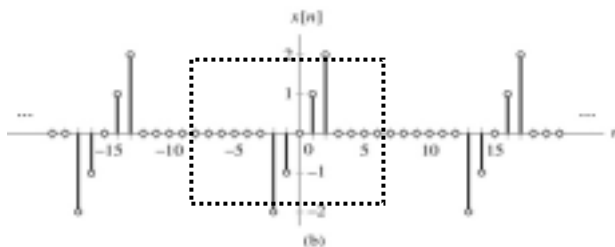
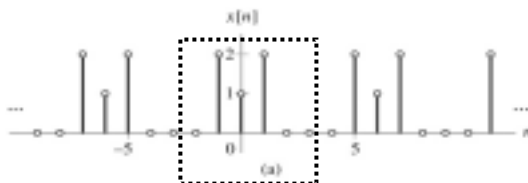


Magnitude and phase of the DTFS coefficients for the signal in Fig. 3.5 are periodic with period of $N=5$.



Problem 3.2: 試求下圖 (a) 和 (b) DTFS 係數 = ?

學生試一試 ?





Example 3.3 : 用審視法 (Method of Inspection) 求出 $x[n] = \cos(\pi n/3 + \phi)$ 的 DTFS 係數 = ?

solution :

$$\therefore \Omega_0 = \frac{2\pi}{N} = \frac{\pi}{3}, \quad \therefore N = 6.$$

$$x[n] = \cos(\pi n/3 + \phi) = \frac{1}{2} \left\{ e^{j\left(\frac{\pi n}{3} + \phi\right)} + e^{-j\left(\frac{\pi n}{3} + \phi\right)} \right\}$$

$$= \frac{1}{2} e^{j\phi} e^{j\frac{\pi n}{3}} + \frac{1}{2} e^{-j\phi} e^{-j\frac{\pi n}{3}}$$

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solution : (cont.)

$$x[n] = \frac{1}{2} e^{j\phi} e^{j\frac{\pi n}{3}} + \frac{1}{2} e^{-j\phi} e^{-j\frac{\pi n}{3}}$$

comparing : $N = 6; n = -2 \sim +3$

$$x[n] = \sum_{k=-2}^3 X[k] e^{jk\frac{\pi}{3}n}$$

$$= X[-2] e^{-j\frac{2\pi}{3}n} + X[-1] e^{-j\frac{\pi}{3}n} + X[0]$$

$$+ X[1] e^{j\frac{\pi}{3}n} + X[2] e^{j\frac{2\pi}{3}n} + X[3] e^{j\frac{3\pi}{3}n}$$

$$\therefore$$

$$X[-2] = X[0] = X[2] = X[3] = 0,$$

$$X[-1] = \frac{1}{2} e^{-j\phi}, \quad X[1] = \frac{1}{2} e^{j\phi}$$

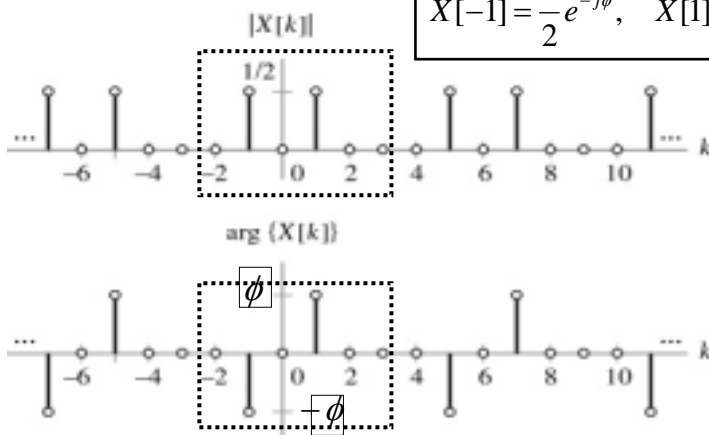
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Magnitude and phase of DTFS coefficients for Example 3.3.

$$X[-2] = X[0] = X[2] = X[3] = 0,$$

$$X[-1] = \frac{1}{2}e^{-j\phi}, \quad X[1] = \frac{1}{2}e^{j\phi}$$



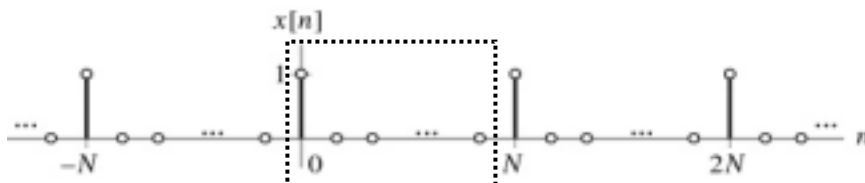
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Find the Magnitude and phase of DTFS coefficients for Example 3.4.

離散時間脈衝列

$$x[n] = \sum_{m=-\infty}^{+\infty} \delta[n - mN]$$



A discrete-time impulse train with period N.

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$$\therefore x[n] = \sum_{m=-\infty}^{+\infty} \delta[n - mN]$$

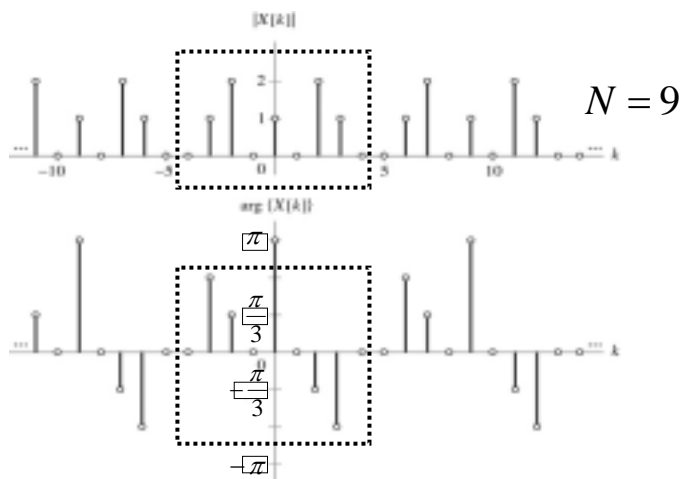
\therefore

$$\begin{aligned}
 X[k] &= \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk \frac{2\pi}{N} n} && \text{Take one cycle:} \\
 &= \frac{1}{N} \sum_{n=0}^{N-1} \sum_{m=-\infty}^{+\infty} \delta[n - mN] e^{-jkn2\pi/N} = \frac{1}{N} \sum_{n=0}^{N-1} \delta[n] e^{-jkn2\pi/N} \\
 &= \frac{1}{N} \left\{ \delta[0] + \delta[1] e^{-jk2\pi/N} + \dots + \delta[N-1] e^{-jk2\pi(N-1)/N} \right\} \\
 &= \frac{\delta[0]}{N} = \frac{1}{N}
 \end{aligned}$$

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EX 3.5: Given the magnitude and phase of DTFS coefficients, please find $x[n] = ?$



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$\because N = 9$, choose $n = -4 \sim +4$,

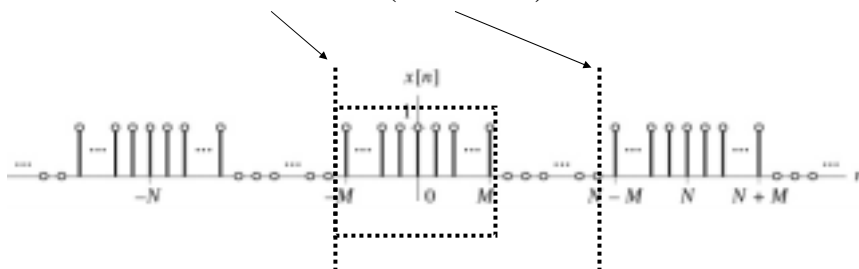
$$\begin{aligned}
 x[n] &= \sum_{k=-4}^{+4} X[k] e^{jk \frac{2\pi}{9} n}, \quad \text{代入已知 } X[k] \text{ 振幅和相位值} \\
 &= X[-3] e^{-jn \frac{2\pi}{9} 3} + X[-2] e^{-jn \frac{2\pi}{9} 2} + X[0] + X[2] e^{jn \frac{2\pi}{9} 2} + X[3] e^{jn \frac{2\pi}{9} 3} \\
 &= e^{j \frac{2\pi}{3}} e^{-jn \frac{6\pi}{9}} + 2e^{j \frac{\pi}{3}} e^{-jn \frac{4\pi}{9}} + e^{-j\pi} + 2e^{-j \frac{\pi}{3}} e^{jn \frac{4\pi}{9}} + e^{-j \frac{2\pi}{3}} e^{jn \frac{6\pi}{9}} \\
 &= \left(e^{-j(n \frac{6\pi}{9} - \frac{2\pi}{3})} + e^{j(n \frac{6\pi}{9} - \frac{2\pi}{3})} \right) + (-1) + 2 \left(e^{-j(n \frac{4\pi}{9} - \frac{\pi}{3})} + e^{j(n \frac{4\pi}{9} - \frac{\pi}{3})} \right) \\
 &= 2 \cos \left(n \frac{6\pi}{9} - \frac{2\pi}{3} \right) + 4 \cos \left(n \frac{4\pi}{9} - \frac{\pi}{3} \right) - 1
 \end{aligned}$$

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EX 3.6: Find the DTFS coefficients of the square wave, or $X[k] = ?$

choose $n = -M \sim (N - M - 1)$



Square wave for Example 3.6.

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Applying DTFS Equation (2)

choose $n = -M \sim N - M - 1$

$$X[k] = \frac{1}{N} \sum_{n=-M}^{N-M-1} x[n] e^{-jk\frac{2\pi}{N}n} = \frac{1}{N} \sum_{n=-M}^M e^{-jk\frac{2\pi}{N}n}$$

let $m = n + M$,

$$\begin{aligned} X[k] &= \frac{1}{N} \sum_{m=0}^{2M} e^{-jk\frac{2\pi}{N}(m-M)} = \frac{1}{N} e^{jk\frac{2\pi}{N}M} \sum_{m=0}^{2M} e^{-jk\frac{2\pi}{N}m} \\ &= \frac{1}{N} e^{jk\frac{2\pi}{N}M} \left(1 + e^{-jk\frac{2\pi}{N}} + e^{-jk\frac{4\pi}{N}} + e^{-jk\frac{6\pi}{N}} + \dots + e^{-jk\frac{4M\pi}{N}} \right) \end{aligned}$$

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if $k = 0, \pm N, \pm 2N, \dots$

$$\begin{aligned} X[k] &= \frac{1}{N} e^{jk\frac{2\pi}{N}M} \left(1 + e^{-jk\frac{2\pi}{N}} + e^{-jk\frac{4\pi}{N}} + \dots + e^{-jk\frac{4M\pi}{N}} \right) \\ &= \frac{1}{N} (1 + 1 + 1 + \dots + 1) \\ &= \frac{1}{N} \left(\sum_{m=0}^{2M} 1 \right) = \frac{2M + 1}{N} \end{aligned}$$

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if $k = \text{otherwise}$,

$$\begin{aligned}
 X[k] &= \frac{1}{N} e^{jk\frac{2\pi}{N}M} \left(1 + e^{-jk\frac{2\pi}{N}} + e^{-jk\frac{4\pi}{N}} + \dots + e^{-jk\frac{4M\pi}{N}} \right) \\
 &= \frac{1}{N} e^{jk\frac{2\pi}{N}M} \left(\sum_{m=0}^{2M} e^{-jk\frac{2\pi}{N}m} \right) \\
 &= \frac{1}{N} e^{jk\frac{2\pi}{N}M} \left(\frac{1 - e^{-jk\frac{2\pi}{N}(2M+1)}}{1 - e^{-jk\frac{2\pi}{N}}} \right)
 \end{aligned}$$

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rewrite :

$$\begin{aligned}
 X[k] &= \frac{1}{N} e^{jk\frac{2\pi}{N}M} \left(\frac{e^{-jk\frac{\pi(2M+1)}{N}} \left(e^{jk\frac{\pi(2M+1)}{N}} - e^{-jk\frac{\pi(2M+1)}{N}} \right)}{e^{-jk\frac{\pi}{N}} \left(e^{jk\frac{\pi}{N}} - e^{-jk\frac{\pi}{N}} \right)} \right) \\
 &= \frac{1}{N} e^{jk\frac{2\pi}{N}M} \left(\frac{e^{-jk\frac{\pi(2M)}{N}} \left(e^{jk\frac{\pi(2M+1)}{N}} - e^{-jk\frac{\pi(2M+1)}{N}} \right)}{\left(e^{jk\frac{\pi}{N}} - e^{-jk\frac{\pi}{N}} \right)} \right)
 \end{aligned}$$

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$$\begin{aligned}
 X[k] &= \frac{1}{N} \left(\frac{\left(e^{jk\frac{\pi(2M+1)}{N}} - e^{-jk\frac{\pi(2M+1)}{N}} \right)}{\left(e^{jk\frac{\pi}{N}} - e^{-jk\frac{\pi}{N}} \right)} \right) \\
 &= \frac{1}{N} \left(\frac{\left(e^{jk\frac{\pi(2M+1)}{N}} - e^{-jk\frac{\pi(2M+1)}{N}} \right) / j2}{\left(e^{jk\frac{\pi}{N}} - e^{-jk\frac{\pi}{N}} \right) / j2} \right) = \frac{1}{N} \frac{\sin\left(k\frac{\pi(2M+1)}{N}\right)}{\sin\left(k\frac{\pi}{N}\right)}
 \end{aligned}$$

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$$\therefore X[k] = \begin{cases} \frac{1}{N} \frac{\sin\left(k\frac{\pi(2M+1)}{N}\right)}{\sin\left(k\frac{\pi}{N}\right)}, & k \neq 0, \pm N, \pm 2N, \dots \\ \frac{2M+1}{N}, & k = 0, \pm N, \pm 2N, \dots \end{cases}$$

$$\text{Ex: } N = 50, M = 12, \quad \frac{2M+1}{N} = \frac{25}{50} = 0.5$$

$$N = 50, M = 4, \quad \frac{2M+1}{N} = \frac{9}{50} = 0.18$$

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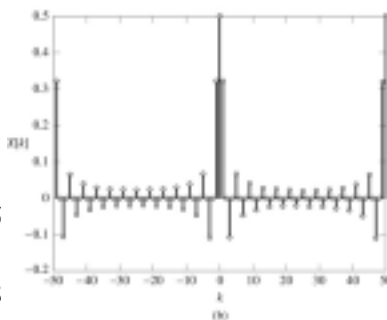
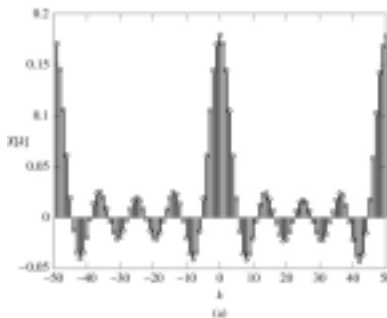
The DTFS coefficients for the square wave shown in Fig. 3.11, assuming a period $N = 50$:

- (a) $M = 4$.
- (b) $M = 12$.

** 討論 M 大小不同
其結果意義為何？

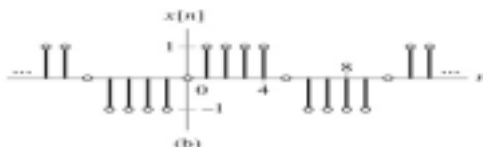
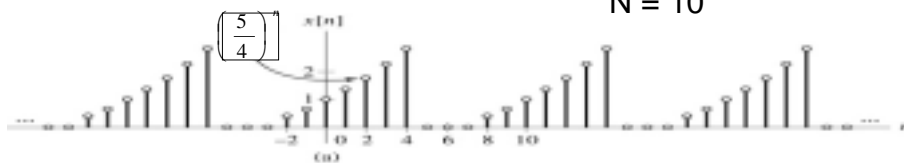
Ex: $N = 50, M = 12, \frac{2M+1}{N} = \frac{25}{50} = 0.5$

$N = 50, M = 4, \frac{2M+1}{N} = \frac{9}{50} = 0.18$



Problem 3.6: Find the DTFS of the $x[n] = ?$

$N = 10$



Signals $x[n]$ for Problem 3.6.



Example: $3.8 \text{ x}[n] \rightarrow X[k]$?

Electrocardiograms for two different heartbeats and the first 60 coefficients of their magnitude spectra.

- Normal heartbeat.
- Ventricular tachycardia.
- Magnitude spectrum for the normal heartbeat.
- Magnitude spectrum for ventricular tachycardia.

心室跳動過速

Or 心律不整 (arrhythmia)

