Audio & DSP Lab.

Fourier Transform of Signals

訊號的傅立葉轉換

Lecture 3-3

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Introduction

- 訊號表示為一組複數弦波的加權疊加
- 把複雜的訊號看成頻率的函數
- 傅立葉表示法
  - 連續時間傅立葉級數 (週期性) : FS
  - 連續時間傅立葉轉換 (非週期性) : FT
  - 離散時間傅立葉級數 (週期性) : DTFS
  - 離散時間傅立葉轉換 (非週期性) : DTFT

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Continuous-Time Periodic Signals (CTPS):
The Fourier Series (FS)

基本週期 T， 基本頻率 ω_0 = 2π/T 的週期訊號 x(t) :

X[k] is discrete spectrum and X[k] is non-periodic.

$$x(t) = \sum_{k=-\infty}^{+\infty} X[k]\, e^{jk\omega_0 t} \qquad (3)$$

x(t) is continuous signal and x(t) is periodic.

$$X[k] = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt \qquad (4)$$

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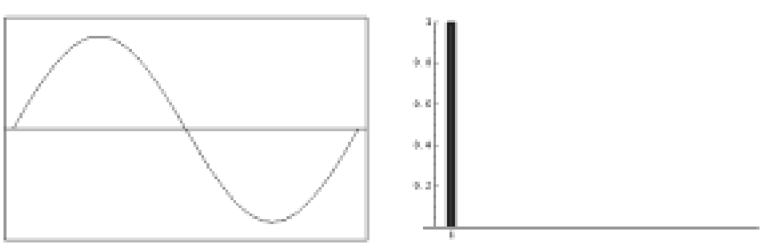
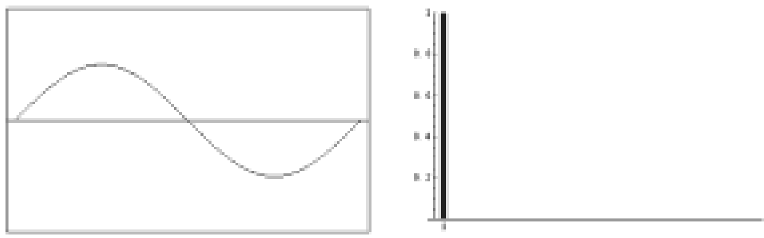
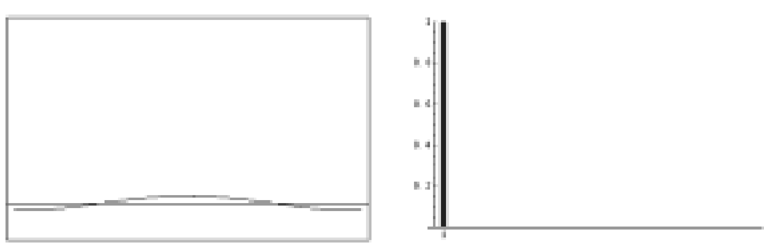
Square Wave & Spectra



Sawtooth wave & Spectrum



Sinc Wave & Spectrum



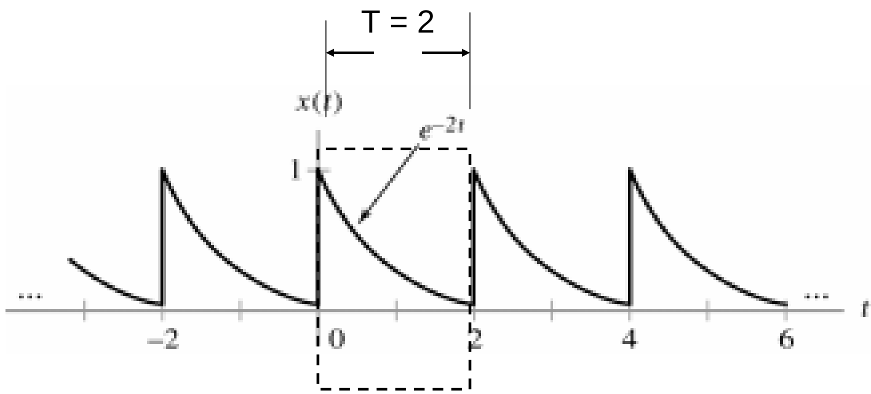
FS – Orthogonal Property

參考 DTFS 範例

請以 正交性特性 推導前頁 FS (4) 公式

Please derive and prove Equation (4)

Example 3.9: Find the FS of the x(t) below:



Time-domain signal for Example 3.9.

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Solution for Ex. 3.9

T=2, $\omega_0 = 2\pi/T = 2\pi/2 = \pi$

$$X[k] = \frac{1}{T}\int_0^T e^{-2t}\ e^{-jk\omega_0 t}\ dt = \frac{1}{2}\int_0^2 e^{-2t}\ e^{-jk\pi t}\ dt$$

$$= \frac{1}{2}\int_0^2 e^{-(2+jk\pi)t}\ dt = -\frac{1}{2(2+jk\pi)}\ e^{-(2+jk\pi)t}\Big|_0^2$$

$$= -\frac{1}{(4+j2k\pi)}e^{-(4+j2k\pi)} + \frac{1}{(4+j2k\pi)}$$

$$= \frac{1}{(4+j2k\pi)}\left(1-e^{-4}e^{-jk2\pi}\right) = \frac{1}{(4+j2k\pi)}\left(1-e^{-4}\right)$$

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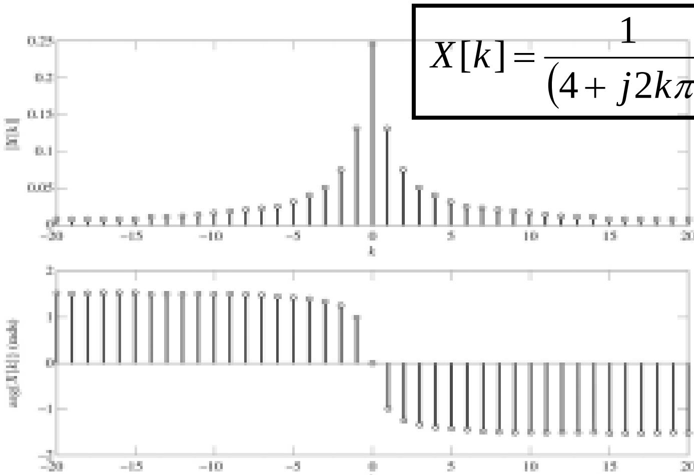
Magnitude and Phase Spectra for Ex. 3.9

$$X[k] = \frac{1}{(4+j2k\pi)}\left(1-e^{-4}\right)$$



*** 學生試寫出 $|X[k]|$ 和 $\arg\{X[k]\}$

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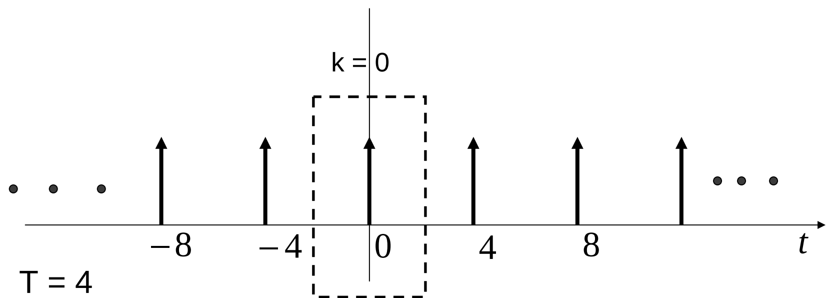
Ex. 3.10 Find the FS for the Impulse Train

$$x(t) = \sum_{k=-\infty}^{+\infty} \delta(t - 4k)$$



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Solution for Ex. 3.10

T = 4

$$\because x(t) = \sum_{k=-\infty}^{+\infty} \delta(t - 4k),$$

$$\therefore X[k] = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk\frac{2\pi}{T}t} dt$$

$$= \frac{1}{4}\int_{-2}^{2} \delta(t)\, e^{-jk\frac{\pi}{2}t} dt = \frac{1}{4}$$

$$\boxed{X[0] = \frac{1}{4}}$$

X[k] is discrete spectrum and X[k] is non-periodic.

|X[k]| is a constant, and the phase spectrum is zero.

Example 3.11:

Use “the method of inspection” to find the FS of x(t)=3cos(πt/2 + π/4).　使用審視法

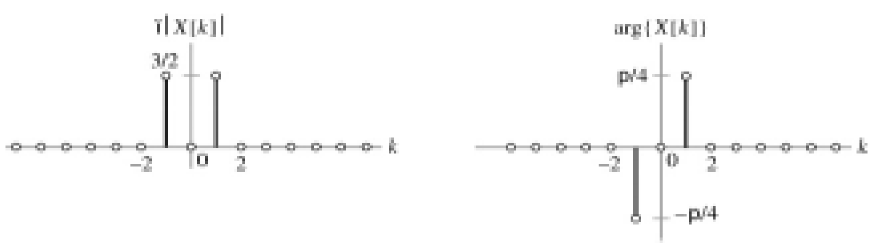
Solution: $\omega_0 = 2\pi/T = \pi/2$, T = 4

$$x(t) = 3\cos\left(\frac{\pi t}{2} + \frac{\pi}{4}\right)$$

$$= 3\cdot\frac{1}{2}\left(e^{j\left(\frac{\pi}{2}t+\frac{\pi}{4}\right)} + e^{-j\left(\frac{\pi}{2}t+\frac{\pi}{4}\right)}\right) = \frac{3}{2}\left(e^{j\frac{\pi}{4}}e^{j\frac{\pi}{2}t} + e^{-j\frac{\pi}{4}}e^{-j\frac{\pi}{2}t}\right)$$

$$= \frac{3}{2}e^{j\frac{\pi}{4}}e^{j\frac{\pi}{2}t} + \frac{3}{2}e^{-j\frac{\pi}{4}}e^{-j\frac{\pi}{2}t}$$

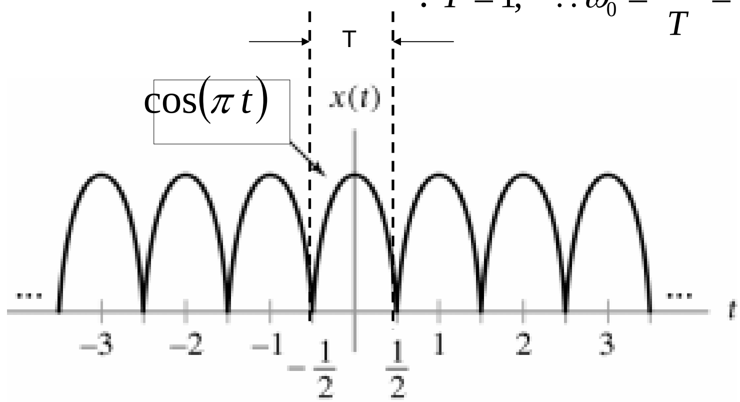
$$X[k] = \begin{cases} \frac{3}{2}e^{j\frac{\pi}{4}}, & k = 1 \\ \frac{3}{2}e^{-j\frac{\pi}{4}}, & k = -1 \\ 0, & otherwise \end{cases}$$



Magnitude and phase spectra for Example 3.11

Problem 3.8: Find the FS of the x(t) below:

$$\because T = 1, \quad \therefore \omega_0 = \frac{2\pi}{T} = 2\pi$$



Full-wave rectified cosine for Problem 3.8

Solution for the P3.8

$$X[k] = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk\omega_0 t} dt = \frac{1}{1}\int_{-1/2}^{1/2} \cos(\pi t)\, e^{-jk2\pi t} dt$$

$$= \frac{1}{2}\int_{-1/2}^{1/2} \left(e^{j\pi t} + e^{-j\pi t}\right) e^{-jk2\pi t} dt = \frac{1}{2}\int_{-1/2}^{1/2} \left(e^{j\pi t} e^{-jk2\pi t} + e^{-j\pi t} e^{-jk2\pi t}\right) dt$$

$$= \frac{1}{2}\int_{-1/2}^{1/2} \left(e^{j\pi t} e^{-jk2\pi t}\right) dt + \frac{1}{2}\int_{-1/2}^{1/2} \left(e^{-j\pi t} e^{-jk2\pi t}\right) dt$$

$$= \frac{1}{2}\int_{-1/2}^{1/2} \left(e^{j(1-2k)\pi t}\right) dt + \frac{1}{2}\int_{-1/2}^{1/2} \left(e^{-j(1+2k)\pi t}\right) dt$$

$$= \frac{1}{2j(1-2k)\pi} e^{j(1-2k)\pi t}\Bigg|_{-\frac{1}{2}}^{+\frac{1}{2}} - \frac{1}{2j(1+2k)\pi} e^{-j(1+2k)\pi t}\Bigg|_{-\frac{1}{2}}^{+\frac{1}{2}}$$

Solution (cont.)

$$X[k]=\frac{1}{2j(1-2k)\pi}e^{j(1-2k)\pi t}\Big|_{-\frac{1}{2}}^{+\frac{1}{2}}-\frac{1}{2j(1+2k)\pi}e^{-j(1+2k)\pi t}\Big|_{-\frac{1}{2}}^{+\frac{1}{2}}$$

$$=\frac{1}{2j(1-2k)\pi}\left(e^{j(1-2k)\pi/2}-e^{-j(1-2k)\pi/2}\right)$$

$$-\frac{1}{2j(1+2k)\pi}\left(e^{-j(1+2k)\pi/2}-e^{j(1-2k)\pi/2}\right)$$

$$=\frac{1}{(1-2k)\pi}\frac{e^{j(1-2k)\pi/2}-e^{-j(1-2k)\pi/2}}{j2}$$

$$+\frac{1}{(1+2k)\pi}\frac{e^{j(1+2k)\pi/2}-e^{-j(1+2k)\pi/2}}{j2}$$

$$X[k]=\frac{1}{(1-2k)\pi}\frac{e^{j(1-2k)\pi/2}-e^{-j(1-2k)\pi/2}}{j2}$$

$$+\frac{1}{(1+2k)\pi}\frac{e^{j(1+2k)\pi/2}-e^{-j(1+2k)\pi/2}}{j2}$$

$$=\frac{1}{(1-2k)\pi}\sin\left(\frac{\pi(1-2k)}{2}\right)+\frac{1}{(1+2k)\pi}\sin\left(\frac{\pi(1+2k)}{2}\right)$$

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Example 3.12:

[Inverse FS] Find the time-domain x(t) from

Solution: $X[k] = (1/2)^{|k|} e^{jk\pi/20}, \quad for\ T = 2.$

$$\omega_0 = 2\pi / T = 2\pi / 2 = \pi$$

$$x(t) = \sum_{k=0}^{+\infty} (1/2)^k e^{jk\pi/20} e^{jk\pi t} + \sum_{k=-1}^{-\infty} (1/2)^{-k} e^{jk\pi/20} e^{jk\pi t}$$

$let\ m = -k,$

$$= \sum_{k=0}^{+\infty} (1/2)^k e^{jk\pi/20} e^{jk\pi t} + \sum_{m=1}^{\infty} (1/2)^m e^{-jm\pi/20} e^{-jm\pi t}$$

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$$\sum_{k=0}^{+\infty} (1/2)^k e^{jk\pi/20} e^{jk\pi t}$$

$$= \sum_{k=0}^{+\infty} (1/2)^k e^{jk(\pi/20+\pi t)} = \sum_{k=0}^{+\infty} \left(\frac{1}{2} e^{j\left(\frac{\pi}{20}+\pi t\right)} \right)^k$$

$$= \frac{1}{1 - \frac{1}{2} e^{j\left(\frac{\pi}{20}+\pi t\right)}}$$

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$$\sum_{m=1}^{\infty} (1/2)^m e^{-jm\pi/20} e^{-jm\pi t}$$

$$= \sum_{m=0}^{\infty} \left(\frac{1}{2} e^{-j\left(\frac{\pi}{20}+\pi t\right)} \right)^m - 1$$

$$= \frac{1}{1-\frac{1}{2} e^{-j\left(\frac{\pi}{20}+\pi t\right)}} - 1$$

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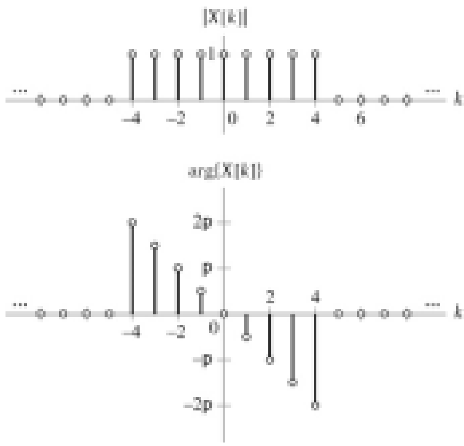
Solution:

$$x(t) = \frac{1}{1-\frac{1}{2} e^{j\left(\frac{\pi}{20}+\pi t\right)}} + \frac{1}{1-\frac{1}{2} e^{-j\left(\frac{\pi}{20}+\pi t\right)}} - 1$$

$$= \frac{1-\frac{1}{2} e^{-j\left(\frac{\pi}{20}+\pi t\right)} + 1 - \frac{1}{2} e^{j\left(\frac{\pi}{20}+\pi t\right)} - \left(1-\frac{1}{2} e^{j\left(\frac{\pi}{20}+\pi t\right)}\right)\left(1-\frac{1}{2} e^{-j\left(\frac{\pi}{20}+\pi t\right)}\right)}{\left(1-\frac{1}{2} e^{j\left(\frac{\pi}{20}+\pi t\right)}\right)\left(1-\frac{1}{2} e^{-j\left(\frac{\pi}{20}+\pi t\right)}\right)}$$

$$= \frac{2-\cos\left(\frac{\pi}{20}+\pi t\right) - \frac{5}{4} + \cos\left(\frac{\pi}{20}+\pi t\right)}{\frac{5}{4} - \cos\left(\frac{\pi}{20}+\pi t\right)} = \frac{\frac{3}{4}}{\frac{5}{4} - \cos\left(\frac{\pi}{20}+\pi t\right)}$$

Problem 3.9: Find the time-domain signal x(t) as the FS coefficients as below.



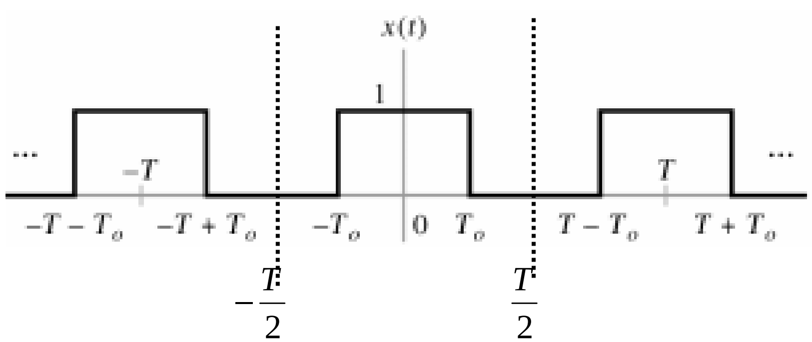
學生自行推算並驗證課本答案

$$x(t) = \sum_{k=-\infty}^{+\infty} X[k]\, e^{jk\omega_0 t}$$

$$X[k] = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

Ex 3.13: Find the FS for the Square wave as below:

$\omega_0 = 2\pi / T$ 積分區間：-T/2 ~ +T/2



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$$for \;\; k \neq 0,$$

$$X[k] = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_{-T_0}^{T_0} 1 \cdot e^{-jk\omega_0 t}\, dt = \frac{-1}{Tjk\omega_0} e^{-jk\omega_0 t}\Big|_{-T_0}^{T_0}$$

$$= \frac{-1}{Tjk\omega_0}\left(e^{-jk\omega_0 T_0} - e^{jk\omega_0 T_0}\right) = \frac{1}{Tjk\omega_0}\left(e^{jk\omega_0 T_0} - e^{-jk\omega_0 T_0}\right)$$

$$= \frac{2}{Tk\omega_0}\left(\frac{e^{jk\omega_0 T_0} - e^{-jk\omega_0 T_0}}{j2}\right) = \frac{2}{Tk\omega_0}\sin\left(k\omega_0 T_0\right)$$

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$$for \;\; k = 0,$$

$$X[0] = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, dt$$

$$= \frac{1}{T}\int_{-T_0}^{T_0} 1\, dt = \frac{1}{T} t\Big|_{-T_0}^{T_0} = \frac{T_0 - (-T_0)}{T} = \frac{2T_0}{T}$$

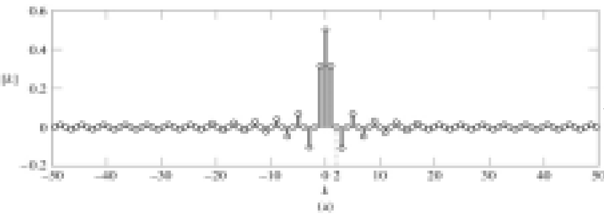
$$for \;\; k \neq 0, \quad \omega_0 = 2\pi / T$$

$$X[k] = \frac{2}{Tk\omega_0}\sin\left(k\omega_0 T_0\right) = \frac{2T}{Tk2\pi}\sin\left(k2\pi T_0 / T\right)$$

$$= \frac{1}{k\pi}\sin\left(2\pi k T_0 / T\right)$$

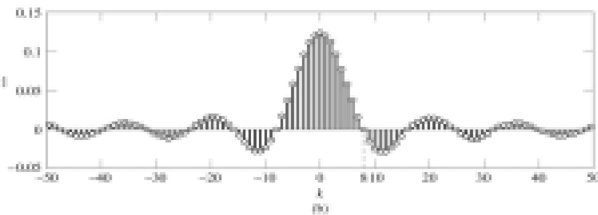
The FS coefficients, $X[k]$, $-50 \le k \le 50$, for three square waves. (a) $T_0/T = 1/4$. (b) $T_0/T = 1/16$.

$$\frac{1}{k\pi}\sin\left(2\pi k T_0 / T\right)$$



EX : 1st zero crossing at $k = 2$

$$\frac{1}{2\pi}\sin\left(2\pi 2 \frac{1}{4}\right) = 0$$

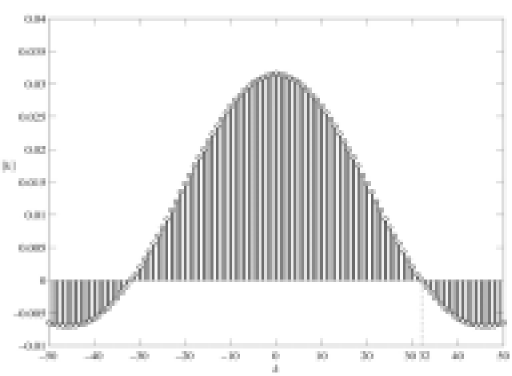


EX : 1st zero crossing at $k = 8$

$$\frac{1}{8\pi}\sin\left(2\pi 8 \frac{1}{16}\right) = 0$$

(c) $T_0/T = 1/64$.

The first zero-crossing point is at k=32.



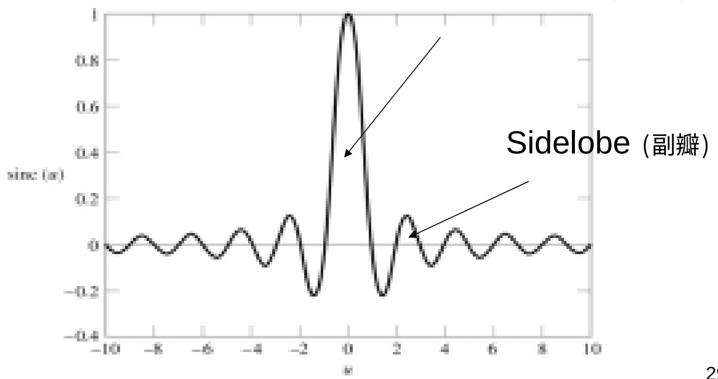
$EX : k = 32$

$$\frac{1}{8\pi}\sin\left(2\pi 32 \frac{1}{64}\right) = 0$$

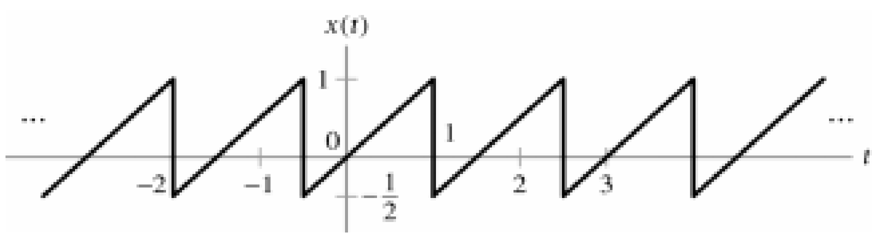
A special function:

Sinc function: sinc(u) = sin(πu)/(πu)

The 1st zero crossing is at u = 1.



Problem 3.10: Find the FS for the Sawtooth wave as below:



Periodic signal for Problem 3.10

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Solution:

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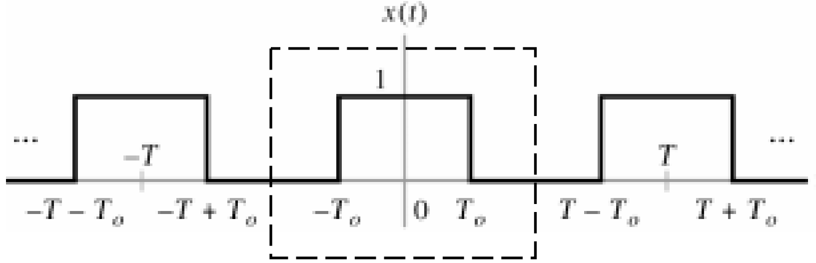
Ex 3.15:

Calculating the RC integrator circuit output by means of FS.

參考： $X[k] = \frac{1}{k\pi}\sin\left(2\pi k T_0 / T\right)$



Solution:

$$impute: \quad x(t) = \sum_{k=-\infty}^{+\infty} X[k]\, e^{jk\omega_0 t},$$

$$output: \quad y(t) = \sum_{k=-\infty}^{+\infty} H(jk\omega_0) X[k]\, e^{jk\omega_0 t},$$

$$y(t) \overset{FS}{\leftrightarrow} Y[k] = H(jk\omega_0)\, X[k]$$

$$\because \quad from \quad Example\ 3.1, \quad \boxed{H(jk\omega_0) = \frac{1/RC}{jk\omega_0 + 1/RC}}$$

$$\Rightarrow \quad for \quad RC = 0.1, \quad \omega_0 = 2\pi, \quad H(jk\omega_0) = \frac{10}{j2\pi k + 10}$$

$$from \quad Example\ 3.13, \quad T_0/T = 1/4, \quad X[k] = \frac{\sin(k\pi/2)}{k\pi}$$

$$\therefore \quad Y[k] = H(jk\omega_0)\, X[k] = \frac{10}{j2\pi k + 10} \frac{\sin(k\pi/2)}{k\pi}$$