



# Fourier Transform of Signals

## 訊號的傅立葉轉換 Lecture 3-4

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## Discrete-Time Non-periodic Signals: The Discrete-Time Fourier Transform (DTFT)

- A discrete-time non-periodic signal is represented by a superposition of complex sinusoids.
- The DTFT involves continuous frequencies on the interval  $-\pi < \Omega < \pi$ . (頻率連續分佈, 週期性頻譜)
- The DTFT pair:

$X(e^{j\Omega})$  is continuous spectrum.

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{+\pi} X(e^{j\Omega}) e^{j\Omega n} d\Omega$$

$x[n]$  is discrete signal.

(離散時間、非週期性訊號)

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\Omega n}$$

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## The Discrete-Time Fourier Transform (DTFT)

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{+\pi} X(e^{j\Omega}) e^{j\Omega n} d\Omega \quad (5)$$

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\Omega n} \quad (6)$$



### Example 3.17:

Find the DTFT of an exponential signal  $x[n] = (\alpha)^n u[n]$ .

Solution:

$$\begin{aligned} X(e^{j\Omega}) &= \sum_{n=-\infty}^{+\infty} \alpha^n u[n] e^{-j\Omega n} = \sum_{n=0}^{+\infty} \alpha^n e^{-j\Omega n} \\ &= \sum_{n=0}^{+\infty} \left( \alpha e^{-j\Omega} \right)^n = \frac{1}{1 - \alpha e^{-j\Omega}}, \quad |\alpha| < 1 \end{aligned}$$



If  $\alpha$  is real valued, we may expand of the denominator as:

$$\therefore X(e^{j\Omega}) = \frac{1}{1 - \alpha e^{-j\Omega}} = \frac{1}{1 - \alpha \cos \Omega + j\alpha \sin \Omega}$$

The magnitude spectrum is denoted by :

$$\begin{aligned} \therefore |X(e^{j\Omega})| &= \frac{1}{\sqrt{(1 - \alpha \cos \Omega)^2 + (\alpha \sin \Omega)^2}} \\ &= \frac{1}{\sqrt{1 - 2\alpha \cos \Omega + \alpha^2 \cos^2 \Omega + \alpha^2 \sin^2 \Omega}} \\ &= \frac{1}{\sqrt{1 - 2\alpha \cos \Omega + \alpha^2}} \end{aligned}$$

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The phase spectrum is denoted by :

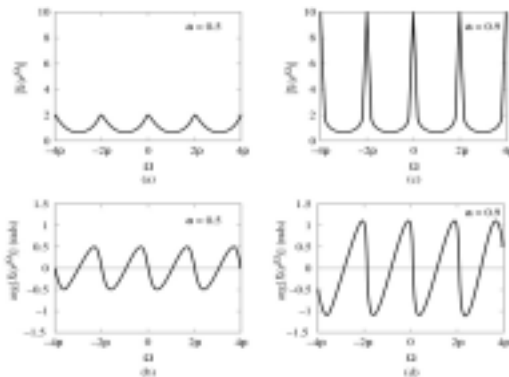
$$\begin{aligned} \therefore X(e^{j\Omega}) &= \frac{1}{1 - \alpha e^{-j\Omega}} = \frac{1}{1 - \alpha \cos \Omega + j\alpha \sin \Omega} \\ \therefore \arg\{X(e^{j\Omega})\} &= -\tan^{-1} \frac{\alpha \sin \Omega}{1 - \alpha \cos \Omega} \end{aligned}$$

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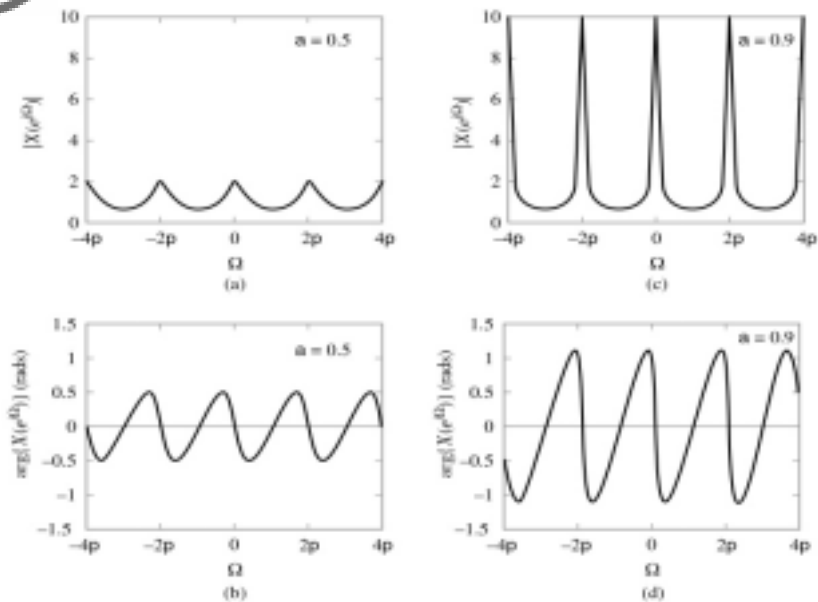


The DTFT of an exponential signal  $x[n] = (\alpha)^n u[n]$ .

- Magnitude spectrum for  $\alpha = 0.5$ .
- Phase spectrum for  $\alpha = 0.5$ .
- Magnitude spectrum for  $\alpha = 0.9$ .
- Phase spectrum for  $\alpha = 0.9$ .



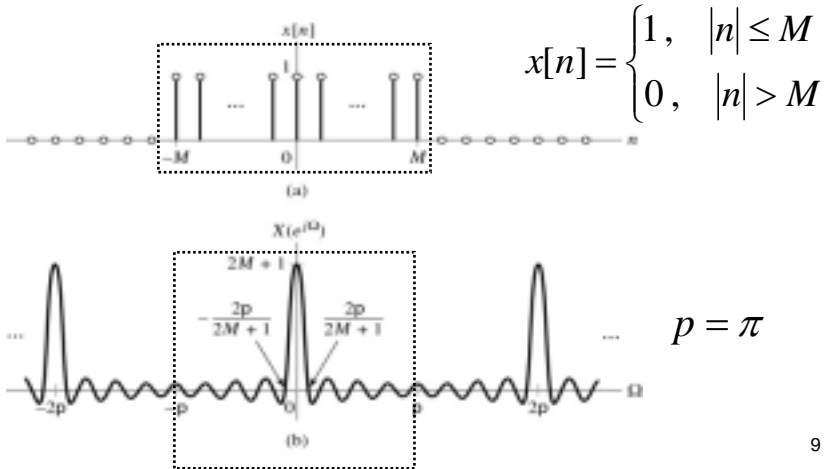
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- Example 3.18. Find the DTFT of  $x[n] = ?$
- Rectangular pulse in the time domain.
  - DTFT in the frequency domain.



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$$\forall \Omega \neq 0, \pm 2\pi, \pm 4\pi, \dots$$

$$X(e^{j\Omega}) = \sum_{n=-M}^{+M} x[n] e^{-j\Omega n} = \sum_{n=-M}^{+M} 1 e^{-j\Omega n}$$

$$\text{let } m = n + M,$$

$$= \sum_{m=0}^{+2M} 1 e^{-j\Omega(m-M)} = e^{j\Omega M} \sum_{m=0}^{+2M} 1 e^{-j\Omega m}$$

$$= e^{j\Omega M} \frac{1 - e^{-j\Omega(2M+1)}}{1 - e^{-j\Omega}}$$

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$$\begin{aligned}
 X(e^{j\Omega}) &= e^{j\Omega M} \frac{1 - e^{-j\Omega(2M+1)}}{1 - e^{-j\Omega}} \\
 &= e^{j\Omega M} \frac{e^{-j\Omega \frac{2M+1}{2}} \left( e^{j\Omega \frac{2M+1}{2}} - e^{-j\Omega \frac{2M+1}{2}} \right)}{e^{-j\frac{\Omega}{2}} \left( e^{j\frac{\Omega}{2}} - e^{-j\frac{\Omega}{2}} \right)} \\
 &= e^{j\Omega M} \frac{e^{-j\Omega \frac{2M+1}{2}} \left( e^{j\Omega \frac{2M+1}{2}} - e^{-j\Omega \frac{2M+1}{2}} \right)}{e^{-j\frac{\Omega}{2}} \left( e^{j\frac{\Omega}{2}} - e^{-j\frac{\Omega}{2}} \right)}
 \end{aligned}$$

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$$\begin{aligned}
 X(e^{j\Omega}) &= e^{j\Omega M} \frac{e^{-j\Omega \frac{2M+1}{2}} \left( e^{j\Omega \frac{2M+1}{2}} - e^{-j\Omega \frac{2M+1}{2}} \right)}{e^{-j\frac{\Omega}{2}} \left( e^{j\frac{\Omega}{2}} - e^{-j\frac{\Omega}{2}} \right)} \\
 &= \frac{\left( e^{j\Omega \frac{2M+1}{2}} - e^{-j\Omega \frac{2M+1}{2}} \right) / j2}{\left( e^{j\frac{\Omega}{2}} - e^{-j\frac{\Omega}{2}} \right) / j2} = \frac{\sin\left(\frac{(2M+1)\Omega}{2}\right)}{\sin\left(\frac{\Omega}{2}\right)}
 \end{aligned}$$

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$$\forall \Omega = 0, \pm 2\pi, \pm 4\pi, \dots$$

$$X(e^{j\Omega}) = \sum_{n=-M}^{+M} x[n] e^{-j\Omega n} = \sum_{n=-M}^{+M} 1 \cdot e^{-j\Omega n}$$

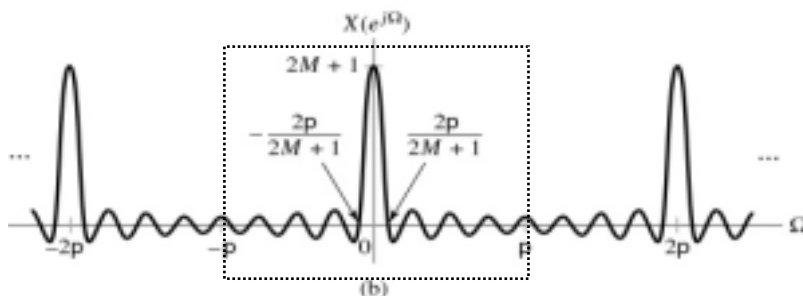
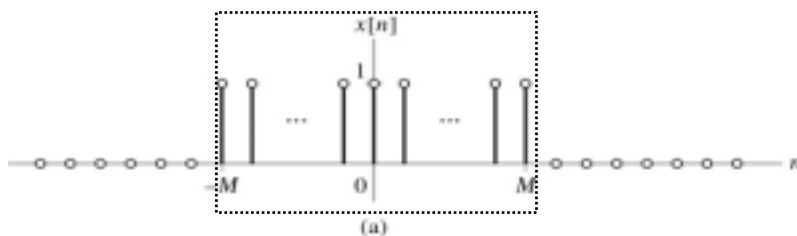
let  $m = n + M$ ,

$$= \sum_{m=0}^{+2M} 1 \cdot e^{-j\Omega(m-M)} = e^{j\Omega M} \sum_{m=0}^{+2M} 1 \cdot e^{-j\Omega m}$$

$$= e^{j2\pi M} \sum_{m=0}^{+2M} 1 \cdot e^{-j2\pi m} = 1 \cdot \sum_{m=0}^{+2M} 1$$

$$= 2M + 1$$

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The first zero crossing is at  $\pm 2\pi/(2M+1)$ .

$$\text{at } \Omega = \frac{2\pi}{2M+1},$$

$$\begin{aligned} X(e^{j\Omega}) &= \frac{\sin\left(\frac{(2M+1)\Omega}{2}\right)}{\sin\left(\frac{\Omega}{2}\right)} = \frac{\sin\left(\frac{(2M+1)}{2} \frac{2\pi}{2M+1}\right)}{\sin\left(\frac{1}{2} \frac{2\pi}{2M+1}\right)} \\ &= \frac{\sin(\pi)}{\sin\left(\frac{\pi}{2M+1}\right)} = 0 \end{aligned}$$

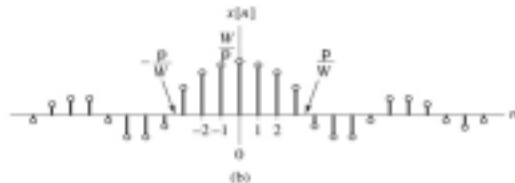
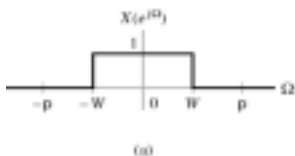
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Example 3.19. Find the  $x[n] = ?$

- Rectangular pulse in the frequency domain.
- Inverse DTFT in the time domain.

$$X(e^{j\Omega}) = \begin{cases} 1, & |\Omega| < W \\ 0, & W < |\Omega| < \pi \end{cases}$$



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Solution:  $n \neq 0$  case

$$\begin{aligned}
 x[n] &= \frac{1}{2\pi} \int_{-\pi}^{+\pi} X(e^{j\Omega}) e^{j\Omega n} d\Omega \\
 &= \frac{1}{2\pi} \int_{-W}^{+W} 1 \cdot e^{j\Omega n} d\Omega \\
 &= \frac{1}{2\pi} \left\{ \frac{1}{jn} e^{j\Omega n} \Big|_{-W}^{+W} \right\} = \frac{1}{\pi n} \frac{e^{jWn} - e^{-jWn}}{j2} \\
 &= \frac{1}{\pi n} \sin(W n)
 \end{aligned}$$

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Solution:  $n = 0$  case

Considering the L'Hospital Rule:

$$x[0] = \lim_{n \rightarrow 0} \frac{1}{\pi n} \sin(W n) = \frac{W}{\pi}$$


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The first zero crossing is at  $n = \pm\pi/W$ .

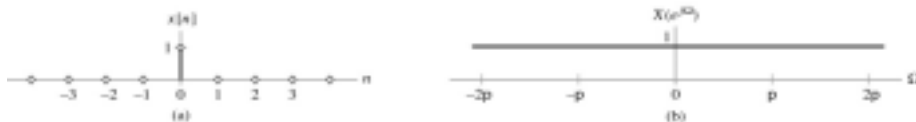
$$x[\pi / W] = \frac{1}{\pi \frac{\pi}{W}} \sin\left(W \frac{\pi}{W}\right) = \frac{W}{\pi^2} \sin(\pi) = 0.$$

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Example 3.20. Find DTFT of the  $x[n] = \delta[n]$  ?

- (a) Unit impulse in the time domain.  
 (b) DTFT of unit impulse in the frequency domain.



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Solution:

$$\because x[n] = \delta[n]$$

$$\therefore X(e^{j\Omega}) = \sum_{n=-\infty}^{+\infty} \delta[n] e^{-j\Omega n} = 1$$

The DTFT vs. Unit Impulse :

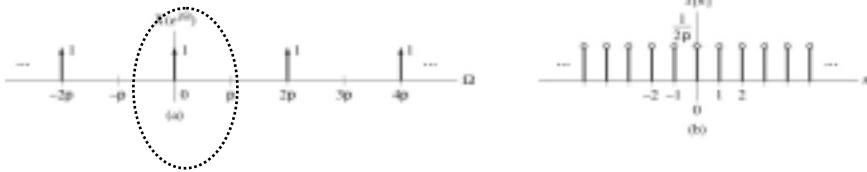
$$\delta[n] \xleftrightarrow{\text{DTFT}} 1$$

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Example 3.21. Find the  $x[n] = ?$   $X(e^{j\Omega}) = \delta(\Omega)$

- (a) Unit impulse in the frequency domain.  
 (b) Inverse DTFT in the time domain.



頻率連續分佈，週期性頻譜

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Solution: defines only one period as,

$$\forall X(e^{j\Omega}) = \delta(\Omega), \quad -\pi < \Omega \leq \pi$$

$$\therefore x[n] = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \delta(\Omega) e^{j\Omega n} d\Omega = \frac{1}{2\pi}$$

The DTFT vs. Unit Impulse :

$$\frac{1}{2\pi} \xleftrightarrow{DTFT} \delta(\Omega)$$

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Similarly, defines all delta functions as:

$$X(e^{j\Omega}) = \sum_{k=-\infty}^{\infty} \delta(\Omega - 2k\pi) \quad \text{頻率連續分佈, 週期性頻譜}$$

The inverse of the DTFT is also impulse train:

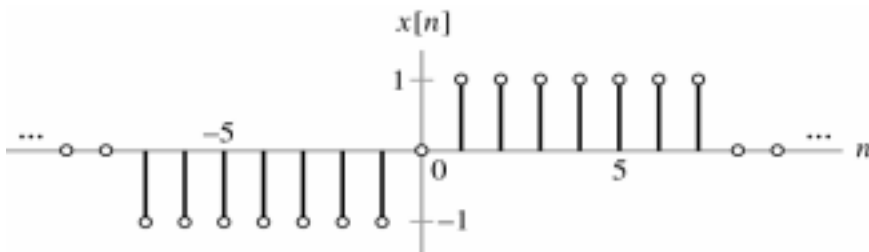
$$x[n] = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \delta[n - k]$$

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Problem 3.12.

Find the DTFT of the  $x[n] = ?$   $X(e^{j\Omega}) = ?$



Signal  $x[n]$  for Problem 3.12.

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Solution:

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} = \sum_{n=-7}^{-1} (-1) e^{-j\Omega n} + \sum_{m=1}^7 (1) e^{-j\Omega m}$$

let  $l = n + 7$ ,  $p = m - 1$ ,

$$= \sum_{l=0}^6 (-1) e^{-j\Omega(l-7)} + \sum_{p=0}^6 (1) e^{-j\Omega(p+1)}$$

$$= -e^{j7\Omega} \sum_{l=0}^6 e^{-j\Omega l} + e^{-j\Omega} \sum_{p=0}^6 e^{-j\Omega p}$$

$$= -e^{j7\Omega} \frac{1 - e^{-j\Omega 7}}{1 - e^{-j\Omega}} + e^{-j\Omega} \frac{1 - e^{-j\Omega 7}}{1 - e^{-j\Omega}}$$

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Solution:

$$\begin{aligned} X(e^{j\Omega}) &= \frac{1 - e^{-j\Omega 7}}{1 - e^{-j\Omega}} (e^{-j\Omega} - e^{j7\Omega}) \\ &= \frac{e^{-j\frac{7\Omega}{2}} (e^{j\frac{7\Omega}{2}} - e^{-j\frac{7\Omega}{2}})}{e^{-j\frac{\Omega}{2}} (e^{j\frac{\Omega}{2}} - e^{-j\frac{\Omega}{2}})} (e^{-j\Omega} - e^{j7\Omega}) \\ &= \frac{e^{-j3} (e^{j\frac{7\Omega}{2}} - e^{-j\frac{7\Omega}{2}})}{(e^{j\frac{\Omega}{2}} - e^{-j\frac{\Omega}{2}})} (e^{-j\Omega} - e^{j7\Omega}) \end{aligned}$$

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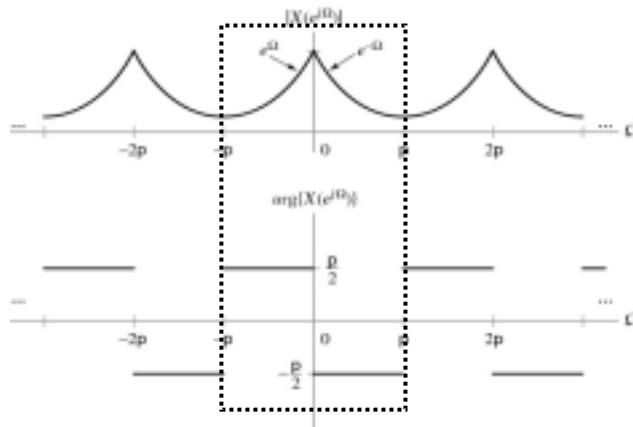
Solution:

$$\begin{aligned}
 X(e^{j\Omega}) &= \frac{(e^{j\frac{7\Omega}{2}} - e^{-j\frac{7\Omega}{2}})}{(e^{j\frac{\Omega}{2}} - e^{-j\frac{\Omega}{2}})} (e^{-j4\Omega} - e^{j4\Omega}) \\
 &= -j2 \frac{(e^{j\frac{7\Omega}{2}} - e^{-j\frac{7\Omega}{2}}) / j2}{(e^{j\frac{\Omega}{2}} - e^{-j\frac{\Omega}{2}}) / j2} (e^{j4\Omega} - e^{-j4\Omega}) / j2 \\
 &= -j2 \frac{\sin\left(\frac{7\Omega}{2}\right) \sin(4\Omega)}{\sin\left(\frac{\Omega}{2}\right)}
 \end{aligned}$$

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Problem 3.13. Find the inverse DTFT of the following frequency-domain signals:  $x[n] = ?$



Frequency-domain signal for Problem 3.13.

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Solution:

$$\begin{aligned}
 x[n] &= \frac{1}{2\pi} \int_{-\pi}^{+\pi} X(e^{j\Omega}) e^{j\Omega n} d\Omega \\
 &= \frac{1}{2\pi} \int_{-\pi}^0 (e^{\Omega} e^{j\frac{\pi}{2}}) e^{j\Omega n} d\Omega + \frac{1}{2\pi} \int_0^{+\pi} (e^{-\Omega} e^{-j\frac{\pi}{2}}) e^{j\Omega n} d\Omega \\
 &= \frac{j}{2\pi} \left\{ \int_{-\pi}^0 \cdot e^{(1+jn)\Omega} d\Omega - \int_0^{+\pi} \cdot e^{(-1+jn)\Omega} d\Omega \right\} \\
 &= \frac{j}{2\pi} \left\{ \frac{1}{1+jn} e^{(1+jn)\Omega} \Big|_{-\pi}^0 + \frac{1}{1-jn} e^{(-1+jn)\Omega} \Big|_0^{\pi} \right\}
 \end{aligned}$$

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$$\begin{aligned}
 x[n] &= \frac{j}{2\pi} \left\{ \frac{1 - e^{-(1+jn)\pi}}{1+jn} + \frac{e^{(-1+jn)\pi} - 1}{1-jn} \right\} \\
 &= \frac{j}{2\pi} \left\{ \frac{1 - e^{-\pi} e^{-jn\pi}}{1+jn} + \frac{e^{-\pi} e^{jn\pi} - 1}{1-jn} \right\} = \frac{j}{2\pi} \left\{ \frac{1 - e^{-\pi} (-1)^n}{1+jn} - \frac{1 - e^{-\pi} (-1)^n}{1-jn} \right\} \\
 &= \frac{j}{2\pi} \left\{ \frac{[1 - e^{-\pi} (-1)^n](1-jn) - [1 - e^{-\pi} (-1)^n](1+jn)}{(1+jn)(1-jn)} \right\} \\
 &= \frac{j}{2\pi} \left\{ \frac{\cancel{1-jn} - e^{-\pi} (-1)^n + jne^{-\pi} (-1)^n - \cancel{1} + e^{-\pi} (-1)^n - jn + jne^{-\pi} (-1)^n}{(1+n^2)} \right\}
 \end{aligned}$$

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$$\begin{aligned}
 x[n] &= \frac{j}{2\pi} \left\{ \frac{-2jn + 2jne^{-\pi}(-1)^n}{(1+n^2)} \right\} \\
 &= \frac{1}{2\pi} \left\{ \frac{-2j^2n + 2j^2ne^{-\pi}(-1)^n}{(1+n^2)} \right\} \\
 &= \frac{1}{2\pi} \left\{ \frac{2n - 2ne^{-\pi}(-1)^n}{(1+n^2)} \right\} = \frac{n}{\pi} \left\{ \frac{1 - e^{-\pi}(-1)^n}{(1+n^2)} \right\}
 \end{aligned}$$



Example 3.22: Consider two moving-average systems described by the equation:

$$y_1[n] = \frac{1}{2}(x[n] + x[n-1]) \quad \text{and} \quad y_2[n] = \frac{1}{2}(x[n] - x[n-1])$$

It implies that the impulse responses of these systems are:

$$h_1[n] = \frac{1}{2}(\delta[n] + \delta[n-1]) \quad \text{and} \quad h_2[n] = \frac{1}{2}(\delta[n] - \delta[n-1])$$

Please find the frequency response of each system and plot the magnitude response.





Solution:

$$H_1(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} h_1[n] e^{-j\Omega n} = \sum_{n=0}^{\infty} \left\{ \frac{1}{2} \delta[n] + \frac{1}{2} \delta[n-1] \right\} e^{-j\Omega n}$$

$$= \frac{1}{2} + \frac{1}{2} e^{-j\Omega}$$


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$$= \frac{1}{2} e^{-j\frac{\Omega}{2}} \left( e^{j\frac{\Omega}{2}} + e^{-j\frac{\Omega}{2}} \right) = e^{-j\frac{\Omega}{2}} \cos\left(\frac{\Omega}{2}\right)$$

$$|H_1(e^{j\Omega})| = \left| \cos\left(\frac{\Omega}{2}\right) \right|$$

$$\arg\{H_1(e^{j\Omega})\} = -\frac{\Omega}{2}$$

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$$H_2(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} h_2[n] e^{-j\Omega n} = \sum_{n=0}^{\infty} \left\{ \frac{1}{2} \delta[n] - \frac{1}{2} \delta[n-1] \right\} e^{-j\Omega n}$$

$$= \frac{1}{2} - \frac{1}{2} e^{-j\Omega}$$


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$$= j \cdot e^{-j\frac{\Omega}{2}} \left( e^{j\frac{\Omega}{2}} - e^{-j\frac{\Omega}{2}} \right) / j2 = j \cdot e^{-j\frac{\Omega}{2}} \sin\left(\frac{\Omega}{2}\right)$$

$$= \begin{cases} j \cdot e^{-j\frac{\Omega}{2}} \sin\left(\frac{\Omega}{2}\right), & \forall \Omega > 0 \\ -j \cdot e^{-j\frac{\Omega}{2}} \sin\left(\frac{\Omega}{2}\right), & \forall \Omega < 0 \end{cases}$$

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$$H_2(e^{j\Omega}) = j \cdot e^{-j\frac{\Omega}{2}} \sin\left(\frac{\Omega}{2}\right), \quad \forall \Omega > 0$$

$$= e^{j\frac{\pi}{2}} \cdot e^{-j\frac{\Omega}{2}} \sin\left(\frac{\Omega}{2}\right) = e^{j\left(\frac{\pi}{2} - \frac{\Omega}{2}\right)} \cdot \sin\left(\frac{\Omega}{2}\right)$$


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$$|H_2(e^{j\Omega})| = \left| \sin\left(\frac{\Omega}{2}\right) \right|$$

$$\arg\{H_2(e^{j\Omega})\} = \frac{\pi}{2} - \frac{\Omega}{2}$$



$$H_2(e^{j\Omega}) = -j \cdot e^{-j\frac{\Omega}{2}} \sin\left(\frac{\Omega}{2}\right), \quad \forall \Omega < 0$$

$$= e^{-j\frac{\pi}{2}} \cdot e^{-j\frac{\Omega}{2}} \sin\left(\frac{\Omega}{2}\right) = e^{j\left(-\frac{\pi}{2} - \frac{\Omega}{2}\right)} \cdot \sin\left(\frac{\Omega}{2}\right)$$


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$$|H_2(e^{j\Omega})| = \left| \sin\left(\frac{\Omega}{2}\right) \right|$$

$$\arg\{H_2(e^{j\Omega})\} = -\frac{\pi}{2} - \frac{\Omega}{2}$$