Audio & DSP Lab.

Fourier Transform of Signals

訊號的傅立葉轉換

Lecture 3-5

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Continuous-Time Non-Periodic Signals

The Fourier Transform (FT)

- 連續時間的非週期性訊號表示為複數弦波的疊加
- 複數弦波所包含的頻率是從 - ∞ 到 + ∞ 連續分佈

X(jω) is continuous spectrum and non-periodic.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{j\omega t}\, d\omega \qquad (7)$$

x(t) is continuous signal and non-periodic.

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \qquad (8)$$

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回憶 FS (if T →∞)

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]\, e^{j\omega_0 kt}$$

$$X[k] = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk\omega_0 t} dt$$

→ FT 討論 ?

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$$

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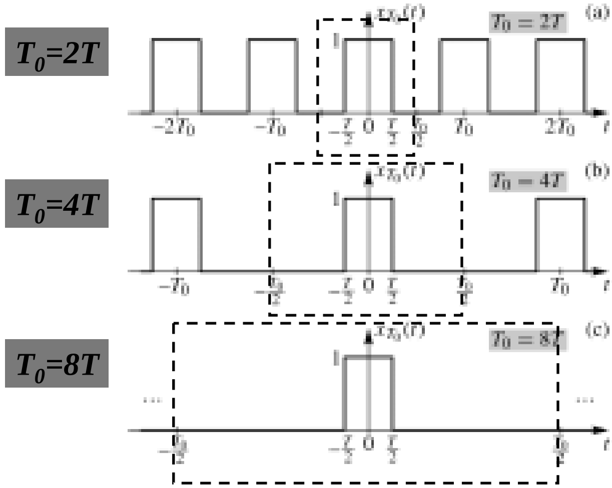
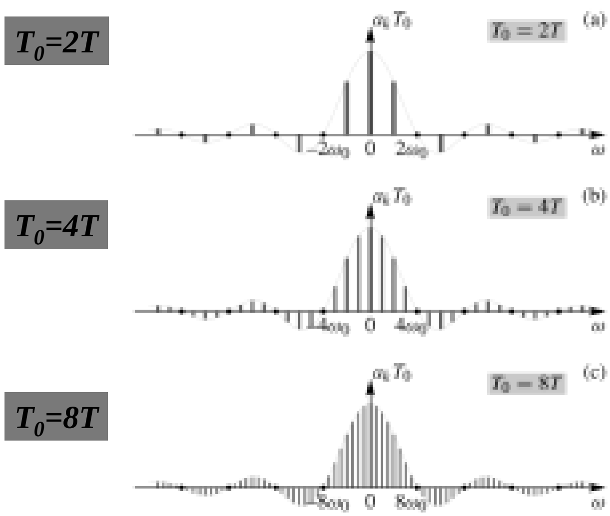
What if Signal *x(t)* is not periodic?

- Still Sum of Sinusoids?
  - Non-harmonically related sinusoids
  - Would not be periodic, but would probably be non-zero for all *t.*
- Need "Fourier Transform" to Find Spectrum?
  - gives a "sum" (actually an **integral**) that involves **ALL** frequencies
  - can represent signals that are identically zero for negative *t*. !!!!!!!!!

Limiting Behavior of FS



Limiting Behavior of Spectrum



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FS in the LIMIT (long period)

Fourier Series Formula: $$x_{T_0}(t) = \sum_{k=-\infty}^{\infty} X[k] e^{j\omega_0 kt}$$

$k\omega_0 = \omega$ variable

$2\pi / T_0 = \omega_0$

$\lim_{T_0 \to \infty} 2\pi / T_0 = d\omega$

$$x_{T_0}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \left[T_0 X[k]\right] e^{j(\omega_0 k)t} \left(\frac{2\pi}{T_0}\right)$$

$$\Updownarrow$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[X(j\omega)\right] e^{j(\omega)t} d\omega$$

$$\lim_{T_0 \to \infty} x_{T_0}(t) = x(t) \quad for\ -\infty < t < \infty$$

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Fourier Synthesis

$$x_{T_0}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \left(T_0 X[k]\right) e^{j\omega_0 kt} \left(\frac{2\pi}{T_0}\right) \mapsto x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

$$\lim_{T_0 \to \infty} \frac{2\pi}{T_0} = d\omega \qquad \lim_{T_0 \to \infty} \frac{2\pi}{T_0} k = \omega \qquad \lim_{T_0 \to \infty} T_0 \cdot X[k] = X(j\omega)$$

$$X[k] = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x_{T_0}(t) e^{-j\omega_0 kt} dt \mapsto X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

FS

Fourier Analysis

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Fourier Transform Defined

- For non-periodic signals

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

Fourier Synthesis

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Fourier Analysis

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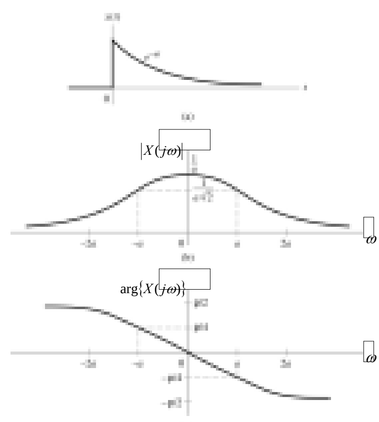
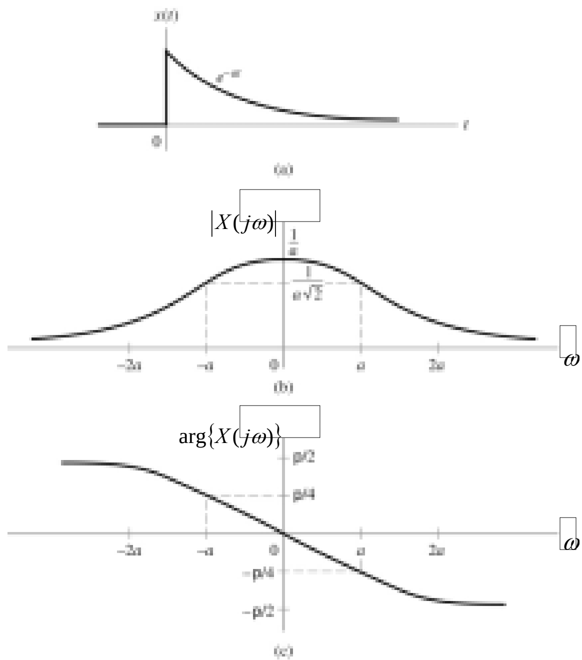


Ex 3.24 Find the FT of $x(t) = e^{-at} u(t)$:

(a) Real time-domain exponential signal.
(b) Magnitude spectrum.
(c) Phase spectrum.



Solution of Ex 3.24:

$case\,1: \quad a \le 0,$ (Growing Exponential Signal)

$$\int_0^{\infty} e^{-at}\, dt = \infty. \qquad \textit{not absolutely summable.}$$

$case\,2: \quad a > 0,$ (Decaying Exponential Signal)

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-at} u(t) e^{-j\omega t}\, dt$$

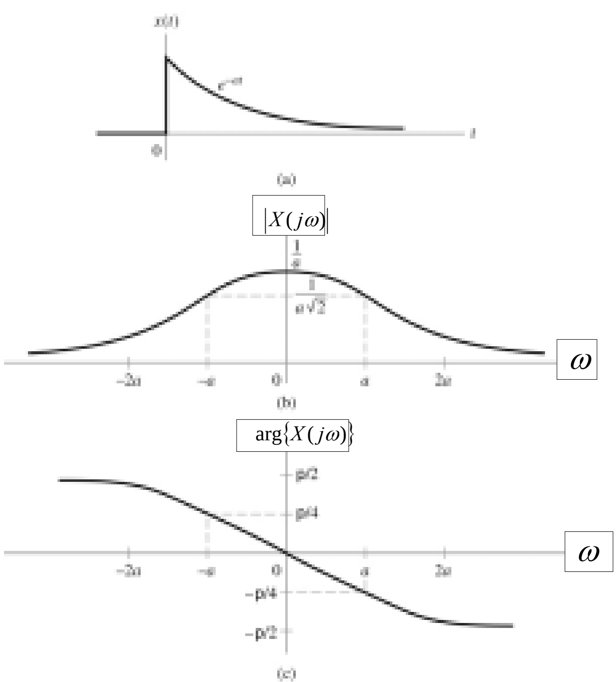
$$= \int_0^{\infty} e^{-(a+j\omega)t}\, dt = \frac{-1}{a+j\omega} e^{-(a+j\omega)t}\Big|_0^{\infty} = \frac{-1}{a+j\omega}(0-1)$$

$$= \frac{1}{a+j\omega}$$

Solution: (cont.)

$$X(j\omega) = \frac{1}{a + j\omega}$$

$$\left|X(j\omega)\right| = \frac{1}{\sqrt{a^2 + \omega^2}}; \qquad \arg\{X(j\omega)\} = -\tan^{-1}\left(\frac{\omega}{a}\right)$$



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Example 3.25:

Consider the rectangular pulse in the figure (a), and find the FT of x(t), X(jω) = ?

$$x(t) = \begin{cases} 1, & -T_0 < t < T_0 \\ 0, & |t| > T_0 \end{cases}$$

(a) Rectangular pulse in the time domain.
(b) FT in the frequency domain.

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Solution of Ex 3.25:

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

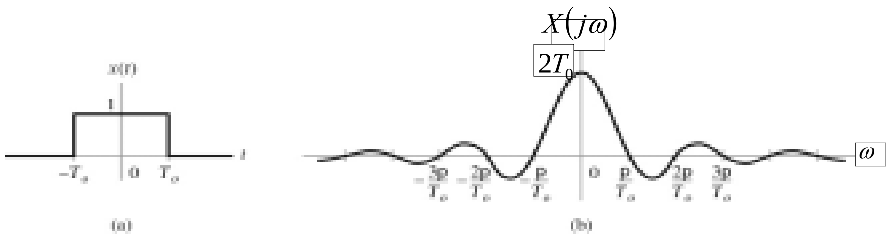
$$= \int_{-T_0}^{+T_0} 1 \cdot e^{-j\omega t}\, dt = -\frac{1}{j\omega} e^{-j\omega t} \Big|_{-T_0}^{T_0}$$

$$= -\frac{1}{j\omega}\left(e^{-j\omega T_0} - e^{j\omega T_0}\right) = \frac{1}{j\omega}\left(e^{j\omega T_0} - e^{-j\omega T_0}\right)$$

$$= \frac{2}{\omega}\left(\frac{e^{j\omega T_0} - e^{-j\omega T_0}}{j2}\right) = \frac{2}{\omega}\sin(\omega T_0), \quad \forall \omega \neq 0$$

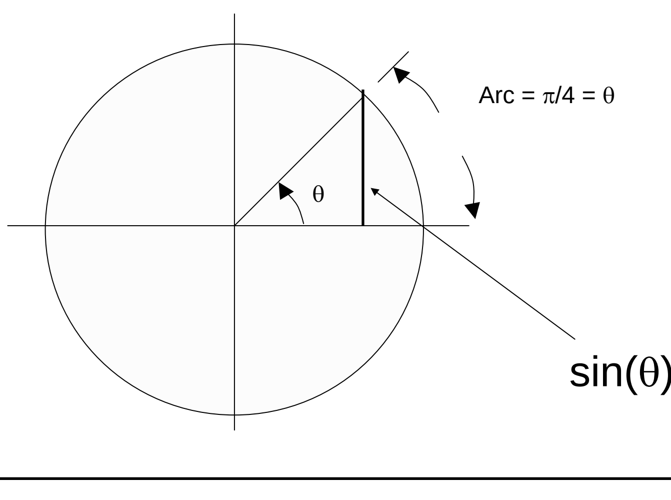
$$\forall \omega = 0,$$

$$\lim_{\omega \to 0} X(j\omega) = \lim_{\omega \to 0} \frac{2}{\omega} \sin(\omega T_0) = \frac{2}{\omega}(\omega T_0) = 2\,T_0$$



參考資料

- sin(θ) ≈ θ if θ is small.



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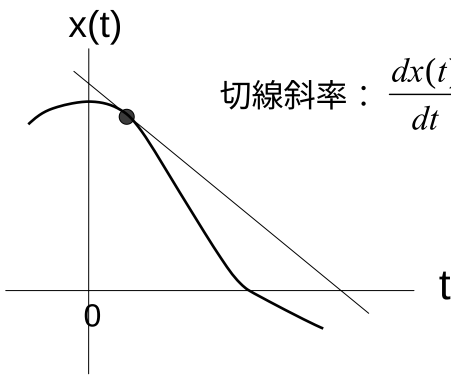
參考資料

- Changing Rate



切線斜率： $\frac{dx(t)}{dt}$ （針對時間變化率）

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Time-domain signals for Problem 3.14.

Find the FT of the parts (a) and (b) = ?

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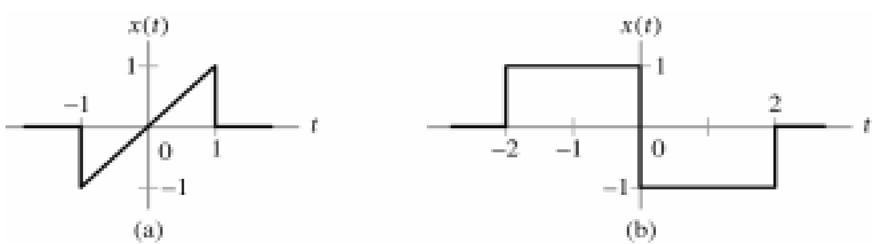
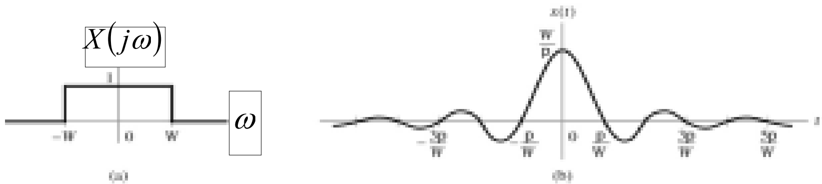


Example 3.26:

(a) Rectangular spectrum in the frequency domain.

(b) Inverse FT in the time domain.

$$X(j\omega)=\begin{cases}1, & -W<\omega<W\\ 0, & |\omega|>W\end{cases}$$



Solution of the Ex 3.26:

$$\mathrm{sinc}(W)=\frac{\sin(\pi W)}{\pi W}$$

$$x(t)=\frac{1}{2\pi}\int_{-\pi}^{\pi} X(j\omega)\,e^{j\omega t}\,d\omega$$

$$=\frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}\,d\omega$$

$$=\frac{1}{2\pi\, jt}e^{j\omega t}\Big|_{-W}^{W}=\frac{1}{2\pi\, j\,t}\left(e^{jW t}-e^{-jW t}\right)$$

$$=\frac{1}{\pi\, t}\,\frac{e^{jW t}-e^{-jW t}}{j2}$$

$$=\frac{1}{\pi\, t}\sin(Wt)$$

$$\frac{W}{\pi}\mathrm{sinc}\left(\frac{Wt}{\pi}\right)=\frac{W}{\pi}\frac{\sin(Wt)}{Wt}=\frac{\sin(W\,t)}{\pi\, t}$$

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Example 3.27: Find the FT of x(t) = δ(t)

Solution:

$$X(j\omega)=\int_{-\infty}^{+\infty}\delta(t)\,e^{-j\omega t}\,dt=1$$

$notation:$

$$X(j\omega)=FT\{\delta(t)\}=1$$

or

$$\delta(t)\overset{FT}{\longleftrightarrow}1$$

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Example 3.28: Find the inv FT of X(jω) = 2πδ(ω) ?

Solution:

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}2\pi\delta(\omega)\,e^{j\omega t}\,d\omega=1$$

$notation:$

意義： 直流訊號頻率為0

$$x(t)=FT^{-1}\{2\pi\delta(\omega)\}=1$$

or

$$1\overset{FT}{\longleftrightarrow}2\pi\delta(\omega)$$

Frequency-domain signals for Problem 3.15.

Find the inverse of FT = ?

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