



Fourier Transform of Signals

訊號的傅立葉轉換

Lecture 3-6



Fourier Representations

- DTFS: $x[n]$: discrete-time and periodic signal
 $X[k]$: discrete and periodic spectrum
- FS: $x(t)$: continuous-time and periodic signal
 $X[k]$: discrete-time and non-periodic spectrum
- DTFT: $x[n]$: discrete-time and non-periodic signal
 $X(e^{j\Omega})$: continuous and periodic spectrum
- FT: $x(t)$: continuous-time and non-periodic signal
 $X(j\omega)$: continuous-time and non-periodic spectrum



Linearity and Symmetry

Linear Properties: (想一想 線性特性的用處?)

$$z(t) = ax(t) + by(t) \xleftrightarrow{FT} Z(j\omega) = aX(j\omega) + bY(j\omega)$$

$$z(t) = ax(t) + by(t) \xleftrightarrow{FS} Z[k] = aX[k] + bY[k]$$

$$z[n] = ax[n] + by[n] \xleftrightarrow{DTFT} Z(e^{j\Omega}) = aX(e^{j\Omega}) + bY(e^{j\Omega})$$

$$z[n] = ax[n] + by[n] \xleftrightarrow{DTFS} Z[k] = aX[k] + bY[k]$$

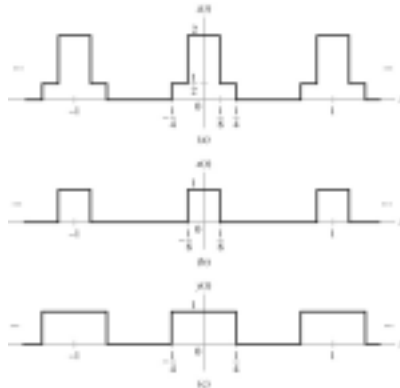
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Representation of the periodic signal $z(t)$ as a weighted sum of periodic square waves:

$$z(t) = (3/2)x(t) + (1/2)y(t).$$

- (a) $z(t)$.
- (b) $x(t)$.
- (c) $y(t)$.

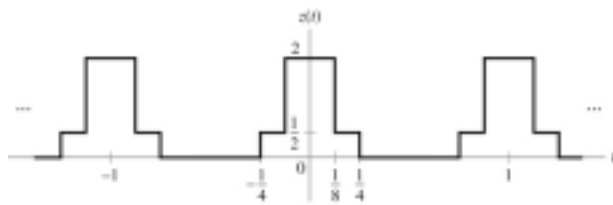


$$Z[k] = (3/2)X[k] + (1/2)Y[k]$$

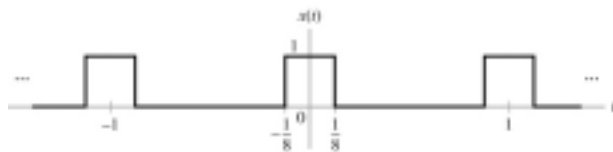
$$X[k] = ?$$

$$Y[k] = ?$$

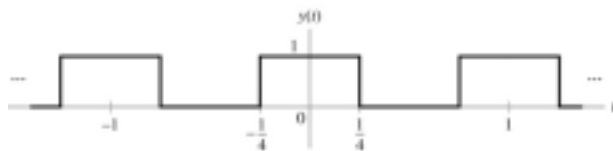
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(a)



(b)



(c)

$$z(t) = (3/2)x(t) + (1/2)y(t).$$

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Problem 3.16:

利用線性性質求取下述訊號傅立葉表示法

$$x(t) = 2e^{-t}u(t) - 3e^{-2t}u(t)$$

Solution:

$$\begin{aligned} X(j\omega) &= 2 \cdot FT\{e^{-t}u(t)\} - 3 \cdot FT\{e^{-2t}u(t)\} \\ &= 2 \left(\frac{1}{1+j\omega} \right) - 3 \left(\frac{1}{2+j\omega} \right) \\ &= \frac{2}{1+j\omega} - \frac{3}{2+j\omega} = ? \end{aligned}$$

請繼續完成 ...

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Symmetry Properties: Real Signals

$\because x(t) = x^*(t)$ $x(t)$ 是實數

$$\begin{aligned} X^*(j\omega) &= \left[\int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \right]^* = \int_{-\infty}^{+\infty} x^*(t) e^{j\omega t} dt \\ &= \int_{-\infty}^{+\infty} x(t) e^{j\omega t} dt = \int_{-\infty}^{+\infty} x(t) e^{-j(-\omega)t} dt \\ &= X(-j\omega) \end{aligned}$$

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$\because x(t) = x^*(t)$

$x(t)$ 是實數

$$X^*(j\omega) = X(-j\omega)$$

$X(j\omega)$ 為 複數共軛對稱 (complex-conjugate symmetric)

$$\because X(j\omega) = \text{Re}\{X(j\omega)\} + j \text{Im}\{X(j\omega)\},$$

$$X^*(j\omega) = \text{Re}\{X(j\omega)\} - j \text{Im}\{X(j\omega)\},$$

$$X(-j\omega) = \text{Re}\{X(-j\omega)\} + j \text{Im}\{X(-j\omega)\},$$

$$\rightarrow \text{Re}\{X(j\omega)\} = \text{Re}\{X(-j\omega)\}$$

$$\rightarrow \text{Im}\{X(j\omega)\} = -\text{Im}\{X(-j\omega)\}$$

$x(t)$ FT後 實數部份是頻率偶函數(Even function)

$x(t)$ FT後 虛數部份是頻率奇函數(Odd function)

振幅頻譜是 偶對稱

相位頻譜是 奇對稱

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Symmetry Properties: Imaginary Signals

$$\because -x(t) = x^*(t)$$

$x(t)$ 是虛數

$$\begin{aligned} X^*(j\omega) &= \left[\int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \right]^* = \int_{-\infty}^{+\infty} x^*(t) e^{j\omega t} dt \\ &= - \int_{-\infty}^{+\infty} x(t) e^{j\omega t} dt = - \int_{-\infty}^{+\infty} x(t) e^{j\omega t} dt \\ &= - \int_{-\infty}^{+\infty} x(t) e^{-j(-\omega)t} dt = -X(-j\omega) \end{aligned}$$

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$$\because -x(t) = x^*(t)$$

$x(t)$ 是虛數

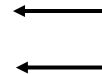
$$X^*(j\omega) = -X(-j\omega)$$

複數共軛對稱 (complex-conjugate symmetric)

$$\because X(j\omega) = \text{Re}\{X(j\omega)\} + j \text{Im}\{X(j\omega)\},$$

$$X^*(j\omega) = \text{Re}\{X(j\omega)\} - j \text{Im}\{X(j\omega)\},$$

$$-X(-j\omega) = -\text{Re}\{X(-j\omega)\} - j \text{Im}\{X(-j\omega)\},$$



$$\rightarrow \text{Re}\{X(j\omega)\} = -\text{Re}\{X(-j\omega)\}$$

$$\rightarrow \text{Im}\{X(j\omega)\} = \text{Im}\{X(-j\omega)\}$$

$x(t)$ FT後實數部份是頻率奇函數(Odd function)

$x(t)$ FT後虛數部份是頻率偶函數(Even function)

振幅頻譜是偶對稱

相位頻譜是奇對稱

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$x(t)$ 是實數

FT後 實數部份是頻率偶函數(Even function)

FT後 虛數部份是頻率奇函數(Odd function)

振幅頻譜是 偶對稱 相位頻譜是 奇對稱

$x(t)$ 是虛數

FT後 實數部份是頻率奇函數(Odd function)

FT後 虛數部份是頻率偶函數(Even function)

振幅頻譜是 偶對稱 相位頻譜是 奇對稱

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Symmetry Property 應用範例

若實數輸入訊號： $x(t) = A \cos(\omega_0 t - \phi) = \frac{Ae^{-j\phi} e^{j\omega_0 t} + Ae^{j\phi} e^{-j\omega_0 t}}{2}$

系統振幅響應： $|H(j\omega)|$

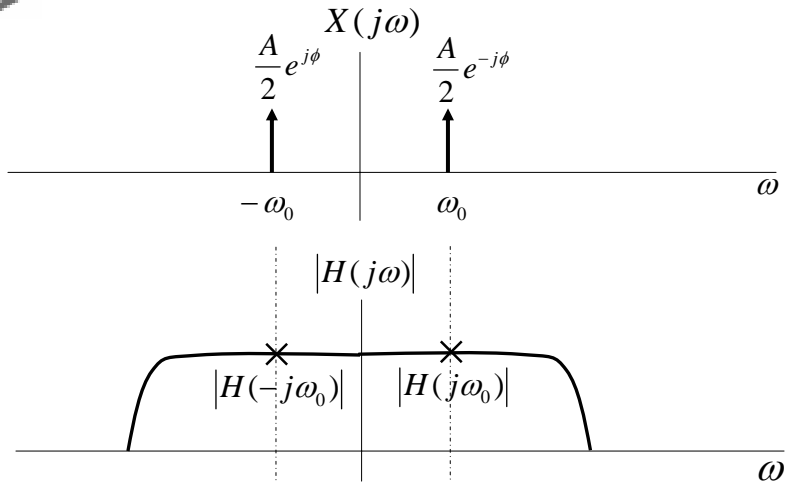
系統相位響應： $\arg[H(j\omega)]$

系統輸出： $y(t) = \frac{A}{2} e^{-j\phi} e^{j\omega_0 t} \cdot H(j\omega_0) + \frac{A}{2} e^{j\phi} e^{-j\omega_0 t} \cdot H(-j\omega_0)$

重新整理系統輸出：

$$y(t) = \frac{A}{2} e^{-j\phi} e^{j\omega_0 t} \cdot \left\{ |H(j\omega_0)| e^{j \arg[H(j\omega_0)]} \right\} \\ + \frac{A}{2} e^{j\phi} e^{-j\omega_0 t} \cdot \left\{ |H(-j\omega_0)| e^{j \arg[H(-j\omega_0)]} \right\}$$

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請補上“相位響應”後，繪出“輸出頻譜” $Y(j\omega)$

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系統振幅響應是偶對稱： $|H(j\omega)| = |H(-j\omega)|$

系統相位響應是奇對稱： $\arg[H(j\omega)] = -\arg[H(-j\omega)]$

整理系統輸出：

$$\begin{aligned}
 y(t) &= \frac{A}{2} e^{-j\phi} e^{j\omega_0 t} \left\{ |H(j\omega_0)| e^{j\arg[H(j\omega_0)]} \right\} + \frac{A}{2} e^{j\phi} e^{-j\omega_0 t} \left\{ |H(-j\omega_0)| e^{j\arg[H(-j\omega_0)]} \right\} \\
 &= \frac{A}{2} e^{-j\phi} e^{j\omega_0 t} \left\{ |H(j\omega_0)| e^{j\arg[H(j\omega_0)]} \right\} + \frac{A}{2} e^{j\phi} e^{-j\omega_0 t} \left\{ |H(-j\omega_0)| e^{-j\arg[H(j\omega_0)]} \right\} \\
 &= \frac{A}{2} |H(j\omega_0)| \left\{ e^{-j\phi} e^{j\omega_0 t} e^{j\arg[H(j\omega_0)]} + e^{j\phi} e^{-j\omega_0 t} e^{-j\arg[H(j\omega_0)]} \right\} \\
 &= A |H(j\omega_0)| \left\{ \left(e^{j(\omega_0 t - \phi + \arg[H(j\omega_0)])} + e^{-j(\omega_0 t - \phi + \arg[H(j\omega_0)])} \right) / 2 \right\} \\
 &= A |H(j\omega_0)| \cos(\omega_0 t - \phi + \arg[H(j\omega_0)])
 \end{aligned}$$

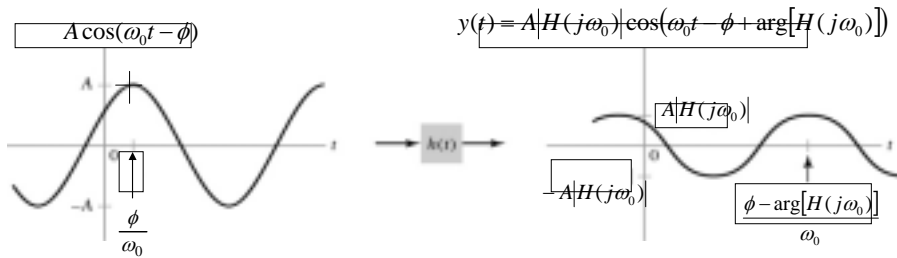
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A sinusoidal input to an LTI system:

It results in a sinusoidal output of the same frequency.

The amplitude and phase are modified by the system's frequency response.



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Symmetry Properties: Even Signals

Even function : $x(-t) = x(t)$

Real Function: $x^*(t) = x(t)$

Even and Real Function: $x(t)^* = x(-t)$

$$\begin{aligned}
 X^*(j\omega) &= \left[\int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \right]^* = \int_{-\infty}^{+\infty} x^*(t) e^{j\omega t} dt \\
 &= \int_{-\infty}^{+\infty} x(-t) e^{j\omega t} dt = \int_{-\infty}^{+\infty} x(-t) e^{-j\omega(-t)} dt, \text{ let } \tau = -t \\
 &= \int_{-\infty}^{+\infty} x(\tau) e^{-j\omega \tau} d\tau \\
 &= X(j\omega)
 \end{aligned}$$

$X(j\omega)$ is real.

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Symmetry Properties: Odd Signals

Odd function : $x(-t) = -x(t)$

Real Function: $x^*(t) = x(t)$

Odd and Real Function: $x(t)^* = -x(-t)$

$$\begin{aligned}
 X^*(j\omega) &= \left[\int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \right]^* = \int_{-\infty}^{+\infty} x^*(t) e^{j\omega t} dt \\
 &= - \int_{-\infty}^{+\infty} x(-t) e^{j\omega t} dt = - \int_{-\infty}^{+\infty} x(-t) e^{-j\omega(-t)} dt, \quad \text{let } \tau = -t \\
 &= - \int_{-\infty}^{+\infty} x(\tau) e^{-j\omega\tau} d\tau \\
 &= -X(j\omega)
 \end{aligned}$$

$X(j\omega)$ is imaginary.

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補充資料: Odd Signals

Odd function : $x(-t) = -x(t)$

Imaginary Function: $x^*(t) = -x(t)$

Odd and Imaginary Function: $x(t)^* = x(-t)$

$$\begin{aligned}
 X^*(j\omega) &= \left[\int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \right]^* = \int_{-\infty}^{+\infty} x^*(t) e^{j\omega t} dt \\
 &= \int_{-\infty}^{+\infty} x(-t) e^{j\omega t} dt = \int_{-\infty}^{+\infty} x(-t) e^{-j\omega(-t)} dt, \quad \text{let } \tau = -t \\
 &= \int_{-\infty}^{+\infty} x(\tau) e^{-j\omega\tau} d\tau \\
 &= X(j\omega)
 \end{aligned}$$

$X(j\omega)$ is real.

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補充資料: Even Signals

Even function : $x(-t) = x(t)$

Imaginary Function: $x^*(t) = -x(t)$

Even and Imaginary Function: $x(t)^* = -x(-t)$

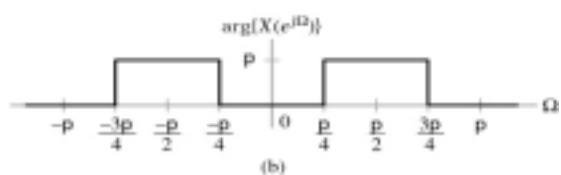
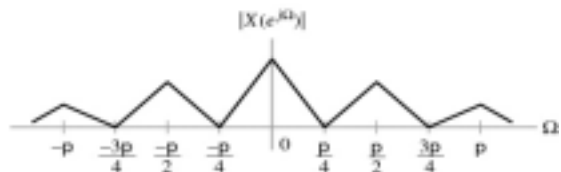
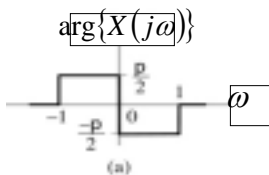
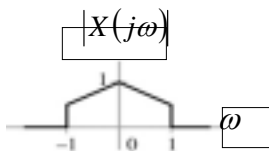
$$\begin{aligned}
 X^*(j\omega) &= \left[\int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \right]^* = \int_{-\infty}^{+\infty} x^*(t) e^{j\omega t} dt \\
 &= - \int_{-\infty}^{+\infty} x(-t) e^{j\omega t} dt = - \int_{-\infty}^{+\infty} x(-t) e^{-j\omega(-t)} dt, \text{ let } \tau = -t \\
 &= - \int_{-\infty}^{+\infty} x(\tau) e^{-j\omega\tau} d\tau \\
 &= -X(j\omega)
 \end{aligned}$$

$X(j\omega)$ is imaginary.

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Problem 3.17. 判定(a)及(b)所對應時域訊號是實、複數以及是偶、奇函數？



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(a) The Magnitude Response is even, and the Phase Response is odd functions.

→ The $x(t)$ is a real and odd function.

時域中(實/奇)訊號 → 偶振幅奇相位/頻域虛數 → 相位 $\pm\pi/2$ 變化

(b) The Magnitude Response is even, and the Phase Response is even functions.

→ The $x[n]$ is a real and even function.

時域中(實/偶)訊號 → 偶振幅偶相位/頻域實數 → 相位 0 和 π 變化



Convolution Property 褶積性質

$$y(t) = h(t) * x(t) \xleftrightarrow{FT} Y(j\omega) = H(j\omega) \cdot X(j\omega)$$

$$y[n] = h[n] * x[n] \xleftrightarrow{DTFT} Y(e^{j\Omega}) = H(e^{j\Omega}) \cdot X(e^{j\Omega})$$



Convolution of Non-periodic Signal 非週期訊號的褶積

$$y(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} h(\tau) x(t - \tau) d\tau$$

$\Downarrow FT$

$$Y(j\omega) = H(j\omega) \cdot X(j\omega)$$

where

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} [Y(j\omega)] e^{j\omega t} d\omega$$

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摺積 vs. 乘積

$$y(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} h(\tau) x(t - \tau) d\tau$$

$$\because x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$$

$$\therefore x(t - \tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega(t - \tau)} d\omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{-j\omega\tau} e^{j\omega t} d\omega$$

$$y(t) = \int_{-\infty}^{+\infty} h(\tau) \left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{-j\omega\tau} e^{j\omega t} d\omega \right] d\tau$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} h(\tau) e^{-j\omega\tau} d\tau \right] X(j\omega) e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} [H(j\omega)] X(j\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} [H(j\omega) \cdot X(j\omega)] e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} [Y(j\omega)] e^{j\omega t} d\omega$$

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Example 3.31:

$$x(t) = (1/\pi t) \sin(\pi t), \quad h(t) = (1/\pi t) \sin(2\pi t),$$

$$y(t) = x(t) * h(t) = ? \quad \text{Solution: } y(t) = (1/\pi t) \sin(\pi t)$$

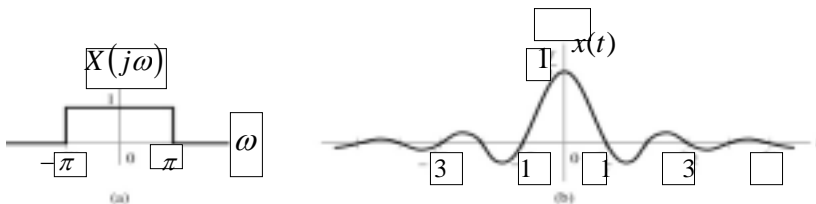
回想

$$x(t) = \frac{1}{\pi t} \sin(Wt) \stackrel{FT}{\leftrightarrow} X(j\omega) = \begin{cases} 1, & -W < \omega < W \\ 0, & |\omega| > W \end{cases}$$



$$x(t) = (1/\pi t) \sin(\pi t),$$

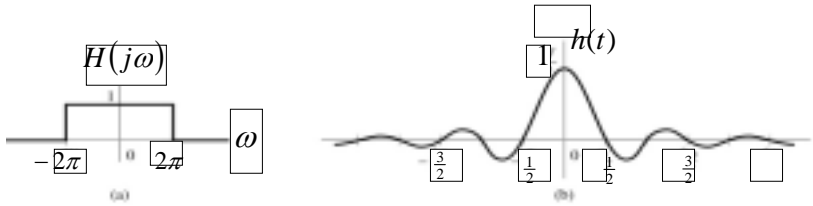
$$x(t) = \frac{1}{\pi t} \sin(\pi t) \leftrightarrow X(j\omega) = \begin{cases} 1, & -\pi < \omega < \pi \\ 0, & |\omega| > \pi \end{cases}$$



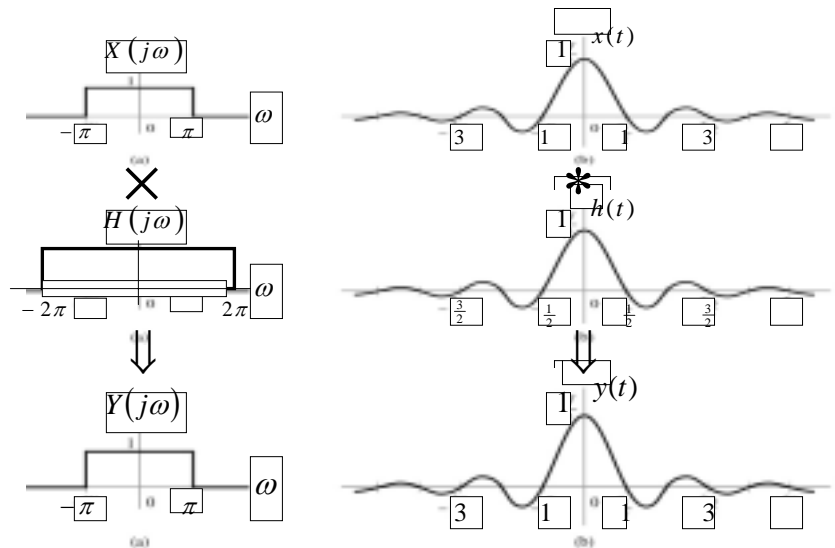


$$h(t) = (1/\pi t) \sin(2\pi t),$$

$$h(t) = \frac{1}{\pi t} \sin(2\pi t) \leftrightarrow H(j\omega) = \begin{cases} 1, & -2\pi < \omega < 2\pi \\ 0, & |\omega| > 2\pi \end{cases}$$



$$y(t) = x(t) * h(t) \leftrightarrow X(j\omega) \cdot H(j\omega) = Y(j\omega)$$

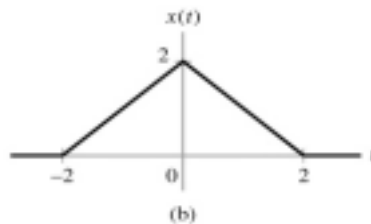
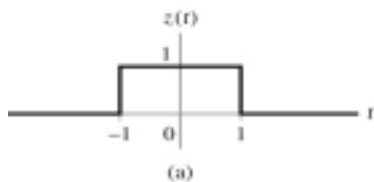




Example 3.32: $X(j\omega) = (4/\omega^2) \sin^2(\omega)$; $x(t) = ?$

Hint: $X(j\omega) = (2/\omega) \sin(\omega) \bullet (2/\omega) \sin(\omega)$

求出 $x(t) = ?$



Example 3.32.

(a) Rectangular pulse $z(t)$.

(b) Convolution of $z(t)$ with itself gives $x(t)$.