# Fourier Transform of Signals

訊號的傅立葉轉換

Lecture 3-7





## Fourier Representation

DTFS: x[n]: discrete-time and periodic signal

X[k]: discrete and periodic spectrum

FS: x(t): continuous-time and periodic signal

X[k]: discrete-time and non-periodic spectrum

DTFT: x[n]: discrete-time and non-periodic signal

$X(e^{j\Omega})$: continuous and periodic spectrum

FT: x(t): continuous-time and non-periodic signal

$X(j\omega)$: continuous-time and non-periodic spectrum

# Filtering　濾波

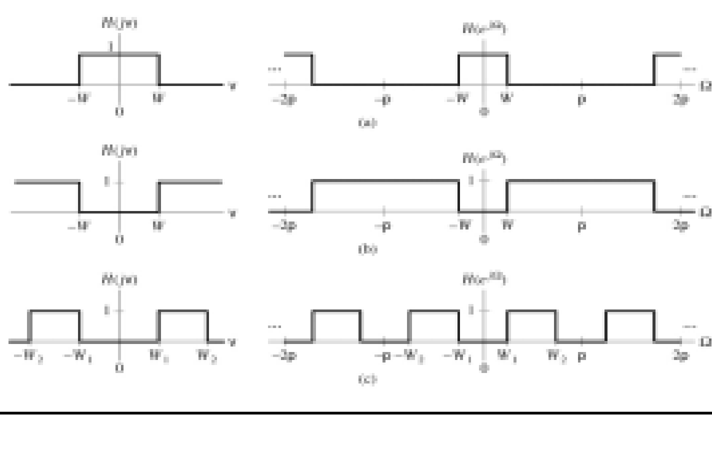
- 任一輸入訊號中含有不同頻率的分量
- 系統透過對輸入訊號中不同頻率的分量做出不同的響應
- 濾波將訊號中部份頻率分量消除，讓其他部份不受影響
  - Low Pass Filter
  - Band Pass Filter
  - High Pass Filter
- 濾波器頻帶
  - 通頻帶 (Pass band)
  - 阻頻帶 (Stop Band)
  - 過渡頻帶 (Transition Band)





Frequency response of ideal continuous- (left panel) and discrete-time (right panel) filters:

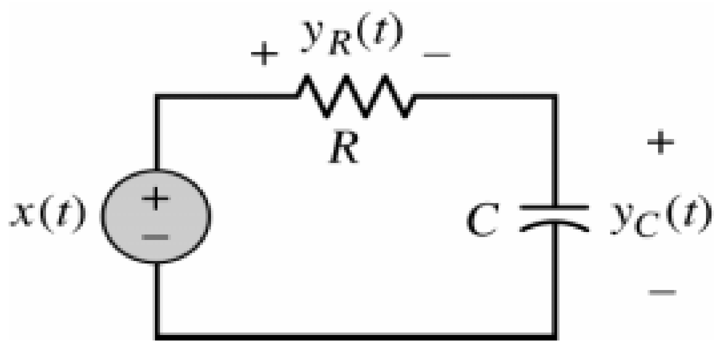
(a) Low-pass characteristic,
(b) High-pass characteristic,
(c) Band-pass characteristic.



Ex 3.33

*RC* circuit with input $x(t)$ and outputs $y_{c(t)}$ and $y_R(t)$.

請分述兩種系統頻率響應與濾波性質








$$\because h_C(t) = \frac{1}{RC} e^{-t/(RC)} u(t)$$

$$\therefore H_C(j\omega) = \frac{1}{1 + j\omega RC}$$

$$|H_C(j\omega)| = \frac{1}{\sqrt{1+\omega^2 R^2 C^2}}, \quad \arg\{H_C(j\omega)\} = -\tan^{-1}(\omega RC)$$

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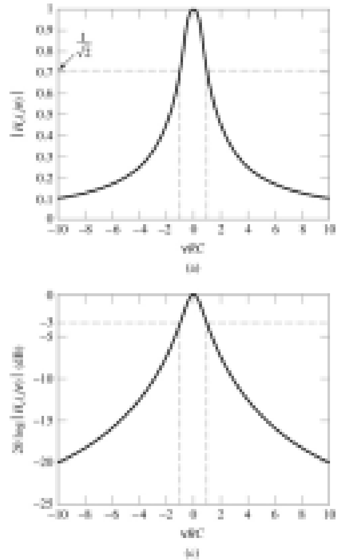
$$\because h_R(t) = \delta(t) - \frac{1}{RC} e^{-t/(RC)} u(t)$$

$$\therefore H_R(j\omega) = 1 - \frac{1}{1 + j\omega RC} = \frac{j\omega RC}{1 + j\omega RC}$$

$$|H_R(j\omega)| = \frac{\omega RC}{\sqrt{1+\omega^2 R^2 C^2}}, \quad \arg\{H_R(j\omega)\} = \frac{\pi}{2} - \tan^{-1}(\omega RC)$$

*RC* circuit magnitude responses as a function of normalized frequency ω*RC*.



(a) Frequency response of the system corresponding to $y_C(t)$, linear scale.

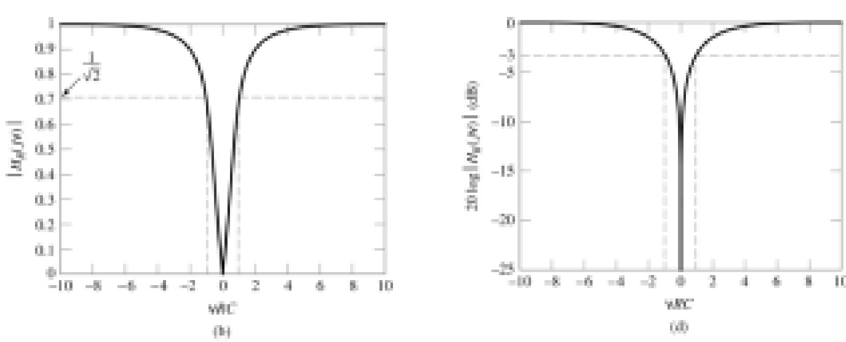
(c) Frequency response of the system corresponding to $y_C(t)$, dB scale.





*RC* circuit magnitude responses as a function of normalized frequency ω*RC*.

(b) Frequency response of the system corresponding to *y*R(*t*), linear scale.
(d) Frequency response of the system corresponding to *y*R(*t*), dB scale, shown on the range from 0 dB to –25 dB.



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# Convolution of Periodic Signal

# 週期訊號的褶積

- 週期性訊號褶積不會出現在使用計算系統輸出上，因為系統脈衝響應若為週期性函數，系統即為不穩定系統。
- 週期性訊號褶積可能會出現在訊號分析與操作上。
- 週期性訊號褶積定義：

$$y(t) = x(t) \circledast z(t) = \int_0^T x(\tau)\, z(t-\tau)\, d\tau$$



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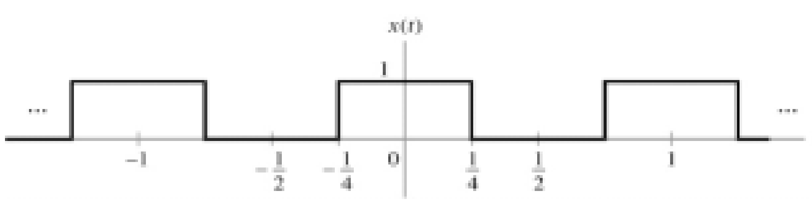
Example 3.36 計算 z(t) = 2cos(2πt)+sin(4πt) 和方波 x(t) 作 週期褶積。



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Solution :

$$z(t) = 2\cos(2\pi t) + \sin(4\pi t)$$

$$= 2\left(\frac{e^{j(2\pi)t} + e^{-j(2\pi)t}}{2}\right) + \left(\frac{e^{2(2\pi)t} - e^{-2(2\pi)t}}{j2}\right)$$

$$= e^{j(2\pi)t} + e^{-j(2\pi)t} + \frac{1}{j2}e^{2(2\pi)t} - \frac{1}{j2}e^{-2(2\pi)t}$$

(FS Spectrum)

$$\therefore Z[k] = \begin{cases} 1, & k = \pm 1 \\ \frac{1}{j2}, & k = +2 \\ -\frac{1}{j2}, & k = -2 \\ 0, & k = others \end{cases}$$



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*Given from example* 3.13

$$X[k] = \frac{2}{2\pi k}\sin\left(\frac{k\pi}{2}\right),$$

$$\therefore Y[k] = Z[k] \cdot X[k]$$

$$= \begin{cases} \frac{2}{2\pi k}\sin\left(\frac{k\pi}{2}\right) = \frac{1}{\pi}, & k = \pm 1 \\ \frac{1}{j2}\frac{2}{2\pi k}\sin\left(\frac{k\pi}{2}\right) = 0, & k = +2 \\ -\frac{1}{j2}\frac{2}{2\pi k}\sin\left(\frac{k\pi}{2}\right) = 0, & k = -2 \\ 0, & others \end{cases}$$

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$$Y[k]=\begin{cases}\frac{2}{2\pi k}\sin\left(\frac{k\pi}{2}\right)=\frac{1}{\pi}, & k=\pm 1\\ 0, & others\end{cases}$$

$\therefore$ [ The fundamental frequency: $2\pi$ ]

$$y(t)=Y[1]e^{j(2\pi)t}+Y[-1]e^{-j(2\pi)t}$$

$$=\frac{1}{\pi}\left(e^{j(2\pi)t}+e^{-j(2\pi)t}\right)$$

$$=\frac{2}{\pi}\left(\frac{e^{j(2\pi)t}+e^{-j(2\pi)t}}{2}\right)=\frac{2}{\pi}\cos(2\pi t)$$



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# Differentiation & Integration Properties

# 微分與積分性質

Differentiation in Time:

$$\because \quad x(t) \overset{FT}{\longleftrightarrow} X(j\omega)$$

$$\frac{d}{dt}x(t) \overset{FT}{\longleftrightarrow} j\omega \cdot X(j\omega)$$

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\,e^{j\omega t}\,d\omega$$

$\therefore$

$$\frac{d}{dt}x(t)=\frac{d}{dt}\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\,e^{j\omega t}\,d\omega\right]$$

$$=\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\left(\frac{d}{dt}e^{j\omega t}\right)d\omega=\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\left(j\omega\cdot e^{j\omega t}\right)\,d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{+\infty} \left(j\omega\cdot X(j\omega)\right)e^{j\omega t}\,d\omega$$

$\therefore$

$$\frac{d}{dt}x(t)\overset{FT}{\longleftrightarrow} j\omega\cdot X(j\omega)$$





# Differentiation in Time (cont.)

$\because$

$$\frac{d}{dt}x(t)\overset{FT}{\longleftrightarrow} j\omega\cdot X(j\omega)$$

物理意義：

微分運算是計算變化量

時域上微分會移除直流(DC)分量

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# Ex. 3.37

驗證

$$\frac{d}{dt}\left(e^{-at}u(t)\right) \overset{FT}{\longleftrightarrow} \frac{j\omega}{a+j\omega}$$

Solution:

$$\frac{d}{dt}\left(e^{-at}u(t)\right)=\left(\frac{d}{dt}e^{-at}\right)u(t)+e^{-at}\left(\frac{d}{dt}u(t)\right)$$

$$=-a\cdot e^{-at}\cdot u(t)+e^{-at}\cdot\delta(t)=-a\cdot e^{-at}u(t)+\delta(t)$$

$$\therefore$$

$$FT\left\{\frac{d}{dt}\left(e^{-at}u(t)\right)\right\}=FT\left\{-a\cdot e^{-at}u(t)+\delta(t)\right\}$$

$$=FT\left\{-a\cdot e^{-at}u(t)\right\}+FT\left\{\delta(t)\right\}$$

$$=\frac{-a}{a+j\omega}+1=\frac{-a+a+j\omega}{a+j\omega}=\frac{j\omega}{a+j\omega}$$ 驗證無誤



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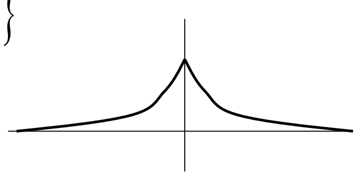
# Problem 3.22: 利用微分性質求取下列訊號的 FT

$$(a)\quad x(t)=\frac{d}{dt}e^{-2|t|};\quad (b)\quad x(t)=\frac{d}{dt}\left(2te^{-2t}u(t)\right)$$

Solution:

$$(a)\quad FT\left\{\frac{d}{dt}e^{-2|t|}\right\}=j\omega\cdot FT\left\{e^{-2|t|}\right\}$$



$$FT\left\{e^{-2|t|}\right\}=\int_{-\infty}^{+\infty}e^{-2|t|}\,e^{-j\omega t}dt$$

$$=\int_{-\infty}^{0}e^{2t}\,e^{-j\omega t}dt+\int_{0}^{+\infty}e^{-2t}\,e^{-j\omega t}dt$$

Solution: (cont.)

$$=\int_{-\infty}^{0} e^{2t}\ e^{-j\omega t}dt+\int_{0}^{+\infty} e^{-2t}\ e^{-j\omega t}dt$$

$$=\int_{-\infty}^{0} e^{(2-j\omega)t}\ dt+\int_{0}^{+\infty} e^{-(2+j\omega)t}\ dt$$

$$=\frac{1}{2-j\omega}e^{(2-j\omega)t}\Big|_{-\infty}^{0}+\frac{-1}{2+j\omega}e^{-(2+j\omega)t}\Big|_{0}^{+\infty}$$

$$=\frac{1}{2-j\omega}(1-0)-\frac{1}{2+j\omega}(0-1)$$

$$=\frac{1}{2-j\omega}+\frac{1}{2+j\omega}=\frac{2+j\omega+2-j\omega}{(2+j\omega)(2-j\omega)}=\frac{4}{4+\omega^2}$$





Solution: (cont.)

$$FT\left\{e^{-2|t|}\right\}=\frac{4}{4+\omega^2},$$

$$\therefore$$

$$FT\left\{\frac{d}{dt}e^{-2|t|}\right\}=j\omega\cdot\frac{4}{4+\omega^2}=\frac{4j\omega}{4+\omega^2}$$

Solution:

$$(b)\quad FT\left\{\frac{d}{dt}2te^{-2t}u(t)\right\} = j\omega \cdot FT\left\{2te^{-2t}u(t)\right\}$$

$$FT\left\{2te^{-2t}u(t)\right\} = \int_0^{+\infty} 2te^{-2t}\, e^{-j\omega t}dt$$

$$= \int_{-\infty}^{0} 2t\, e^{-(2+j\omega)t}\, dt\ ;\quad let\ u = 2t, dv = e^{-(2+j\omega)t}dt$$

$$\therefore v = \frac{-1}{2+j\omega}e^{-(2+j\omega)t}, du = 2dt$$

$$= uv - \int vdu = 2t\frac{-1}{2+j\omega}e^{-(2+j\omega)t} - \int_0^{+\infty}\frac{-1}{2+j\omega}e^{-(2+j\omega)t}2dt$$





Solution: (cont.)

$$FT\left\{2te^{-2t}u(t)\right\}$$

$$= 2t\frac{-1}{2+j\omega}e^{-(2+j\omega)t} - 2\int_0^{+\infty}\frac{-1}{2+j\omega}e^{-(2+j\omega)t}dt$$

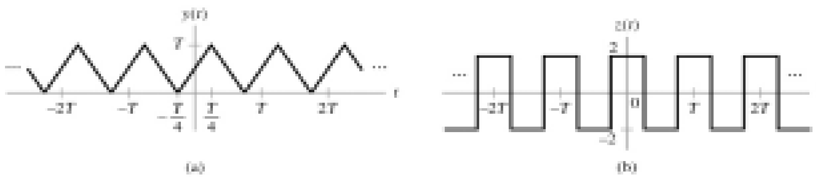
$$= \frac{-2t}{2+j\omega}e^{-(2+j\omega)t}\Bigg|_0^{+\infty} - \frac{2}{(2+j\omega)^2}e^{-(2+j\omega)t}\Bigg|_0^{+\infty}$$

$$= (0-0) - \frac{2}{(2+j\omega)^2}(0-1) = \frac{2}{(2+j\omega)^2}$$

$$\therefore FT\left\{\frac{d}{dt}2te^{-2t}u(t)\right\} = \frac{2j\omega}{(2+j\omega)^2}$$

# Ex. 3.39 利用微分性質求取 y(t) 訊號的 FT

想一想 怎麼做 ？ 直接求 FT of y(t) 是否比較難 ？



(a) Triangular wave $y(t)$.

(b) The derivative of $y(t)$ is the square wave $z(t)$.





# Solution:

學生試一試

# Differentiation in Frequency

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

$\therefore$

$$\frac{d}{d\omega} X(j\omega) = \frac{d}{d\omega}\left[\int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt\right]$$

$$= \int_{-\infty}^{+\infty} x(t)\left(\frac{d}{d\omega} e^{-j\omega t}\right) dt = \int_{-\infty}^{+\infty} x(t)\left(-jt \cdot e^{-j\omega t}\right) dt$$

$$= \int_{-\infty}^{+\infty} \left(-jt \cdot x(t)\right) e^{-j\omega t}\, dt$$

$\therefore$

$$\frac{d}{d\omega} X(j\omega) \overset{FT}{\longleftrightarrow} -jt \cdot x(t)$$





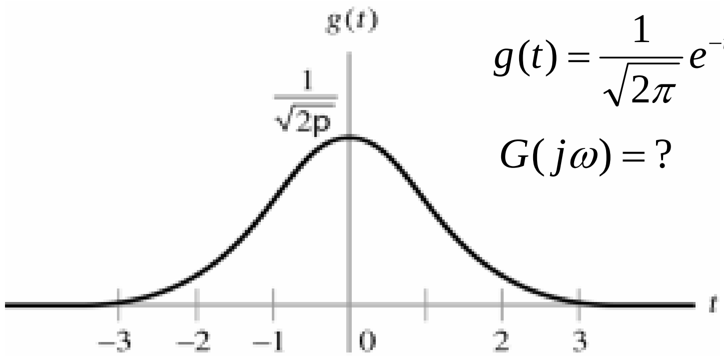
## Differentiation in Frequency:

$\because$

$$\boxed{\frac{d}{d\omega} X(j\omega) \overset{FT}{\longleftrightarrow} -jt \cdot x(t)}$$

- 頻域上微分 對應在時域上把訊號乘上 $-jt$

Ex. 3.40: Find the FT of the Gaussian pulse *g*(*t*)?




$$g(t)=\frac{1}{\sqrt{2\pi}}e^{-t^2/2}$$

$$G(j\omega)=?$$

利用對時間與對頻率微分性質求取 高斯脈波 (Gaussian pulse), g(t) 訊號的 FT.





Solution:

$$\text{given} \quad g(t)=\frac{1}{\sqrt{2\pi}}e^{-t^2/2},$$

$$\therefore \frac{d}{dt}g(t)=(-t)\frac{1}{\sqrt{2\pi}}e^{-t^2/2}=-t\cdot g(t)$$

應用對時間微分性質： $\frac{d}{dt}g(t) \overset{FT}{\leftrightarrow} j\omega\cdot G(j\omega)$

代入上面推導式：

$$\therefore \; -t\cdot g(t) \overset{FT}{\leftrightarrow} j\omega\cdot G(j\omega)$$

應用對頻率微分性質：

$$-jt\cdot g(t) \overset{FT}{\leftrightarrow} \frac{d}{d\omega}G(j\omega) \Rightarrow -t\cdot g(t) \overset{FT}{\leftrightarrow} \frac{1}{j}\frac{d}{d\omega}G(j\omega)$$

Solution: (cont.)

$\therefore$

$$-t \cdot g(t) \overset{FT}{\leftrightarrow} j\omega \cdot G(j\omega); \qquad -t \cdot g(t) \overset{FT}{\leftrightarrow} \frac{1}{j}\frac{d}{d\omega}G(j\omega)$$

$$\therefore \quad j\omega \cdot G(j\omega) = \frac{1}{j}\frac{d}{d\omega}G(j\omega)$$

$$\Rightarrow \quad (j)\,j\omega \cdot G(j\omega) = (j)\frac{1}{j}\frac{d}{d\omega}G(j\omega)$$

$$\Rightarrow \quad \boxed{-\omega \cdot G(j\omega) = \frac{d}{d\omega}G(j\omega)}$$





Solution: (cont.)

$$\boxed{\frac{d}{dt}g(t) = -t \cdot g(t)} \qquad \boxed{-\omega \cdot G(j\omega) = \frac{d}{d\omega}G(j\omega)}$$

對照上述關係式，g(t) 和 G(jω) 應該有相同 **函數式**：

$$\therefore \quad G(j\omega) = c\, e^{-\omega^2/2}$$

當 ω = 0 時，G(j0) = c 同時又可得：

$$G(j0) = \int_{-\infty}^{+\infty} g(t)\, dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-t^2/2}\, dt = 1 = c$$

因此 G(jω)： $G(j\omega) = e^{-\omega^2/2}$

如何 = 1 ?試推導 ！

$$\boxed{\because \int_{-\infty}^{+\infty} e^{-t^2/2}\, dt = \sqrt{2\pi}}$$

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**範例** $\int_{-\infty}^{+\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}$ ???

$$\int_{-\infty}^{+\infty} e^{-x^2/2}\,dx = \sqrt{\int_{-\infty}^{+\infty} e^{-x^2/2}\,dx \int_{-\infty}^{+\infty} e^{-x^2/2}\,dx}, \quad let\ y = x,$$

$$= \sqrt{\int_{-\infty}^{+\infty} e^{-x^2/2}\,dx \int_{-\infty}^{+\infty} e^{-y^2/2}\,dy} = \sqrt{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(x^2+y^2)/2}\,dx\,dy},$$

改用 polar form $\because dA = dx\cdot dy = r d\theta \cdot dr \quad \& \quad r^2 = x^2 + y^2$

$$= \sqrt{\int_0^{2\pi}\int_0^{+\infty} e^{-(r^2)/2}\, r\,dr\,d\theta} = \sqrt{\int_0^{2\pi}\left(\int_0^{+\infty} r\cdot e^{-(r^2)/2}\,dr\right)d\theta}$$

$$= \sqrt{\int_0^{2\pi}\left(-e^{-(r^2)/2}\Big|_0^{+\infty}\right)d\theta} = \sqrt{\int_0^{2\pi} -(0-1)\,d\theta} = \sqrt{\int_0^{2\pi} d\theta}$$

$$= \sqrt{\theta\Big|_0^{2\pi}} = \sqrt{2\pi - 0} = \sqrt{2\pi}$$



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# Integration　積分

積分運算僅適用於連續的應變數

(continuous dependent variables )。

積分定義式： $y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$

y 在時間 t 的值為 x 對所有在 t 以前時間的積分。

因此： $\frac{d}{dt}y(t) = x(t)$

經由微分性質，右式可進一步推論：

$$\frac{d}{dt}y(t) = x(t)$$

$$\because FT\{y(t)\} = Y(j\omega)$$

$$\because FT\left\{\frac{d}{dt}y(t)\right\} = j\omega \cdot Y(j\omega)$$

$$\therefore FT\left\{\frac{d}{dt}y(t)\right\} = FT\{x(t)\} = X(j\omega)$$

$$\therefore j\omega \cdot Y(j\omega) = X(j\omega)$$

特例：

未定義 $\omega = 0$ 處

$$Y(j\omega) = \frac{1}{j\omega}X(j\omega)$$





增加項目定義 ω = 0

ω = 0 此項為零

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau \quad \overset{FT}{\longleftrightarrow} \quad \frac{1}{j\omega}X(j\omega) + \pi X(j0)\delta(\omega)$$

ω = 0 時出現

驗證積分性質：

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau$$

$$\delta(t) \overset{FT}{\longleftrightarrow} 1$$

$$1 \overset{FT}{\longleftrightarrow} 2\pi\, \delta(\omega)$$

or

$$FT\{\delta(t)\} = 1$$

$$FT\{1\} = 2\pi\delta(\omega)$$

$$\therefore U(j\omega) = FT\{u(t)\} = FT\left\{\int_{-\infty}^{t} \delta(\tau)\, d\tau\right\}$$

$$= \frac{1}{j\omega}\underbrace{\Delta(j\omega)}_{1} + \pi \cdot \underbrace{\Delta(j0)}_{1}\, \delta(\omega)$$

$$= \frac{1}{j\omega} + \pi \cdot \delta(\omega)$$

$$\delta(t) \overset{FT}{\leftrightarrow} \Delta(j\omega)$$

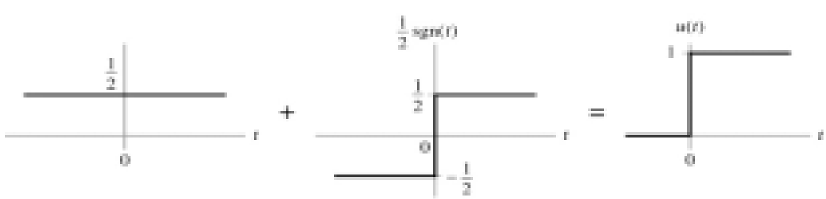




A step function can be denoted as the sum of a constant and a signum function.

$$u(t) = \frac{1}{2} + \frac{1}{2}\mathrm{sgn}(t)$$

$$\mathrm{sgn}(t) = \begin{cases} -1, & t < 0 \\ 0, & t = 0 \\ +1, & t > 0 \end{cases}$$



## Another way to derive out U(jω) = ? .

$$\because u(t) = \frac{1}{2} + \frac{1}{2}\operatorname{sgn}(t)$$

$$\therefore U(j\omega) = FT\{u(t)\} = FT\left\{\frac{1}{2}\right\} + FT\left\{\frac{1}{2}\operatorname{sgn}(t)\right\}$$

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Considering:

$$\because FT\{1\} = 2\pi\,\delta(\omega)$$

$$\therefore FT\left\{\frac{1}{2}\right\} = \pi\,\delta(\omega)$$





Considering: $FT\{\operatorname{sgn}(t)\} = S(j\omega) = ?$

利用微分性質：

$$\because \frac{d}{dt}\operatorname{sgn}(t) = \frac{d}{dt}\left(-1 + 2u(t)\right) = 2\delta(t)$$

$\therefore$

$$FT\left\{\frac{d}{dt}\operatorname{sgn}(t)\right\} = j\omega \cdot FT\{\operatorname{sgn}(t)\} = j\omega \cdot S(j\omega)$$

$$= FT\{2\delta(t)\} = 2FT\{\delta(t)\} = 2$$

$$\therefore \quad S(j\omega) = \frac{2}{j\omega}$$

$\because \operatorname{sgn}(t)$ *is an odd function,*
*the average is* 0.
$\therefore S(j0) = 0,$ *is defined.*

$$\therefore U(j\omega) = FT\{u(t)\} = FT\left\{\frac{1}{2}\right\} + \frac{1}{2} FT\{\mathrm{sgn}(t)\}$$

$$= \pi \cdot \delta(\omega) + \frac{1}{j\omega}$$

*or*

$$U(j\omega) = \begin{cases} \dfrac{1}{j\omega}, & \omega \neq 0 \\ \pi\,\delta(\omega), & \omega = 0 \end{cases}$$