



Fourier Transform of Signals

訊號的傅立葉轉換

Lecture 3-8

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Fourier Representations

- DTFS: $x[n]$: discrete-time and periodic signal
 $X[k]$: discrete and periodic spectrum
- FS: $x(t)$: continuous-time and periodic signal
 $X[k]$: discrete-time and non-periodic spectrum
- DTFT: $x[n]$: discrete-time and non-periodic signal
 $X(e^{j\Omega})$: continuous and periodic spectrum
- FT: $x(t)$: continuous-time and non-periodic signal
 $X(j\omega)$: continuous-time and non-periodic spectrum

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Time- & Frequency Shifted Properties

時間平移與頻率平移性質

Time-Shift Property :

把訊號 $x(t)$ 做時間平移 t_0 的結果，是把它的FT上 $e^{-j\omega t_0}$ 。

$$x(t - t_0) \xleftrightarrow{FT} e^{-j\omega t_0} X(j\omega)$$

$$x(t - t_0) \xleftrightarrow{FS; \omega_0} e^{-jk\omega_0 t_0} X[k]$$

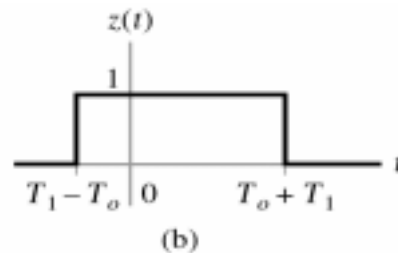
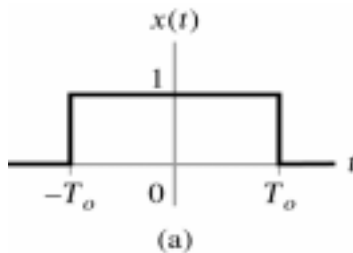
$$x[n - n_0] \xleftrightarrow{DTFT} e^{-j\Omega n_0} X(e^{j\Omega})$$

$$x[n - n_0] \xleftrightarrow{DTFS; \Omega_0} e^{-jk\Omega_0 n_0} X[k]$$

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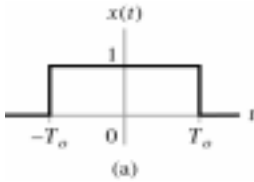
Verify the time-shift property for Example 3.41, where $z(t) = x(t - T_1)$. Find the FT of $x(t)$ and $z(t)$ =?



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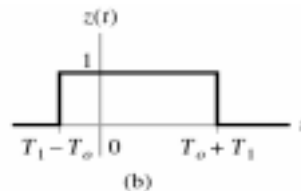
$$\begin{aligned}
 X(j\omega) &= \int_{-\infty}^{+\infty} x(\tau) e^{-j\omega\tau} d\tau = \int_{-T_0}^{+T_0} e^{-j\omega\tau} d\tau \\
 &= \frac{-1}{j\omega} e^{-j\omega\tau} \Big|_{-T_0}^{T_0} = \frac{-1}{j\omega} (e^{-j\omega T_0} - e^{j\omega T_0}) \\
 &= \frac{1}{j\omega} (e^{j\omega T_0} - e^{-j\omega T_0}) = \frac{1}{j\omega} (e^{j\omega T_0} - e^{-j\omega T_0}) \\
 &= \frac{2}{\omega} \left(\frac{e^{j\omega T_0} - e^{-j\omega T_0}}{j2} \right) = \frac{2}{\omega} \sin(\omega T_0)
 \end{aligned}$$



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$$\begin{aligned}
 Z(j\omega) &= \int_{-\infty}^{+\infty} z(\tau) e^{-j\omega\tau} d\tau = \int_{-T_0+T_1}^{+T_0+T_1} e^{-j\omega\tau} d\tau \\
 &= \frac{-1}{j\omega} e^{-j\omega\tau} \Big|_{-T_0+T_1}^{T_0+T_1} = \frac{-1}{j\omega} (e^{-j\omega(T_0+T_1)} - e^{-j\omega(-T_0+T_1)}) \\
 &= \frac{1}{j\omega} (e^{-j\omega(-T_0+T_1)} - e^{-j\omega(T_0+T_1)}) = \frac{1}{j\omega} (e^{j\omega T_0} e^{-j\omega T_1} - e^{-j\omega T_0} e^{-j\omega T_1}) \\
 &= e^{-j\omega T_1} \frac{2}{\omega} \left(\frac{e^{j\omega T_0} - e^{-j\omega T_0}}{j2} \right) = e^{-j\omega T_1} \frac{2}{\omega} \sin(\omega T_0)
 \end{aligned}$$

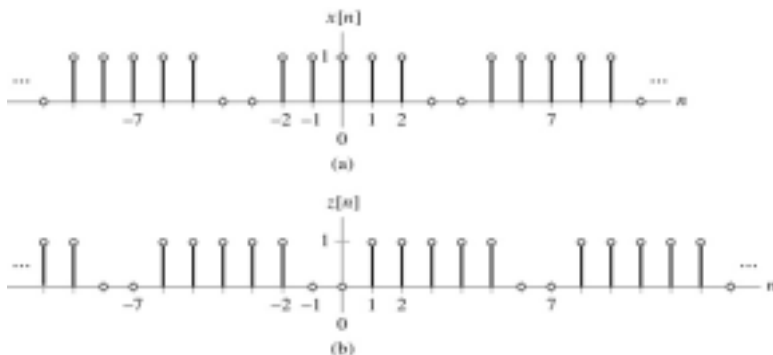


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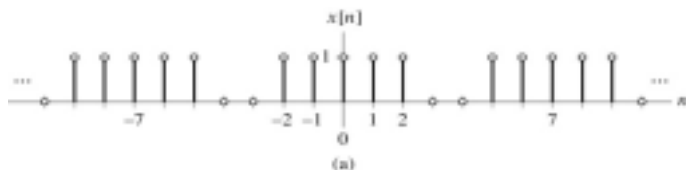


Problem 3.30

利用時間平移特性，使用圖(a)週期性方波的DTFS，尋求圖(b)週期性方波的DTFS。



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$$\begin{aligned}
 X[k] &= \frac{1}{N} \sum_{n=-M}^{N-M-1} x[n] e^{-jk\Omega_0 n}; < N = 7, M = 2, \Omega_0 = 2\pi/7, m = n + 2 > \\
 &= \frac{1}{7} \sum_{n=-2}^2 e^{-jk\Omega_0 n} = \frac{1}{7} \sum_{m=0}^4 e^{-jk\Omega_0 (m-2)} = e^{j2k\Omega_0} \frac{1}{7} \sum_{m=0}^4 e^{-jk\Omega_0 m} \\
 &= e^{j2k\Omega_0} \frac{1}{7} \frac{1 - e^{-j5k\Omega_0}}{1 - e^{-jk\Omega_0}} = \frac{1}{7} e^{j2k\Omega_0} \frac{e^{-j5k\Omega_0/2} (e^{j5k\Omega_0/2} - e^{-j5k\Omega_0/2})}{e^{-jk\Omega_0/2} (e^{jk\Omega_0/2} - e^{-jk\Omega_0/2})} \\
 &= \frac{1}{7} \frac{(e^{j5k\Omega_0/2} - e^{-j5k\Omega_0/2}) / j2}{(e^{jk\Omega_0/2} - e^{-jk\Omega_0/2}) / j2} = \frac{1}{7} \frac{\sin(5k\Omega_0/2)}{\sin(k\Omega_0/2)} \\
 &= \frac{1}{7} \frac{\sin(k(\frac{5\pi}{7}))}{\sin(k(\frac{\pi}{7}))}
 \end{aligned}$$

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利用時間平移特性：

$$Z[k] = e^{-jk\Omega_0 n_0} X[k], \quad \Omega_0 = \frac{2\pi}{7}, n_0 = 3$$

$$Z[k] = e^{-jk\frac{2\pi}{7}(3)} \left[\frac{1}{7} \frac{\sin\left(\frac{k5\pi}{7}\right)}{\sin\left(\frac{k\pi}{7}\right)} \right]$$

** 學生請直接對 $z[n]$ 做DTFS，並驗證答案是否如上。

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Frequency-Shift Property :

頻率平移 γ 相當於在時域中把訊號乘上一個頻率與頻率平移量相等的複數弦波。

$$Z(j\omega) = X(j(\omega - \gamma)) \quad \xleftrightarrow{FT} \quad z(t) = e^{j\gamma t} x(t)$$

$$e^{j\gamma t} x(t) \quad \xleftrightarrow{FT} \quad X(j(\omega - \gamma))$$

$$e^{jk_0\omega_0 t} x(t) \quad \xleftrightarrow{FS; \omega_0} \quad X[k - k_0]$$

$$e^{j\Gamma n} x[n] \quad \xleftrightarrow{DTFT} \quad X(e^{(\Omega - \Gamma)})$$

$$e^{jk_0\Omega_0 n} x[n] \quad \xleftrightarrow{DTFS; \Omega_0} \quad X[k - k_0]$$

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Proof:

$$\begin{aligned}
 z(t) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} Z(j\omega) e^{j\omega t} d\omega, \quad \langle \text{if } Z(j\omega) = X(j(\omega - \gamma)) \rangle \\
 &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j(\omega - \gamma)) e^{j\omega t} d\omega, \quad \langle \text{let } \eta = \omega - \gamma \rangle \\
 &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\eta) e^{j(\eta + \gamma)t} d\eta \\
 &= e^{j\gamma t} \left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\eta) e^{j\eta t} d\eta \right] \\
 &= e^{j\gamma t} x(t)
 \end{aligned}$$

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Example 3.42 利用頻率平移性質求 $z(t)$ 的FT = ?

$$z(t) = \begin{cases} e^{j10t}, & |t| < \pi \\ 0, & |t| > \pi \end{cases}$$

Solution:

$$\text{let } z(t) = e^{j10t} x(t), \quad \therefore x(t) = \begin{cases} 1, & |t| < \pi \\ 0, & |t| > \pi \end{cases}$$

\therefore

$$Z(j\omega) = X(j(\omega - 10))$$

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Solution: (cont.)

$$X(j\omega) = \int_{-\pi}^{+\pi} e^{-j\omega t} dt = \frac{-1}{j\omega} e^{-j\omega t} \Big|_{-\pi}^{+\pi} = \frac{-1}{j\omega} (e^{-j\omega\pi} - e^{j\omega\pi})$$

$$= \frac{1}{j\omega} (e^{j\omega\pi} - e^{-j\omega\pi}) = \frac{2}{\omega} \left(\frac{e^{j\omega\pi} - e^{-j\omega\pi}}{j2} \right) = \frac{2}{\omega} \sin(\omega\pi)$$

\therefore

$$Z(j\omega) = X(j(\omega-10)) = \frac{2}{\omega-10} \sin((\omega-10)\pi)$$



Problem 3.34(b) : $X(j\omega) = \frac{1}{2+j(\omega-3)} + \frac{1}{2+j(\omega+3)}$

Solution: $x(t) = ?$

$$X(j\omega) = \frac{1}{2+j(\omega-(+3))} + \frac{1}{2+j(\omega-(-3))}$$

For a function of $Z(j\omega) = \frac{1}{2+j\omega} \xleftrightarrow{FT} z(t) = e^{-2t}u(t)$

The frequency shifts are $\omega = +3$ and $\omega = -3$, hence

$$x(t) = e^{j3t} z(t) + e^{-j3t} z(t) = z(t) (e^{j3t} + e^{-j3t})$$

$$= 2e^{-2t} u(t) \left(\frac{e^{j3t} + e^{-j3t}}{2} \right) = 2e^{-2t} \cos(3t) u(t)$$



Using Partial-Fraction Expansions to Find the Inverse FT

利用部分分式展開求逆傅立葉轉換

$$X(j\omega) = \frac{b_M(j\omega)^M + b_{M-1}(j\omega)^{M-1} + \dots + b_1(j\omega) + b_0}{(j\omega)^N + a_{N-1}(j\omega)^{N-1} + \dots + a_1(j\omega) + a_0}$$

- 因為是 FT 是線性
- 部分分式展開把 $X(j\omega)$ 表示為含已知逆FT轉換項之和



Using Partial-Fraction Expansions to Find the Inverse DTFT

利用部分分式展開求逆DT傅立葉轉換

$$X(e^{j\Omega}) = \frac{\beta_M e^{-j\Omega M} + \beta_{M-1} e^{-j\Omega(M-1)} + \dots + \beta_1 e^{-j\Omega} + \beta_0}{\alpha_N e^{-j\Omega N} + \alpha_{N-1} e^{-j\Omega(N-1)} + \dots + \alpha_1 e^{-j\Omega} + 1}$$

- 因為是 DTFT 是線性
- 部分分式展開把 $X(e^{j\Omega})$ 表示為含已知逆DTFT轉換項之和



Multiplication Property 乘法性質

乘法性質決定兩個時域訊號乘積的傅立葉表示法

$$y(t) = x(t)z(t) \xleftrightarrow{FT} Y(j\omega) = X(j\omega) * Z(j\omega)$$

其中

$$X(j\omega) * Z(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\nu) Z(j(\omega - \nu)) d\nu$$

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乘法性質決定兩個時域訊號乘積的傅立葉表示法

$$y[n] = x[n] \cdot z[n] \xleftrightarrow{DTFT} Y(e^{j\Omega}) = X(e^{j\Omega}) \circledast Z(e^{j\Omega})$$

其中 $X(e^{j\Omega})$ 和 $Z(e^{j\Omega})$ 為 2π 週期性頻譜:

$$X(e^{j\Omega}) \circledast Z(e^{j\Omega}) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} X(e^{j\theta}) Z(e^{j(\Omega - \theta)}) d\theta$$

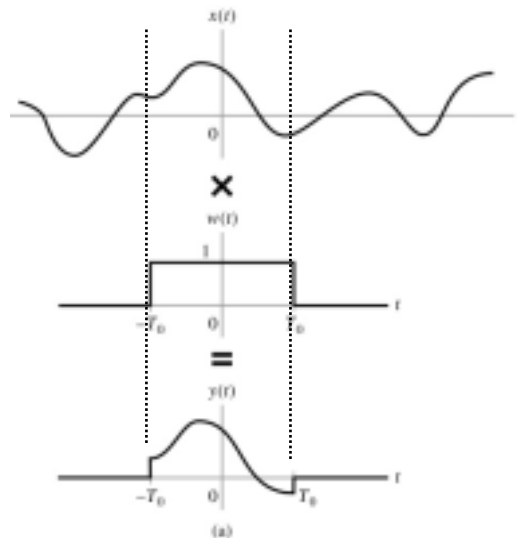
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Windowing

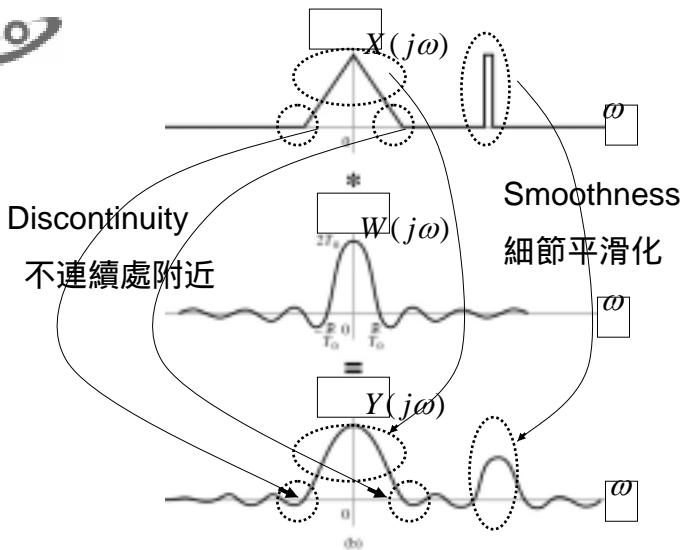
- 截斷 (truncate) 一個時域訊號 $x(t)$ 的過程被稱為視窗法 (windowing)。
- 將一個時域訊號 $x(t)$ 乘上一個視窗函數 (window function) $w(t)$, $y(t) = x(t) w(t)$ 。
- If $w(t)$ is a rectangular shaped window, $w(j\omega) = (2/\omega) \sin(\omega T_0)$, where the windowing is from $-T_0$ to $+T_0$ 。
- 視窗主瓣 (main lobe) $2\pi/T_0$ 把 $X(j\omega)$ 中細節平滑化, 旁瓣 (side lobe) 把不連續處附近產生振盪。

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The effect of windowing. (a) Truncating a signal in time by using a window function $w(t)$.

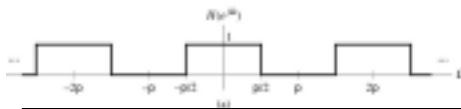
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(b) Convolution of the signal and window FT's resulting from truncation in time.



Ex. 3.46:



上圖為一理想離散系統頻率響應，若另一未知系統的脈衝響應為這理想系統的脈衝響應加以截斷於 $-M \leq n \leq +M$ 區間，請描述其未知系統的頻率響應 = ?

Solution:

這理想系統的脈衝響應可由上圖已知系統頻率響應做逆 DTFT 求出:

$$h[n] = \frac{1}{n\pi} \sin\left(\frac{n\pi}{2}\right)$$



Solution: (cont.)

未知系統的脈衝響應可由已知系統脈衝響應做截斷：

$$h_i[n] = \begin{cases} h[n], & |n| < M \\ 0, & \text{otherwise} \end{cases}$$

截斷的脈衝響應 $h_i[n]$ 可由已知系統脈衝響應 $h[n]$ 和一視窗函數 $w[n]$ (window function) 相乘積： $h_i[n] = h[n] \cdot w[n]$

其中

$$w[n] = \begin{cases} 1, & |n| < M \\ 0, & \text{otherwise} \end{cases}$$

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Solution: (cont.)

應用乘法性質 (Multiplication Property)：

$$h_i[n] = h[n] \cdot w[n] \xleftrightarrow{DFT} H_i(e^{j\Omega}) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} H(e^{j\theta}) W(e^{j(\Omega-\theta)}) d\theta$$

其中

$$H(e^{j\theta}) = \begin{cases} 1, & |\theta| \leq \pi/2 \\ 0, & \pi/2 < |\theta| < \pi \end{cases}$$

參考範例 3.18 可得：

$$X(e^{j\Omega}) = \frac{\sin(\Omega(2M+1)/2)}{\sin(\Omega/2)}$$

$$W(e^{j(\Omega-\theta)}) = \frac{\sin((\Omega-\theta)(2M+1)/2)}{\sin((\Omega-\theta)/2)}$$

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Solution: (cont.)

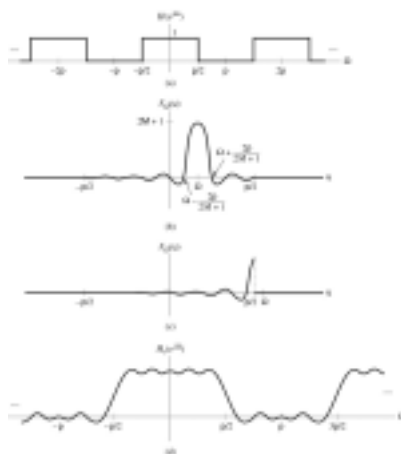
結果

$$H_t(e^{j\Omega}) = \frac{1}{2\pi} \int_{-\pi/2}^{+\pi/2} F_{\Omega}(e^{j\theta}) d\theta$$

其中

$$F_{\Omega}(e^{j\theta}) = H(e^{j\theta}) W(e^{j(\Omega-\theta)}) = \begin{cases} W(e^{j(\Omega-\theta)}), & |\theta| < \frac{\pi}{2} \\ 0, & |\theta| > \frac{\pi}{2} \end{cases}$$

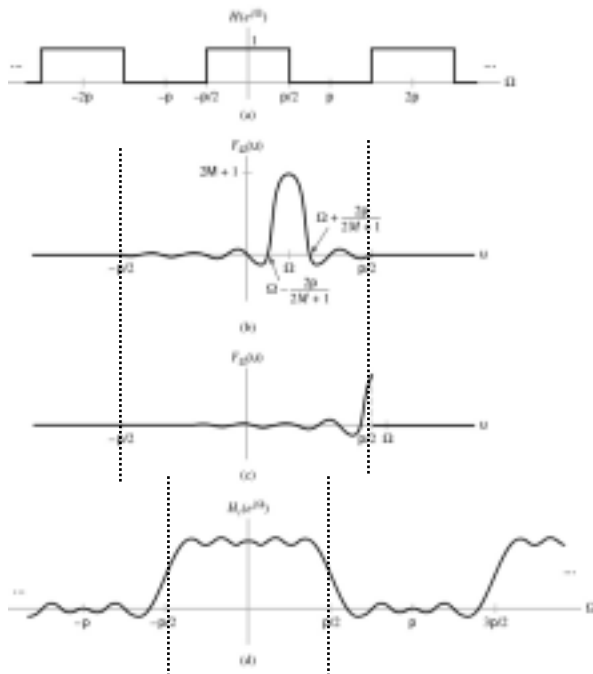
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The effect of truncating the impulse response of a discrete-time system.

- Frequency response of ideal system.
- $F_{\Omega}(\theta)$ for Ω near zero.
- $F_{\Omega}(\theta)$ for Ω slightly greater than $\pi/2$.
- Frequency response of system with truncated impulse response.

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Problem 3.39

應用乘法性質求出訊號 $x(t)$ 的 FT = ?

$$x(t) = \frac{4}{\pi^2 t^2} \sin^2(2t)$$

Solution:

$$x(t) = \frac{2}{\pi t} \sin(2t) \cdot \frac{2}{\pi t} \sin(2t) = w(t) \cdot w(t)$$

所以

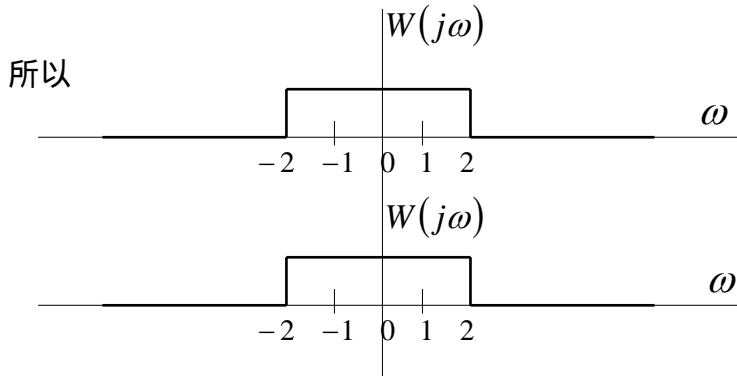
$$X(j\omega) = W(j\omega) * W(j\omega)$$

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Solution: (cont.)

$$\therefore w(t) = \frac{2}{\pi t} \sin(2t); \quad \therefore W(j\omega) = \begin{cases} 2, & |\omega| \leq 2 \\ 0, & |\omega| > 2 \end{cases}$$

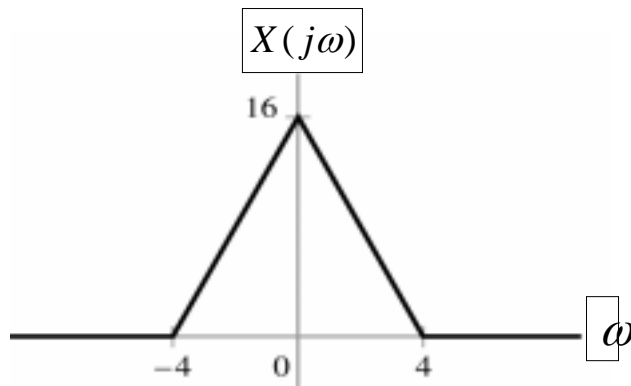


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Solution to Problem 3.39:

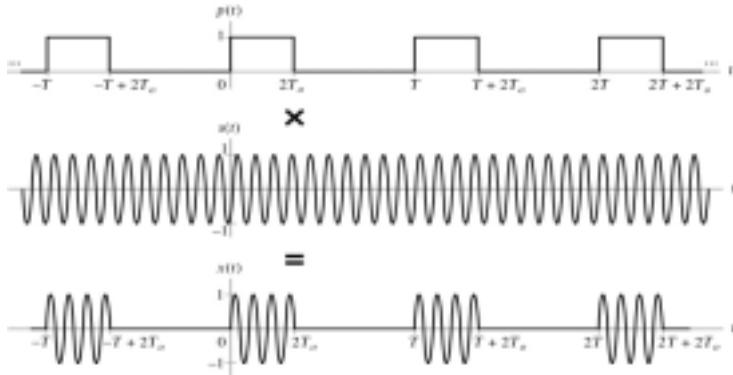
$$X(j\omega) = W(j\omega) * W(j\omega)$$



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Example 3.47: $x(t) = s(t) \bullet p(t)$, $s(t) = \sin(1000\pi t/T)$.
Please find the FT of $x(t) = ?$



The RF pulse is expressed as the product of a periodic square wave and a sine wave.

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Solution: Since $s(t)$ and $p(t)$ are continuous-time periodic signals, the spectra are discrete and non-periodic.

The fundamental frequency of $p(t)$: $\omega_0 = 2\pi/T$,

$s(t) = \sin(1000 \pi t / T) = \sin(\omega t) = \sin(500 \omega_0 t)$,

$\therefore \omega = 500 \omega_0 = k \omega_0$, $k = 500^{\text{th}}$ harmonics.

$$\begin{aligned} s(t) &= \frac{1}{j2} e^{j500\omega_0 t} - \frac{1}{j2} e^{-j500\omega_0 t} \\ &= S[500]e^{j500\omega_0 t} + S[-500]e^{-j500\omega_0 t} \end{aligned}$$

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$$\therefore S[k] = \left. \begin{cases} \frac{1}{j2}, & k = 500 \\ 0, & \text{others} \\ \frac{-1}{j2}, & k = -500 \end{cases} \right\} = \frac{1}{j2} \delta(k - 500) - \frac{1}{j2} \delta(k + 500)$$

參考範例 3.13 並應用時間平移：

$$P[k] = e^{-jkT_0\omega_0} \frac{\sin(k\omega_0 T_0)}{k\pi}$$

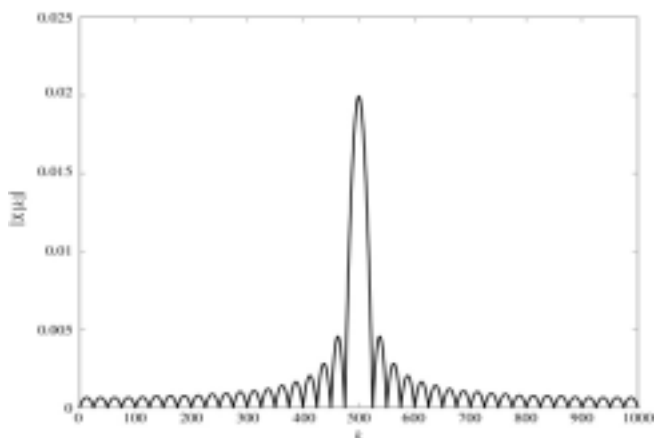


$$\begin{aligned} X[k] &= S[k] * P[k] \\ &= \left(\frac{1}{j2} \delta[k - 500] - \frac{1}{j2} \delta[k + 500] \right) * \left(e^{-jkT_0\omega_0} \frac{\sin(k\omega_0 T_0)}{k\pi} \right) \\ &= \frac{1}{j2} e^{-j(k-500)T_0\omega_0} \frac{\sin((k-500)\omega_0 T_0)}{(k-500)\pi} \\ &\quad - \frac{1}{j2} e^{-j(k+500)T_0\omega_0} \frac{\sin((k+500)\omega_0 T_0)}{(k+500)\pi} \end{aligned}$$

應用脈衝函數的篩選性質



FS magnitude spectrum for $0 \leq k \leq 1000$. The result is depicted as a continuous curve, due to the difficulty of displaying 1000 stems.



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Scaling Properties 比例變換性質

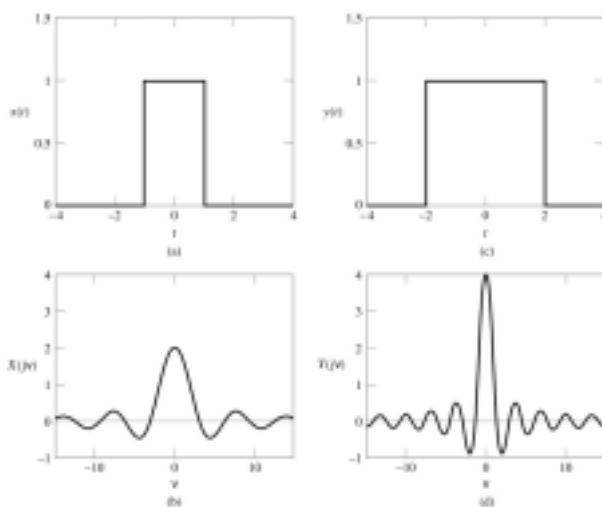
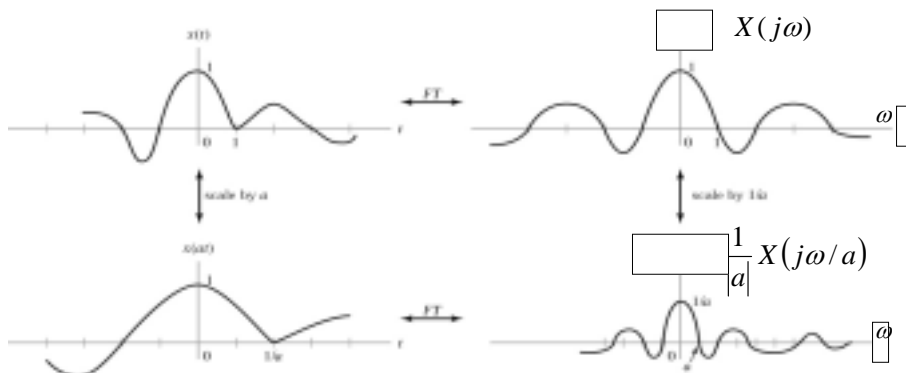
- 顯示出當變換時間變數的比例對訊號的頻域表示

$$z(t) = x(at) \xleftrightarrow{FT} \frac{1}{|a|} X\left(j\frac{\omega}{a}\right)$$

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The FT scaling property.
 The figure assumes that $0 < a < 1$. a



Application of the FT scaling property in Example 3.48.

- (a) Original time signal. (b) Original FT.
- (c) Scaled time signal $y(t) = x(t/2)$.
- (d) Scaled FT $Y(j\omega) = 2X(j2\omega)$.



Parseval Relationship 巴賽瓦關係

訊號時域(time domain)表示法中所包含的能量(energy)或功率(power)等同於它得頻域(frequency domain)表示法中所包含的能量或功率。

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

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$$\begin{aligned} W_x &= \int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} x(t) \cdot x^*(t) dt \\ &= \int_{-\infty}^{+\infty} x(t) \cdot \left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega \right]^* dt \\ &= \int_{-\infty}^{+\infty} x(t) \cdot \left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} X^*(j\omega) e^{-j\omega t} d\omega \right] dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X^*(j\omega) \left[\int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \right] d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X^*(j\omega) [X(j\omega)] d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega \end{aligned}$$

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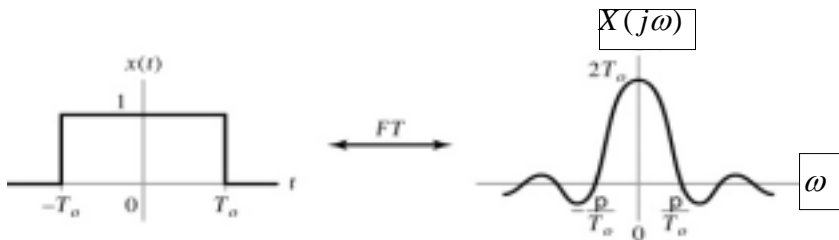
Time-Bandwidth Production 時間頻寬乘積

- 在時域中把訊號壓縮，導致訊號在頻域中擴張。
- 在時域中把訊號擴張，導致訊號在頻域中壓縮。
- “測不準原理”說明無法精確的決定一個電子的位置和動量。
- 新“測不準原理”被用來說明時域解析度和頻域解析度不能同時要求
 - 頻率範圍和時間範圍的乘積永遠有一個常數的下線。

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Rectangular pulse illustrating the inverse relationship between the time and frequency extent of a signal.



頻率範圍： $-\infty \sim +\infty$ 但能量集中於主瓣 (main lobe)。

集中於 $|\omega| \leq \pi / T_0$ 訊號時間範圍 T_0 和頻率範圍相反。

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Duality 對偶性

- 時域/頻域中矩形脈波，在頻域/時域中為 sinc 函數。
- 時域中脈衝函數/常數，在頻域中為常數/脈衝函數。
- 時域中褶積運算/乘法運算，在頻域中為乘法運算/褶積運算。
- 時域/頻域中微分運算，在頻域/時域中為乘 $j\omega$ / $-jt$ 。
- ...
- 上述時間和頻率間互換性質，稱為對偶性。



FT 轉換以一通式表示：

$$y(v) = \frac{1}{2\pi} \int_{-\infty}^{\infty} z(\eta) e^{jv\eta} d\eta$$

若 $\eta = \omega$, $v = t$:

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} z(\omega) e^{j\omega t} d\omega$$

若 $\eta = t$, $v = -\omega$:

$$2\pi y(-\omega) = \int_{-\infty}^{\infty} z(t) e^{-j\omega t} dt$$



通式表示：
$$y(v) = \frac{1}{2\pi} \int_{-\infty}^{\infty} z(\eta) e^{jv\eta} d\eta$$

若已知一 FT pair：
$$f(t) \xleftrightarrow{FT} F(j\omega)$$

$\eta = \omega$, $v = t$ ：
$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) e^{j\omega t} d\omega$$

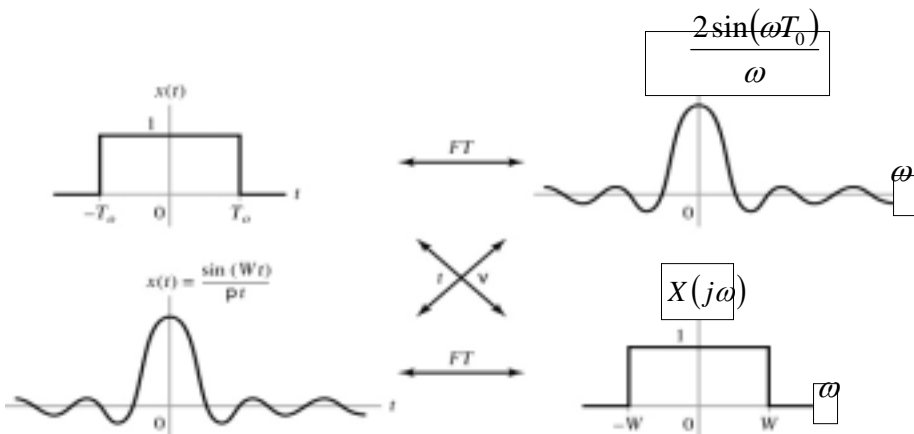
$\eta = t$, $v = -\omega$ ：
$$2\pi f(-\omega) = \int_{-\infty}^{\infty} F(jt) e^{-j\omega t} dt$$

可對應出另一 FT pair：
$$F(jt) \xleftrightarrow{FT} 2\pi f(-\omega)$$

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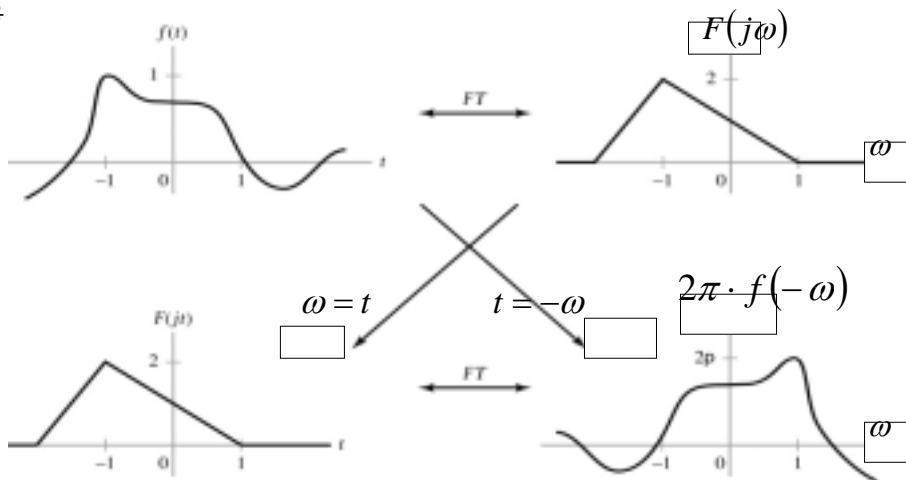
Duality of rectangular pulses and sinc functions



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The FT duality property



Example 3.53.

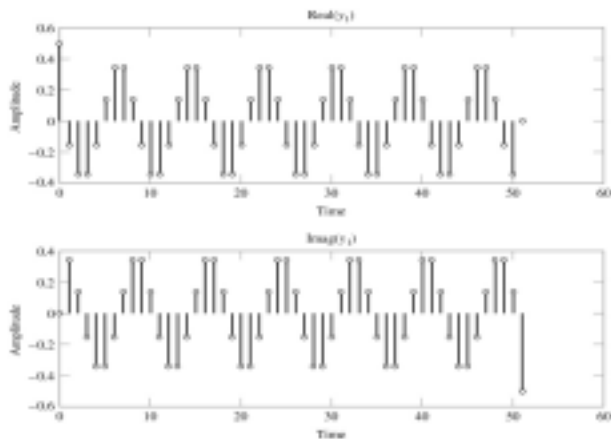
(a) Triangular spectrum.

(b) Inverse DTFT.





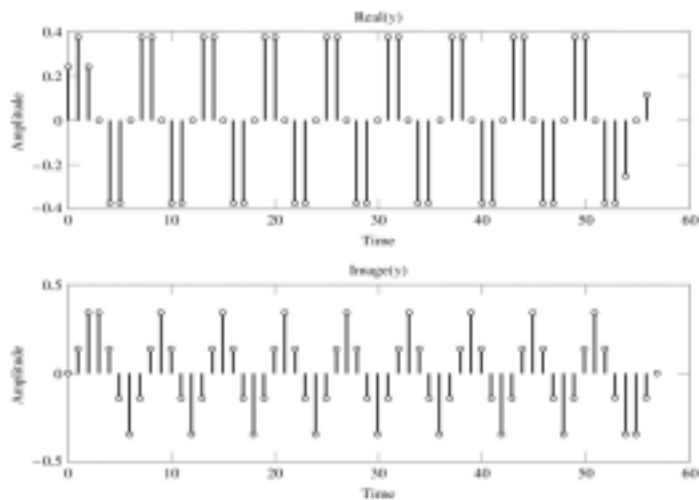
Sinusoidal steady-state response computed with the use of MATLAB. The values at times 1 through 50 represent the sinusoidal steady-state response.



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Solution to Problem 3.46



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