Signals and Systems
信號與系統

Lecture 1-2

Elementary Signals

- Exponential Signals 指數型訊號
- Sinusoidal Signals 弦波型訊號
- Exponentially Damped Sinusoidal Signals
  指數型弦波型訊號
- Step (Function) Signals 步階訊號
- Impulse (Function) Signals 脈衝訊號
- Ramp Function of Unit Slop 單位斜率斜訊號

Exponential Signals

$$x(t) = A\,e^{at}$$

$a < 0$: Decaying Exponential Signal

$a > 0$: Growing Exponential Signal

$$e = 2.71828183\ldots$$

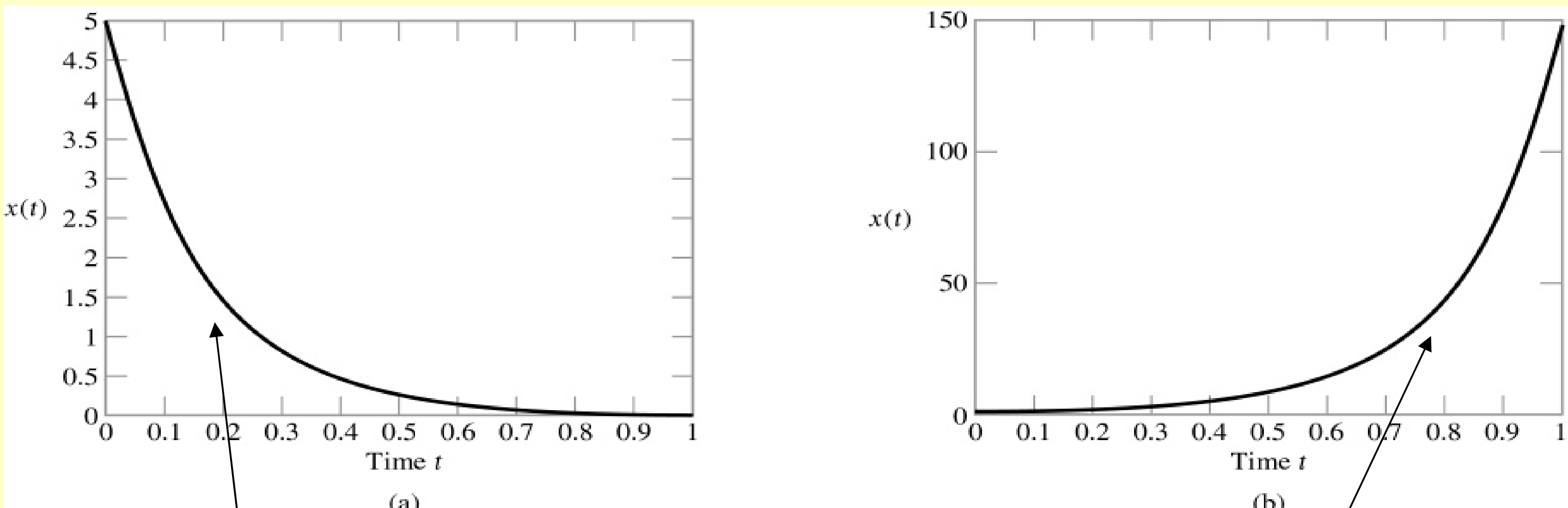
$$e^0 = 1$$

$$e^1 = 2.71828183\ldots$$

$$\ldots$$

$e^t =$以 t 為自變數的指數函數

(a) Decaying exponential form of continuous-time signal.
(b) Growing exponential form of continuous-time signal.



衰退指數性訊號

成長指數性訊號

Example:

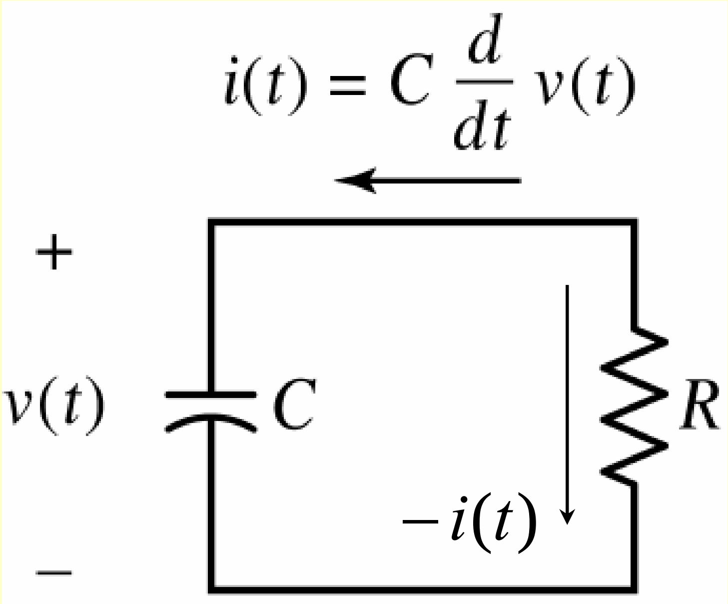
Lossy capacitor, with the loss represented by shunt resistance *R*.

$$\boxed{v(t)=\frac{1}{C}\int_{-\infty}^{t} i(\tau)\,d\tau}$$

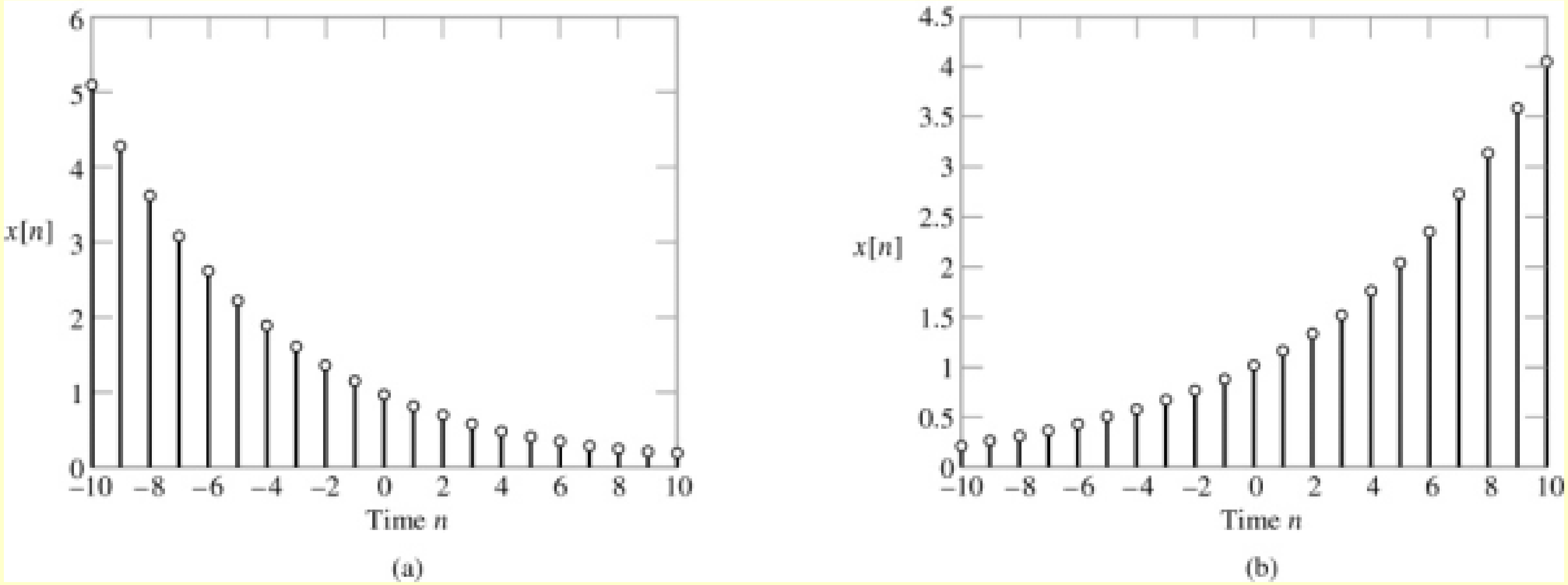
$$\because\ v(t)=-i(t)\cdot R=-RC\frac{d}{dt}v(t)$$

$$\therefore\ RC\frac{d}{dt}v(t)+v(t)=0$$

$$\therefore\quad v(t)=V_0\,e^{-t/(RC)}$$



(a) Decaying exponential form of discrete-time signal.
(b) Growing exponential form of discrete-time signal.



(a)
(b)

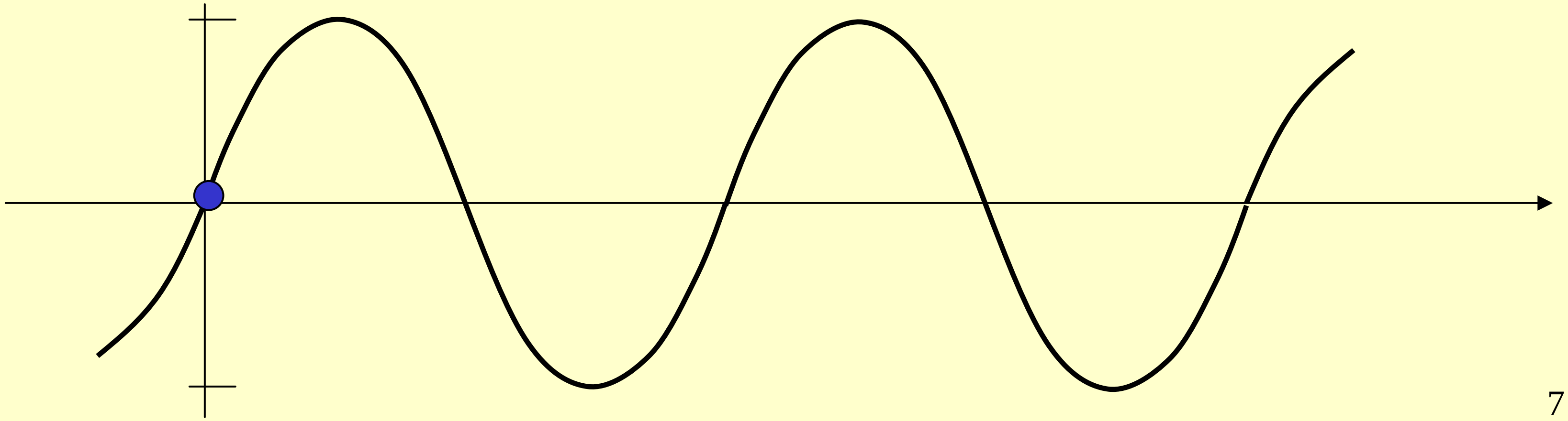
Sinusoidal Signals

$$\boxed{x(t) = A\sin(\omega t + \phi)}$$

$A:\quad amplitude$ 訊號振幅

$\omega:\quad angular\ \ frequency(radians/\mathrm{sec})$ 訊號角頻率

$\phi:\quad phase$ 訊號相位移



Sinusoidal Signals (cont.)

$$\boxed{x(t) = A\sin(\omega t + \phi)}$$

$A:$ *amplitude* 訊號振幅

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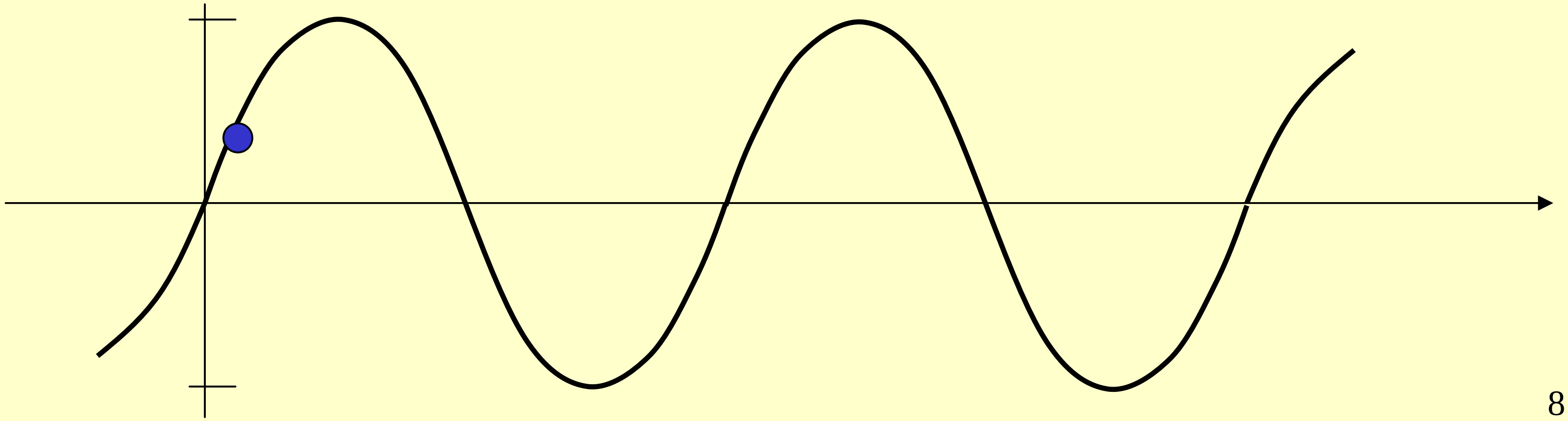

Sinusoidal Signals (cont.)

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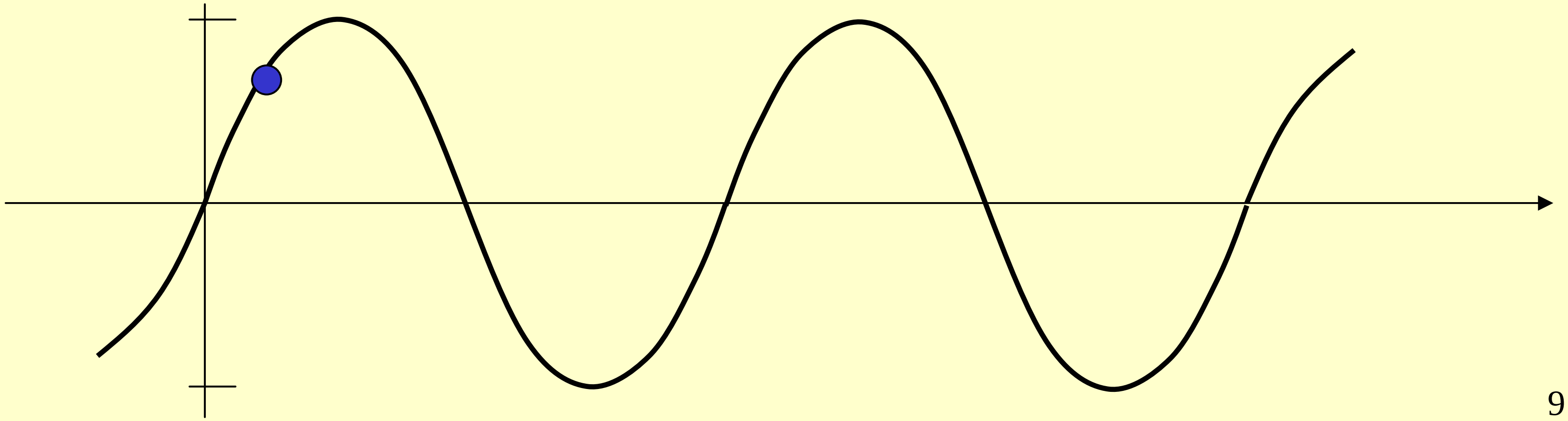

Sinusoidal Signals (cont.)

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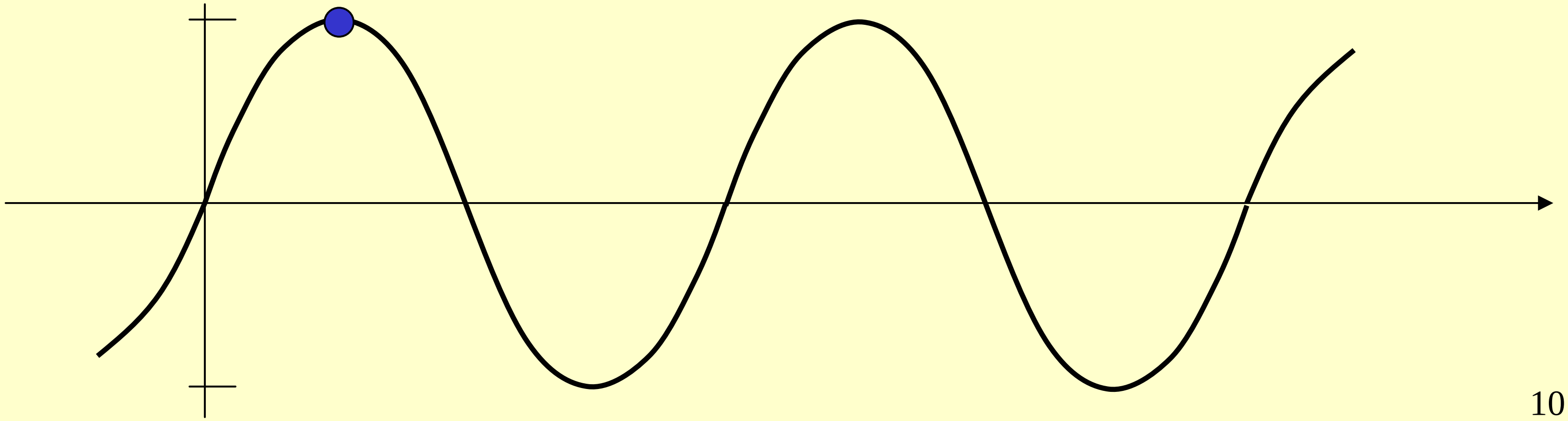

Sinusoidal Signals (cont.)

$$\boxed{x(t) = A\sin(\omega t + \phi)}$$

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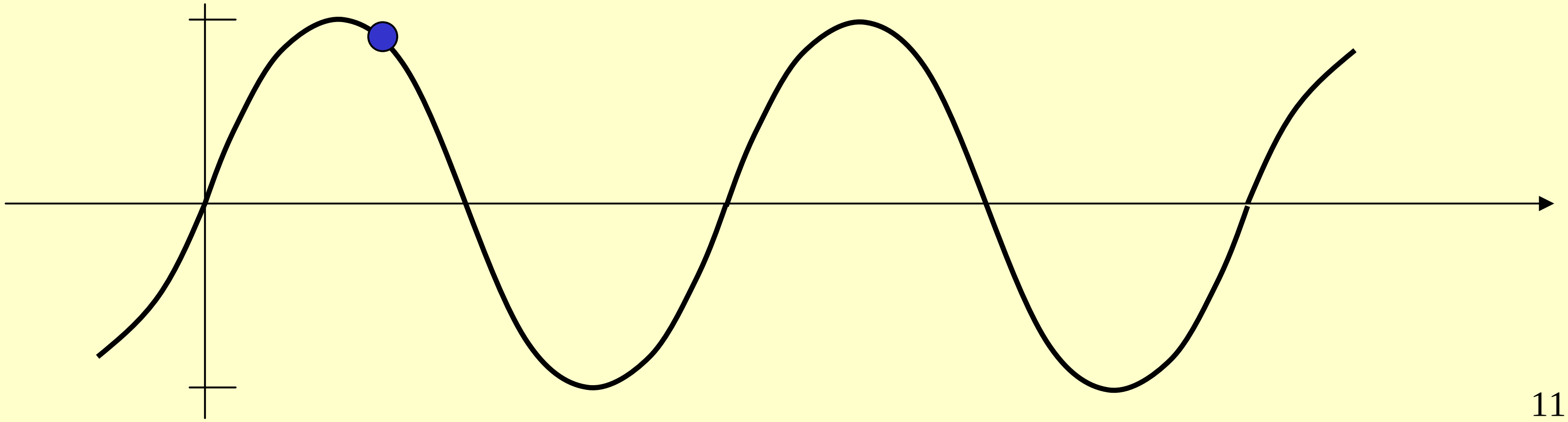

Sinusoidal Signals (cont.)

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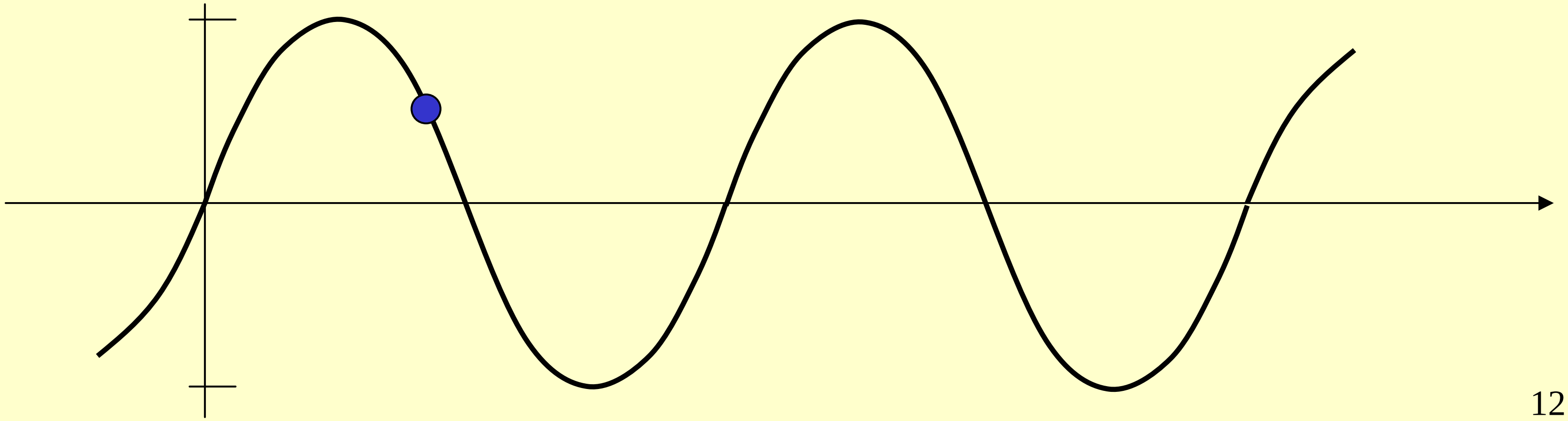

Sinusoidal Signals (cont.)

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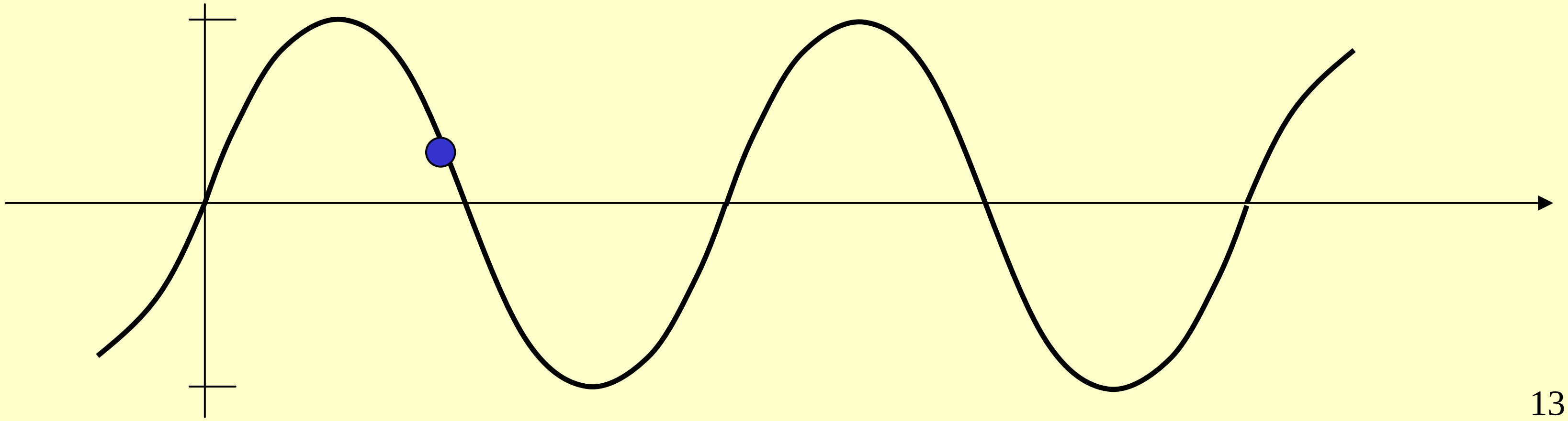

Sinusoidal Signals (cont.)

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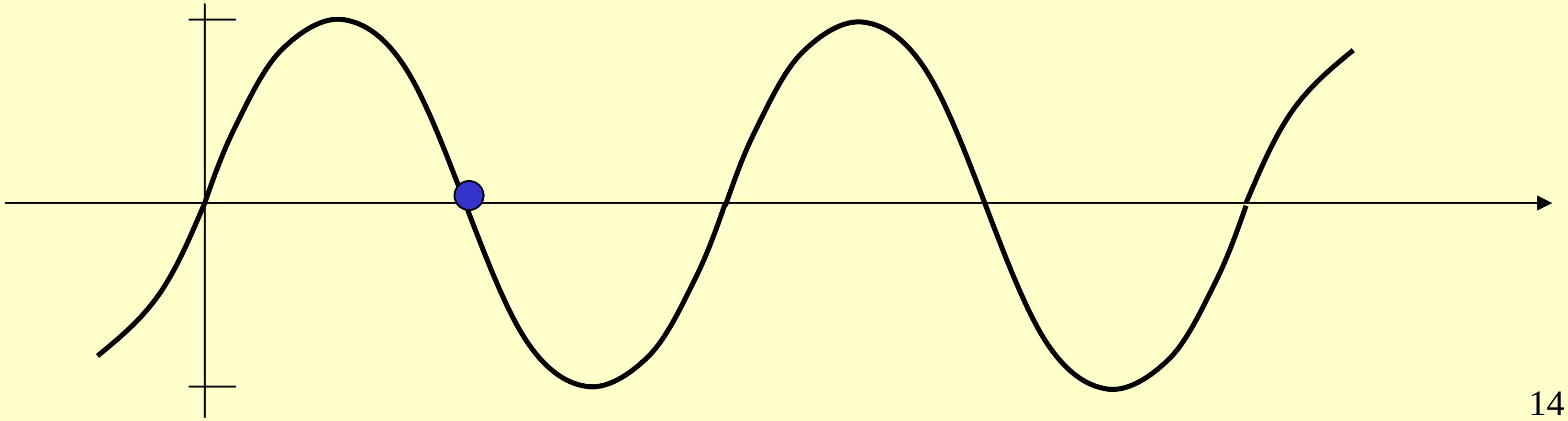
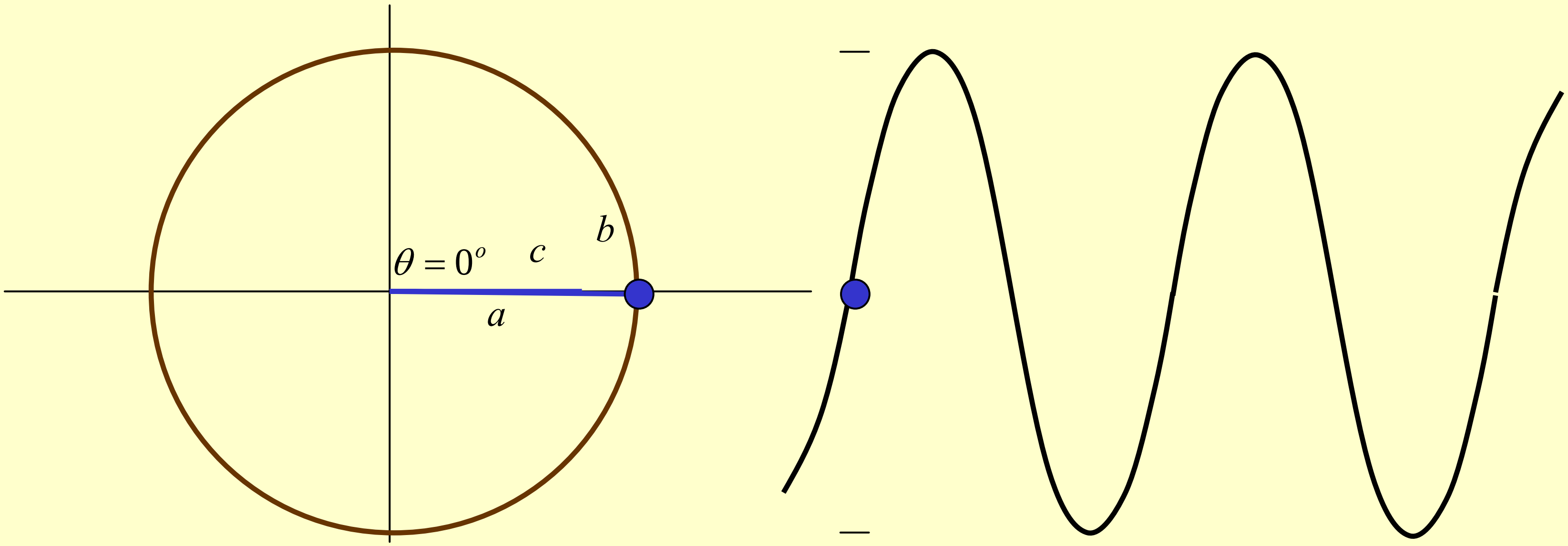
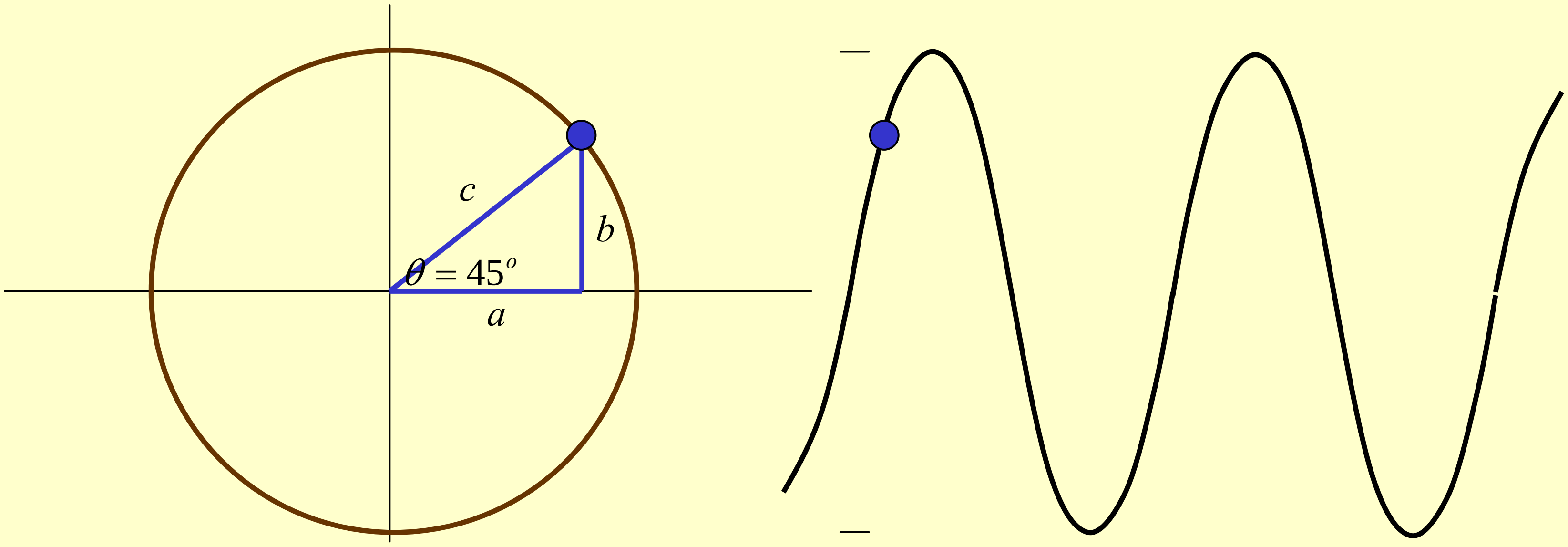


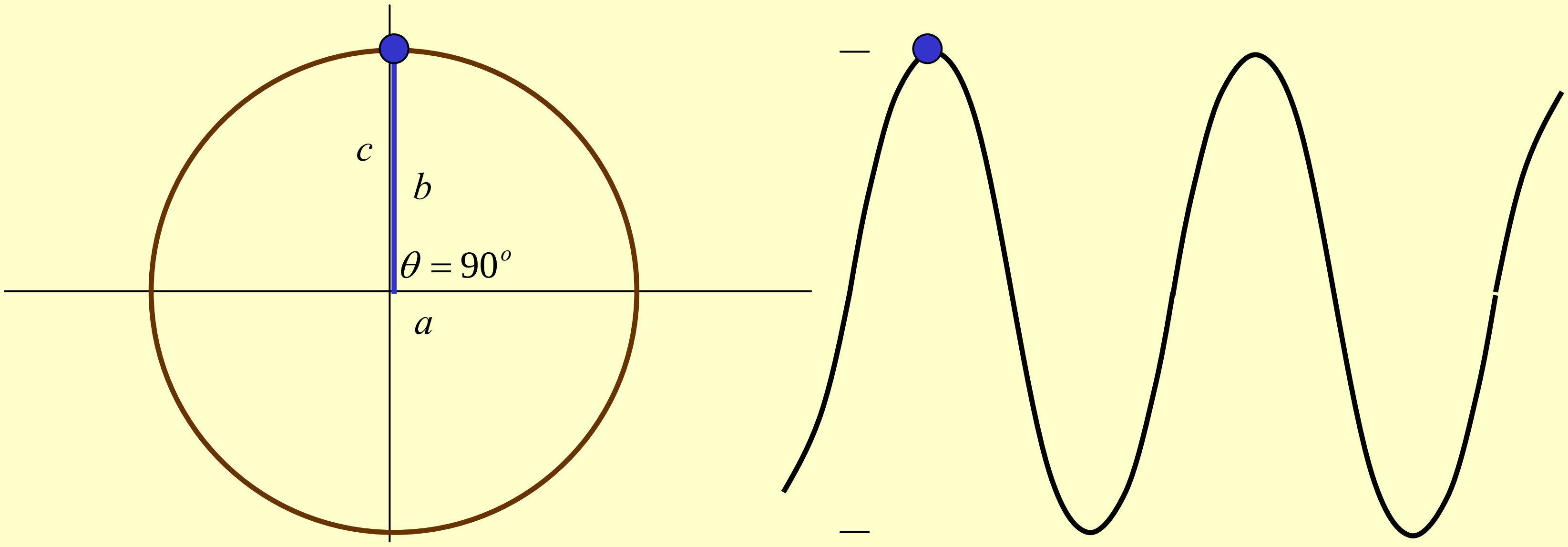
Phasor and Sinusoids



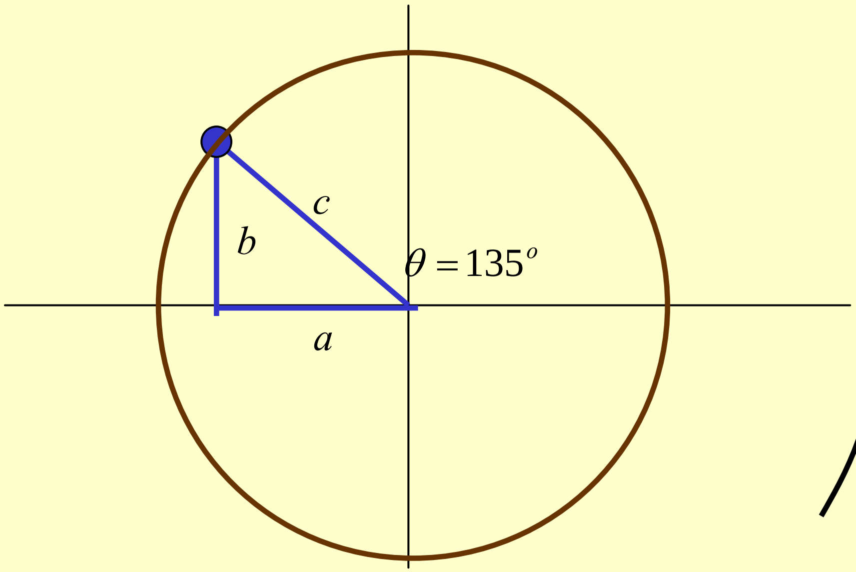
Phasor at 45º

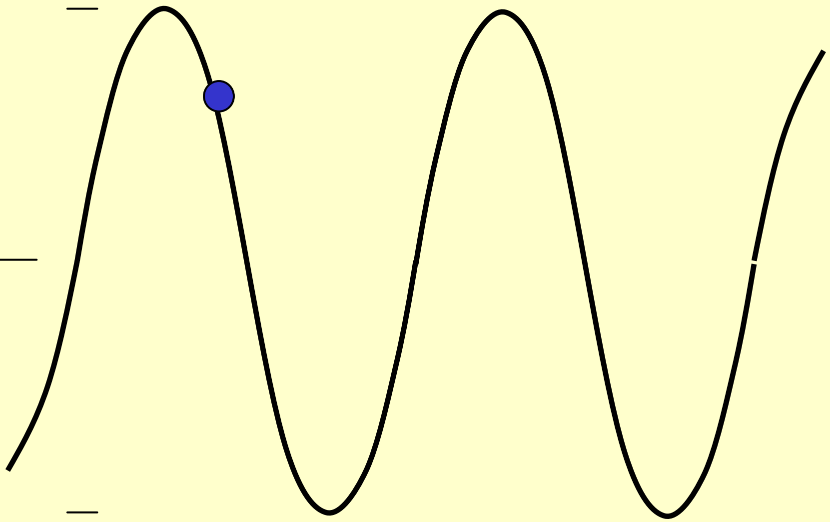


Phasor at 90º

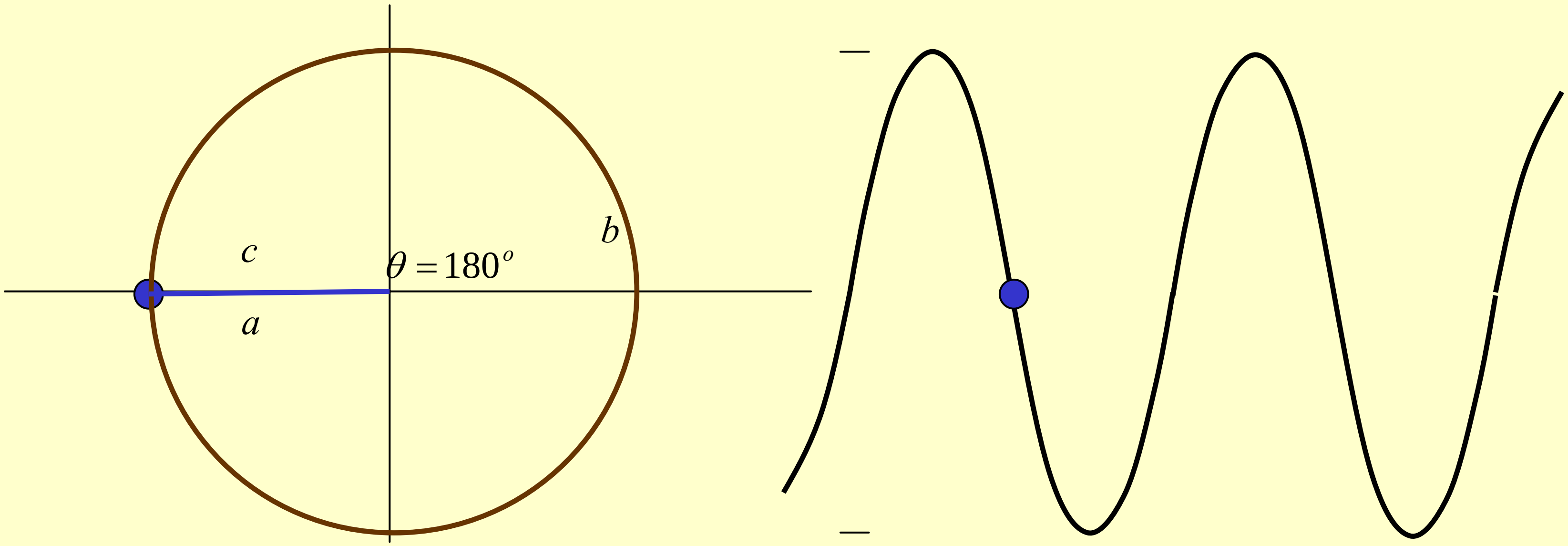


Phasor at 135º

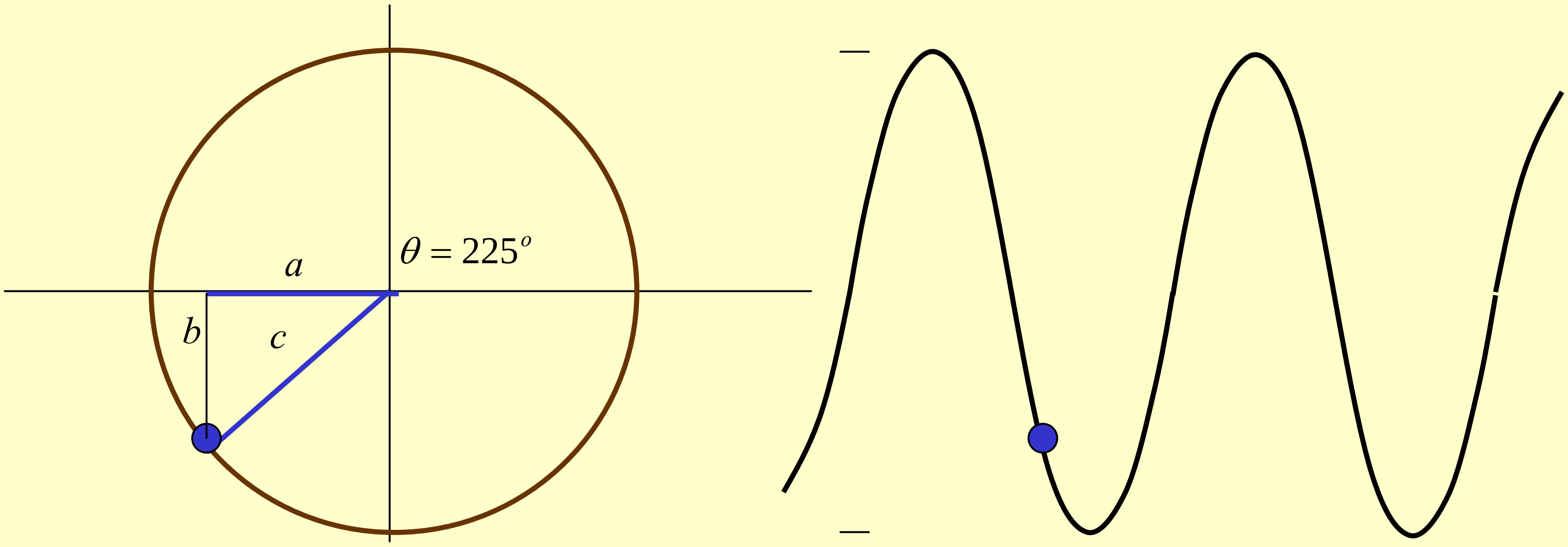




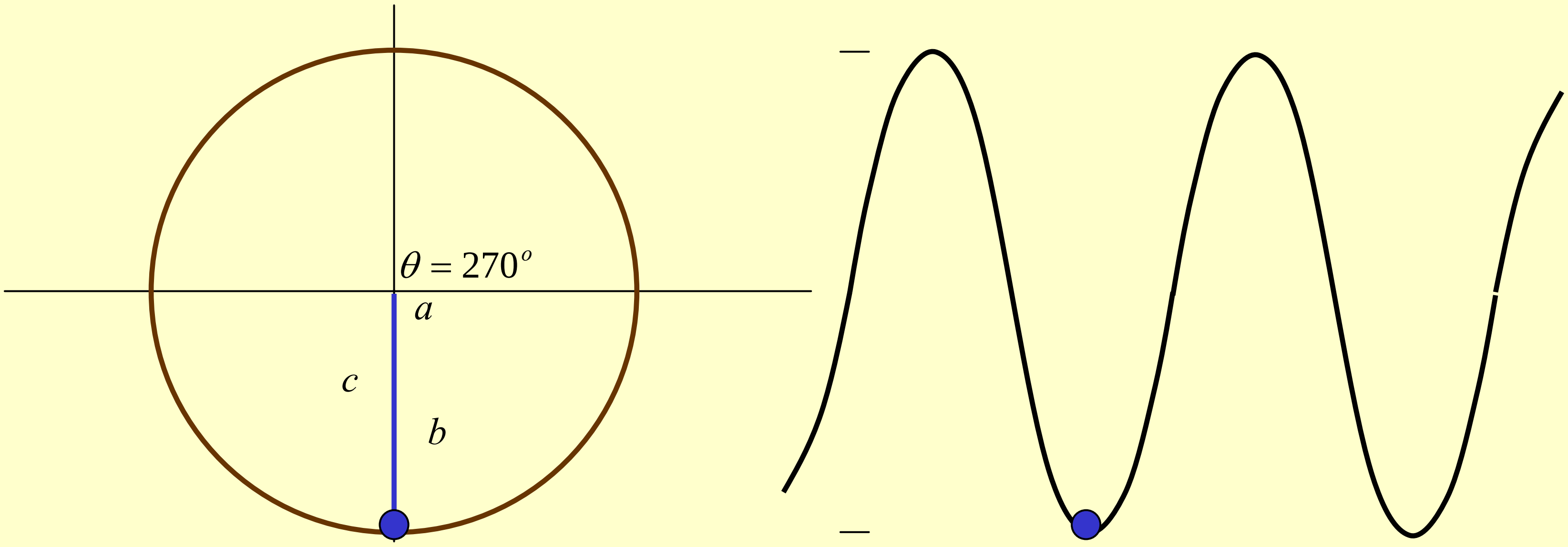
Phasor at 180º



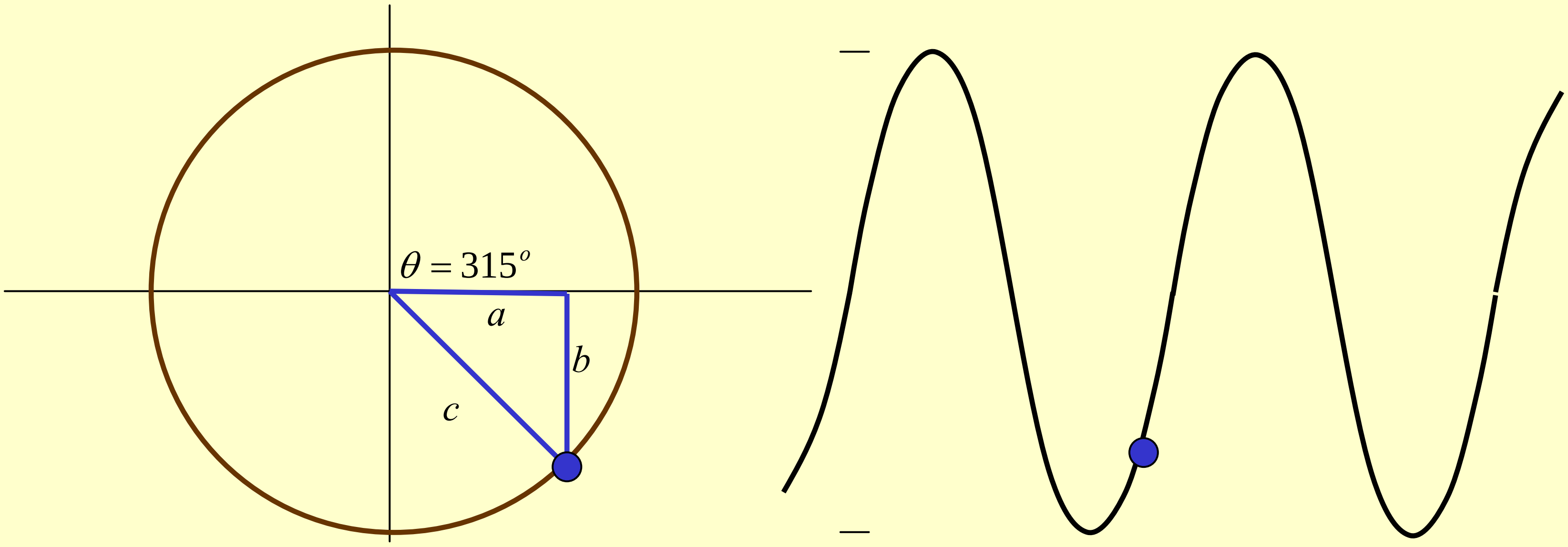
Phasor at 225º



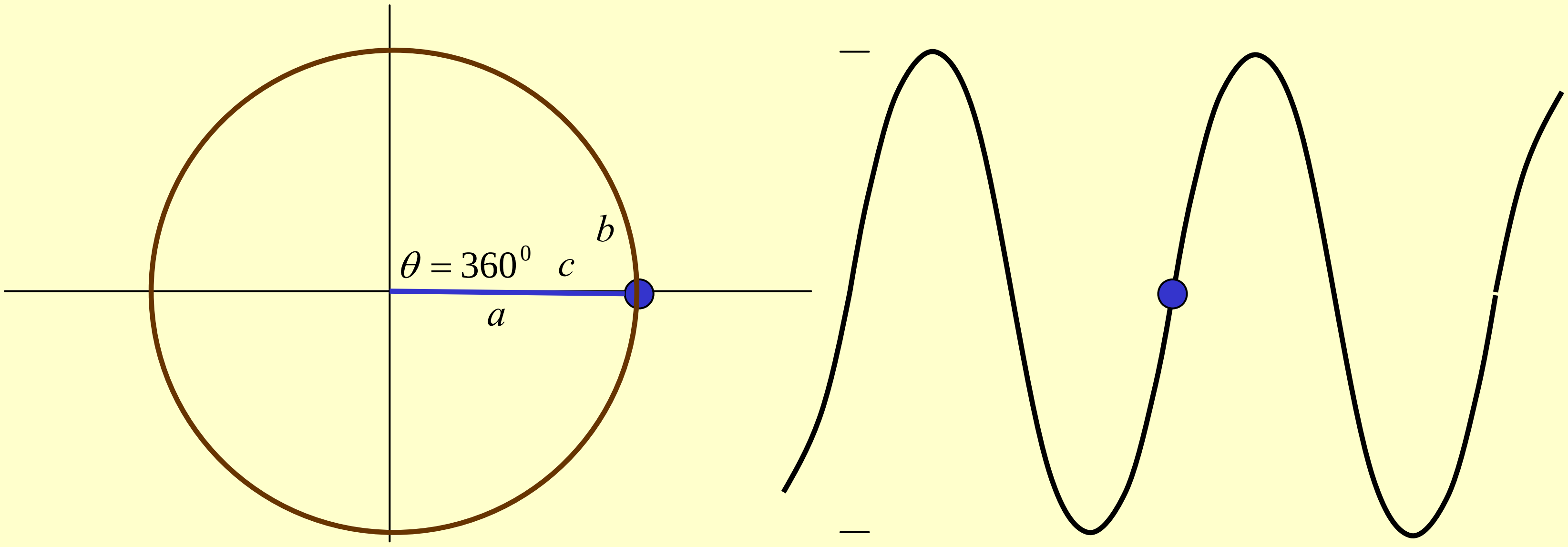
Phasor at 270º



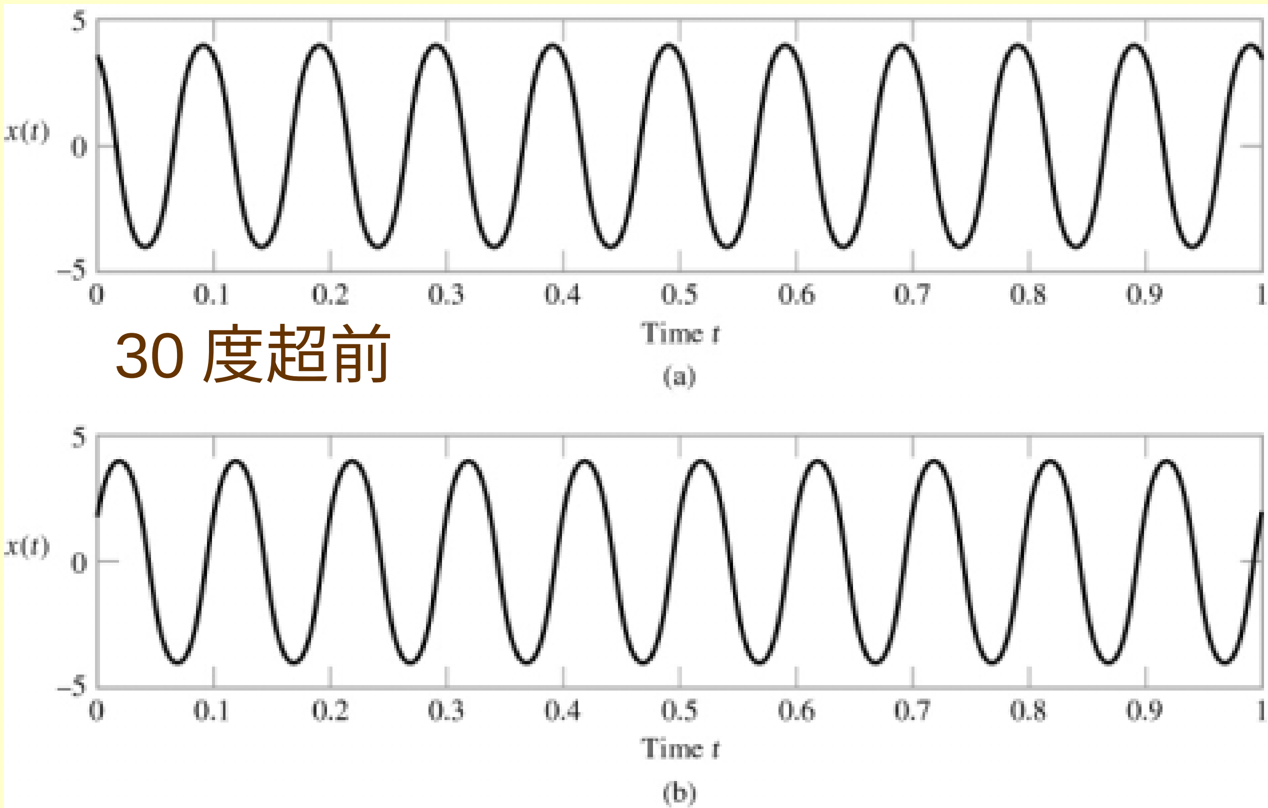
Phasor at 315^o



Phasor at 360º



30 度超前 (T = 0.1 sec., ω=2π/T=20π)



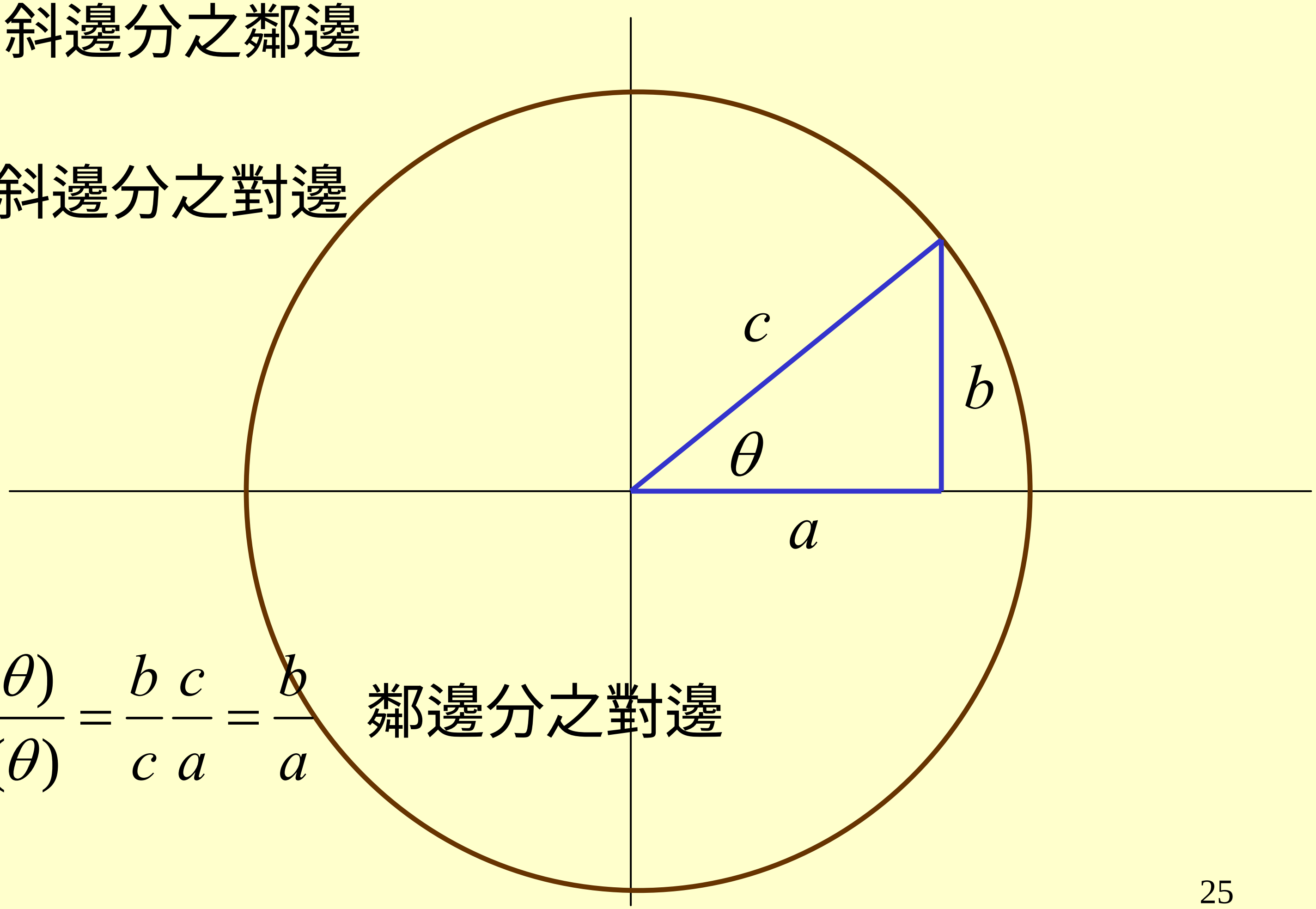
30 度超前

(a) Sinusoidal signal *A* cos(ωt + Φ) with phase Φ = +π/6 radians.
(b) Sinusoidal signal *A* sin (ωt + Φ) with phase Φ = +π/6 radians.

Triangular Function

$$\cos(\theta) = \frac{a}{c}$$ 斜邊分之鄰邊

$$\sin(\theta) = \frac{b}{c}$$ 斜邊分之對邊



$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{b}{c}\frac{c}{a} = \frac{b}{a}$$ 鄰邊分之對邊

Sinusoidal Function

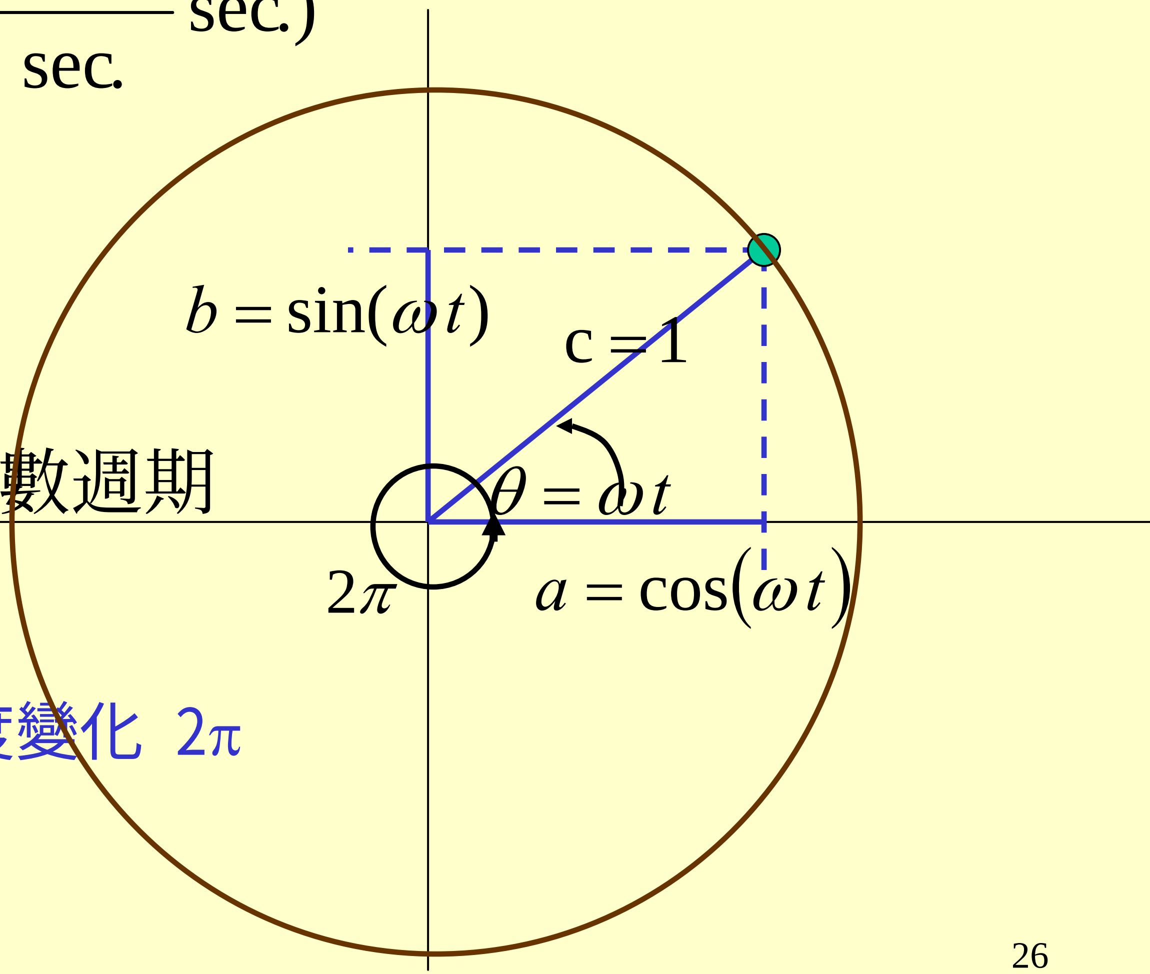
$$\theta = \omega t \qquad (redians = \frac{redians}{\text{sec.}}\,\text{sec.})$$

$\omega: \quad 2\pi / T$

$\theta: \quad 0 \;\sim\; 2\pi$

$t: \quad 0 \;\sim\; T$

$T: \quad Period$ (訊號)函數週期

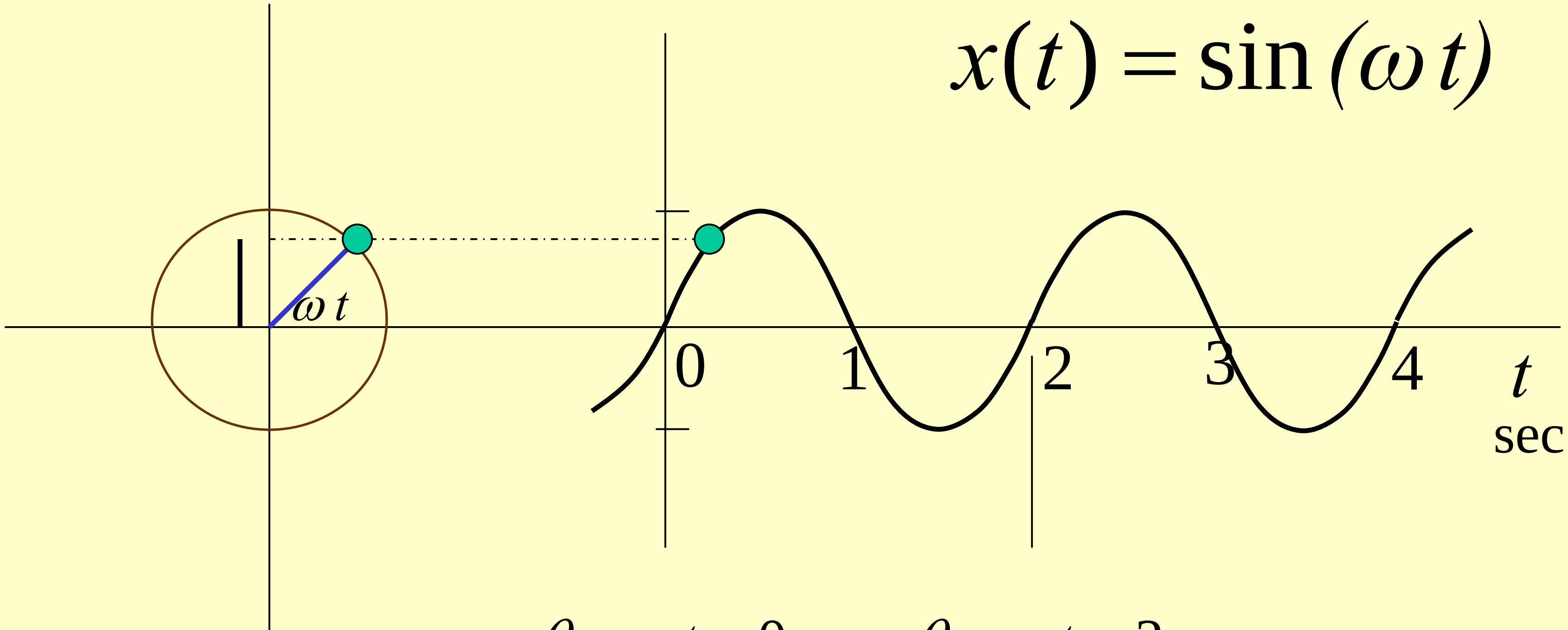


逆時針方向繞一圈角度變化 2π

Counter-clockwise

Sine

$$x(t) = \sin(\omega t)$$



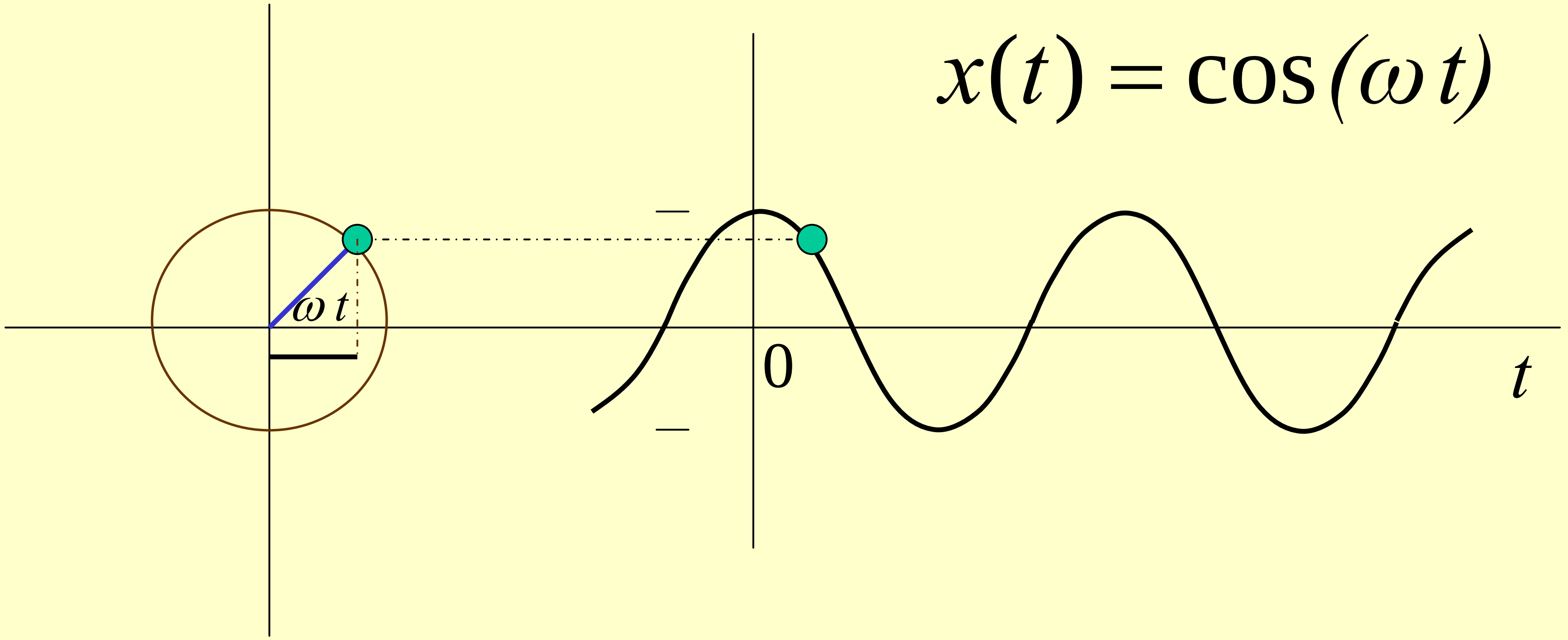
$$\theta = \omega t = 0 \qquad \theta = \omega t = 2\pi$$

$$= \frac{2\pi}{T}(2)$$

週期 T = 2 秒

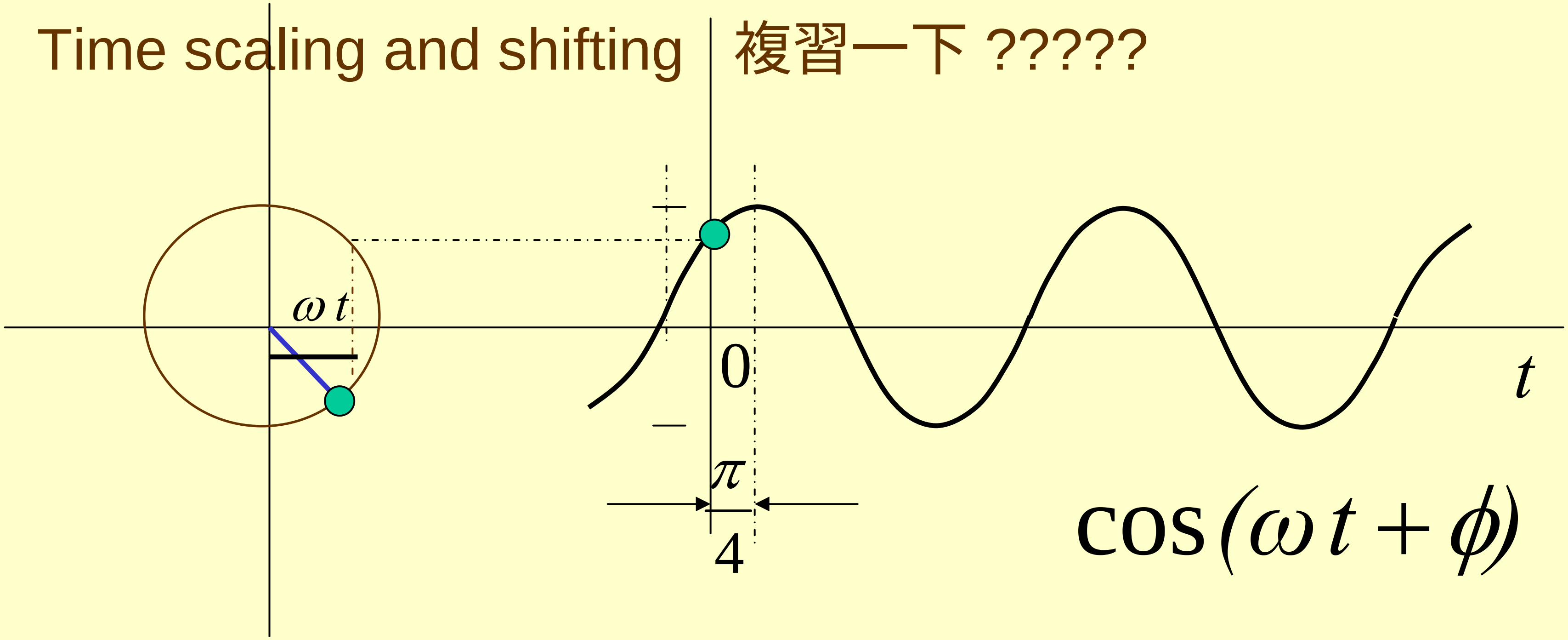
Cosine

$$x(t) = \cos(\omega t)$$



Cosine with Phase

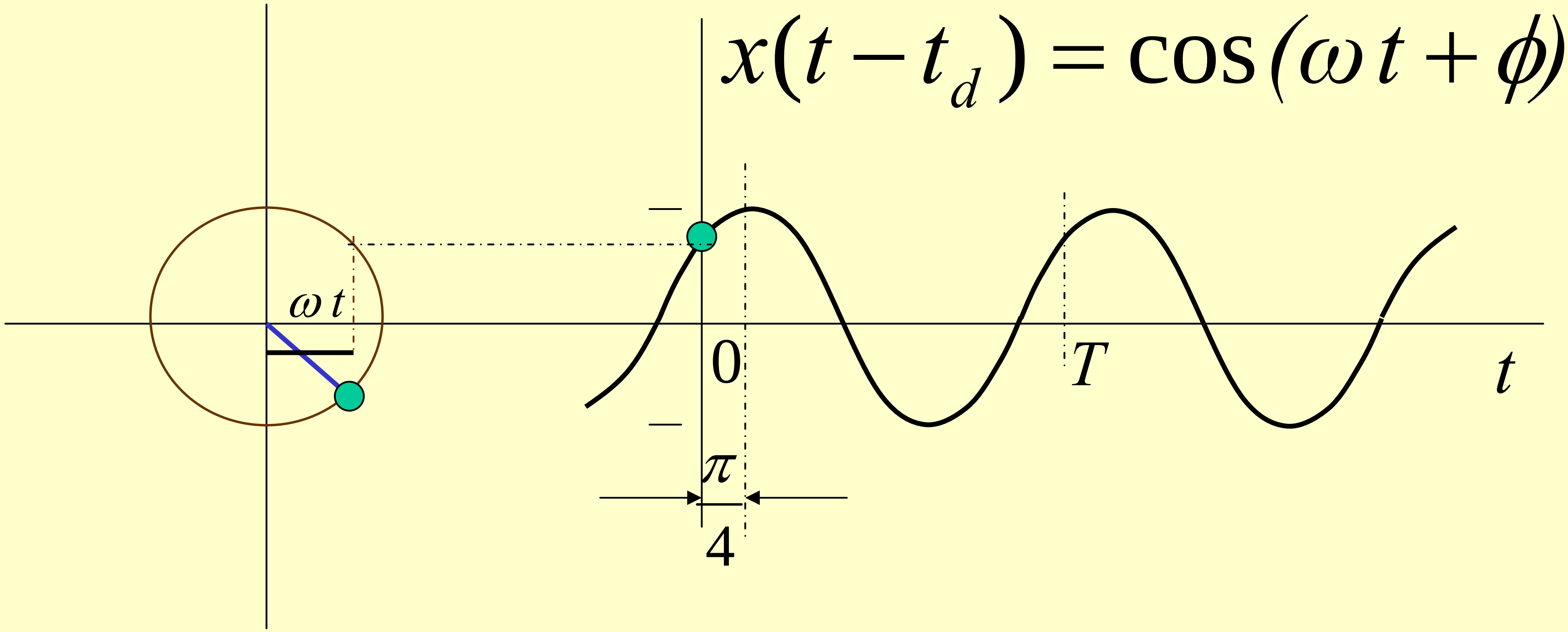
Time scaling and shifting 複習一下 ?????



$$\cos(\omega t + \phi)$$

Cosine 訊號延遲了多少時間 ？ t_d = ?

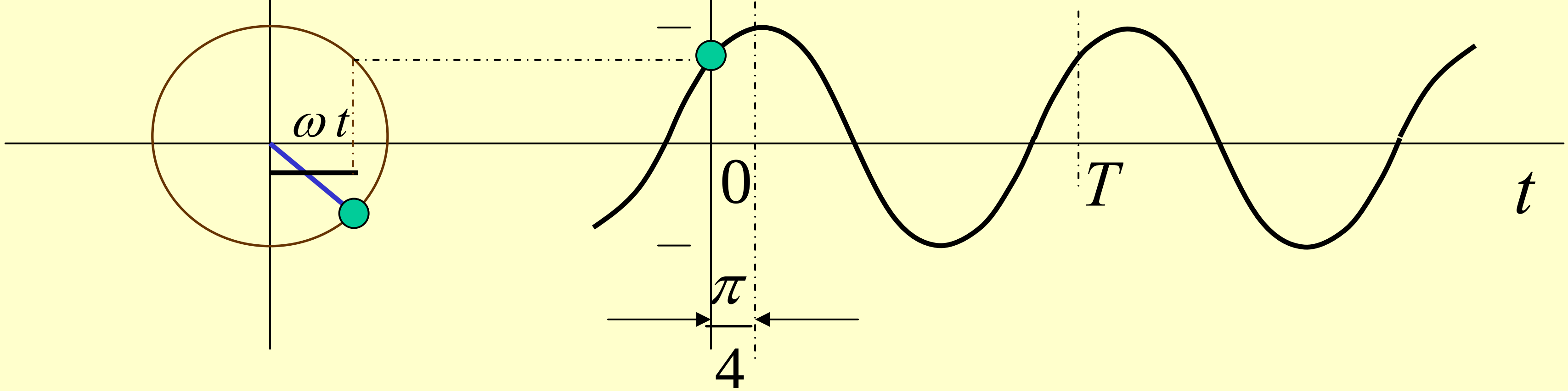
Cosine with Time Shift



Cosine 訊號延遲了多少時間 ? t_d = ?

$$\because \omega t_d = \frac{2\pi}{T} t_d = \frac{\pi}{4}, \qquad \therefore t_d = \frac{\pi}{4}\frac{T}{2\pi} = \frac{T}{8}$$

Cosine with Time Shift (Cont.)

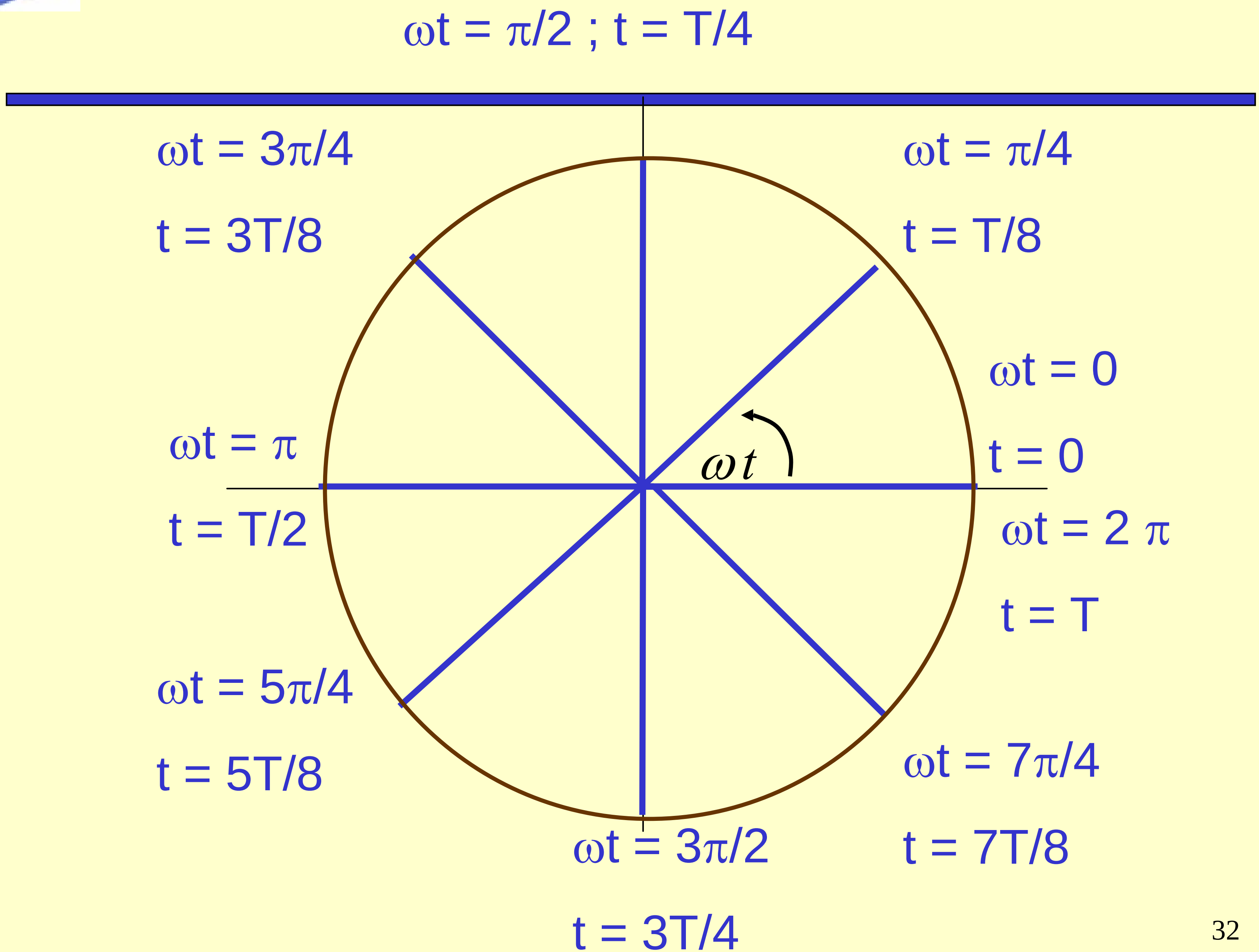


$$x(t-t_d)=\cos(\omega(t-t_d))=\cos(\omega t-\omega t_d)$$

$$=\cos(\omega t+\phi)$$

$$\therefore$$

$$\phi=-\omega t_d \quad =-\frac{\pi}{4}$$


ωt = π/2 ; t = T/4
ωt = 3π/4
t = 3T/8
ωt = π/4
t = T/8
ωt = 0
t = 0
ωt = π
t = T/2
ωt
ωt = 2 π
t = T
ωt = 5π/4
t = 5T/8
ωt = 7π/4
t = 7T/8
ωt = 3π/2
t = 3T/4

Example:

Parallel *LC* circuit, assuming ideal inductor *L* and capacitor *C*

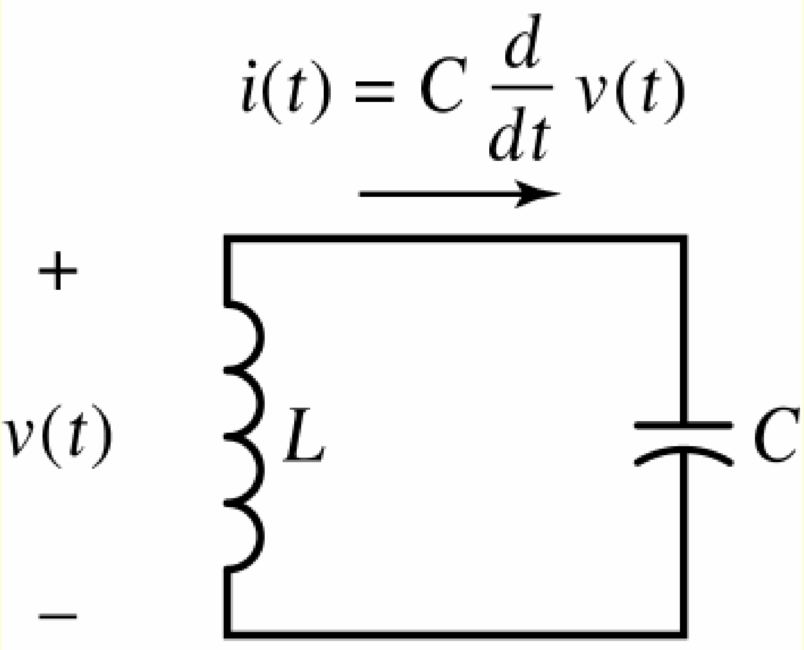
(弦波訊號電路範例)

$$\therefore LC\frac{d^2}{dt^2}v(t)+v(t)=0$$

$$solution:$$

$$v(t)=\cos(\omega_0 t),\quad t\geq 0$$

其中 $\omega_0=\dfrac{1}{\sqrt{LC}}.$



$$\because \quad v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)\, d\tau \Rightarrow i(t) = C\frac{d}{dt}v(t)$$

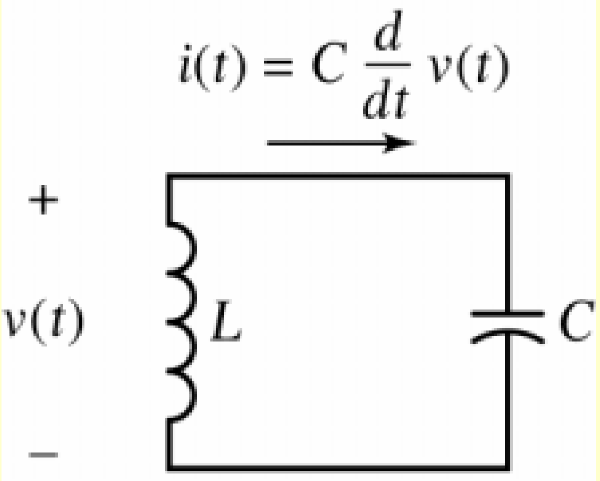
$$\therefore \quad v(t) = -L\frac{d}{dt}i(t) = -L\frac{d}{dt}\left(C\frac{d}{dt}v(t)\right) = -LC\frac{d^2}{dt^2}v(t)$$

$$\therefore LC\frac{d^2}{dt^2}v(t) + v(t) = 0$$

$solution:$

$$v(t) = \cos(\omega_0 t), \quad t \geq 0$$

其中 $\omega_0 = \dfrac{1}{\sqrt{LC}}.$



下一頁說明過程

$$LCs^2 + 1 = 0,$$

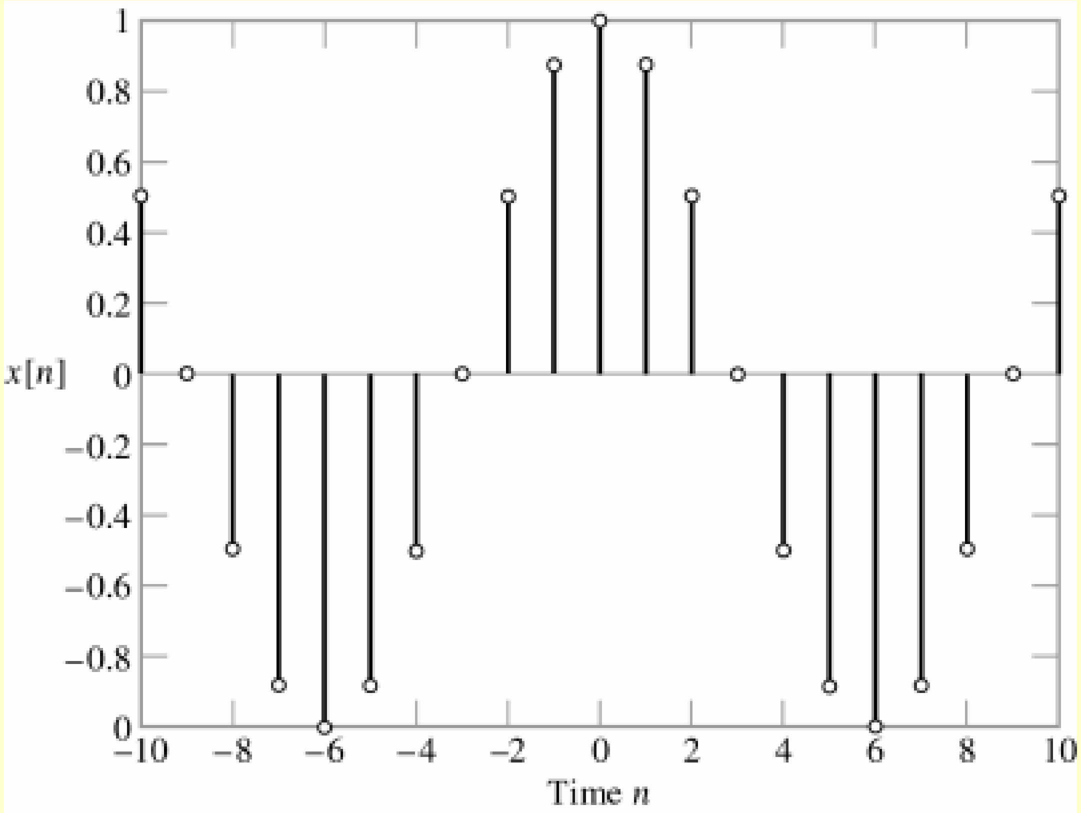
$$\Rightarrow LCs^2 = -1, \quad \Rightarrow s^2 = \frac{1}{-LC}, \Rightarrow s = \pm j\frac{1}{\sqrt{LC}},$$

solution :

$$\frac{1}{2}e^{s_1 t} + \frac{1}{2}e^{s_2 t} = \frac{1}{2}\left(e^{j\frac{t}{\sqrt{LC}}} + e^{-j\frac{t}{\sqrt{LC}}}\right) = \left(\frac{e^{j\frac{t}{\sqrt{LC}}} + e^{-j\frac{t}{\sqrt{LC}}}}{2}\right)$$

$$= \cos\left(\frac{t}{\sqrt{LC}}\right)$$

Discrete-time sinusoidal signal



Sinusoids & Complex Exponential Signals

Euler's Identity:

$$\boxed{e^{j\theta} = \cos(\theta) + j\sin(\theta)}$$

Complex Exponential Signal: 複數指數訊號

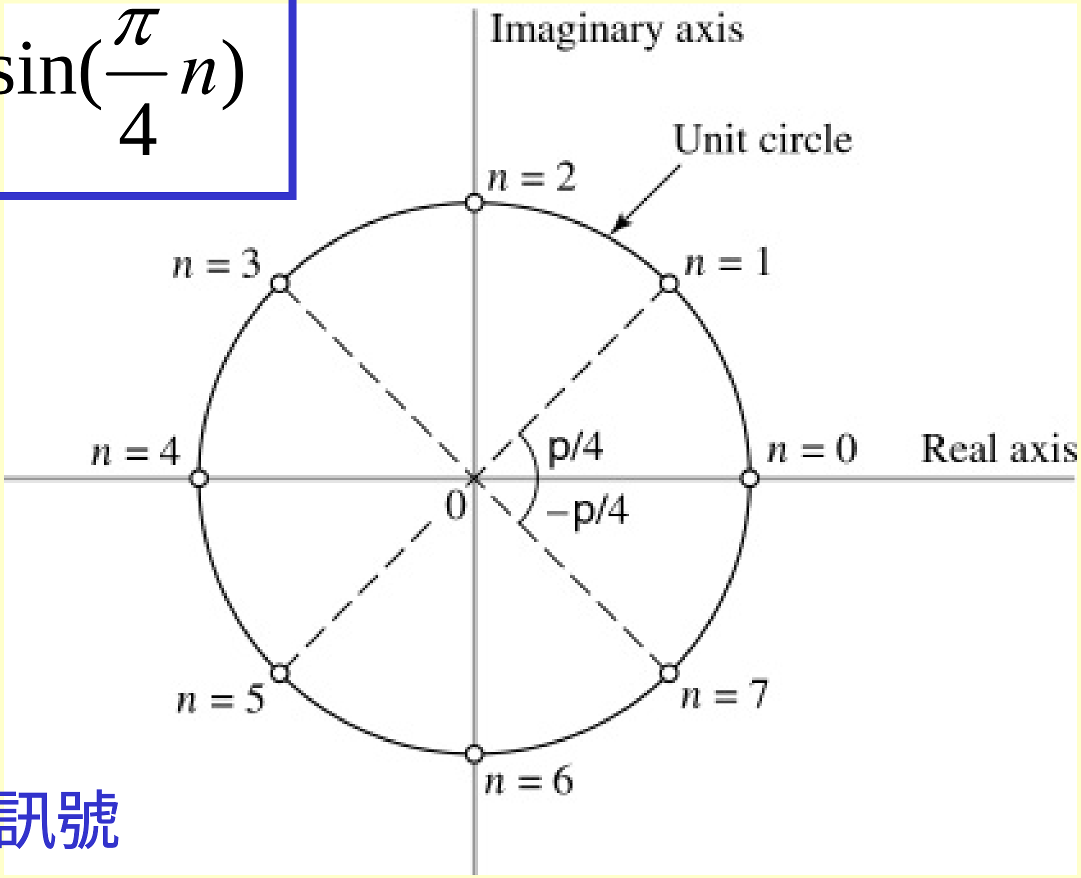
$$e^{j(\omega t+\phi)} = \cos(\omega t + \phi) + j\sin(\omega t + \phi)$$

$$\therefore \mathrm{Re}\left\{e^{j(\omega t+\phi)}\right\} = \cos(\omega t + \phi) \quad \text{實部}$$

$$\therefore \mathrm{Im}\left\{e^{j(\omega t+\phi)}\right\} = \sin(\omega t + \phi) \quad \text{虛部}$$

Sinusoids on the unit circle

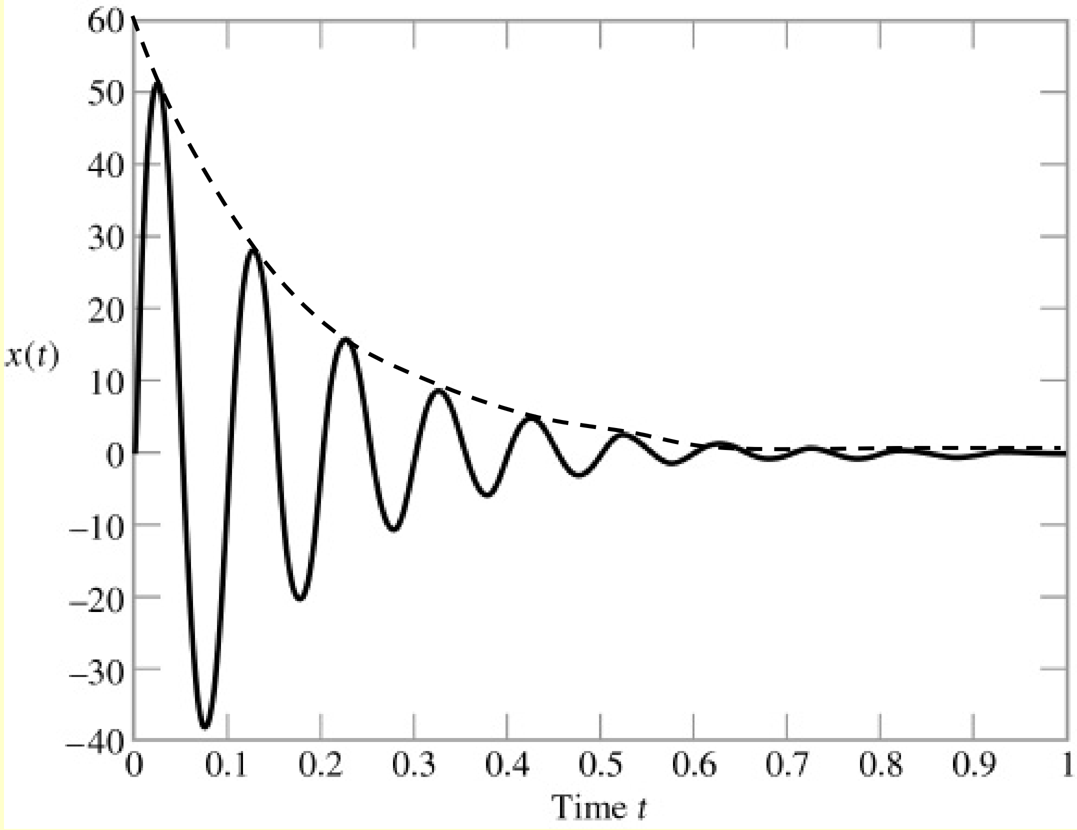
$$e^{j\frac{\pi}{4}n} = \cos\left(\frac{\pi}{4}n\right) + j\sin\left(\frac{\pi}{4}n\right)$$



複數平面單位圓與弦波訊號

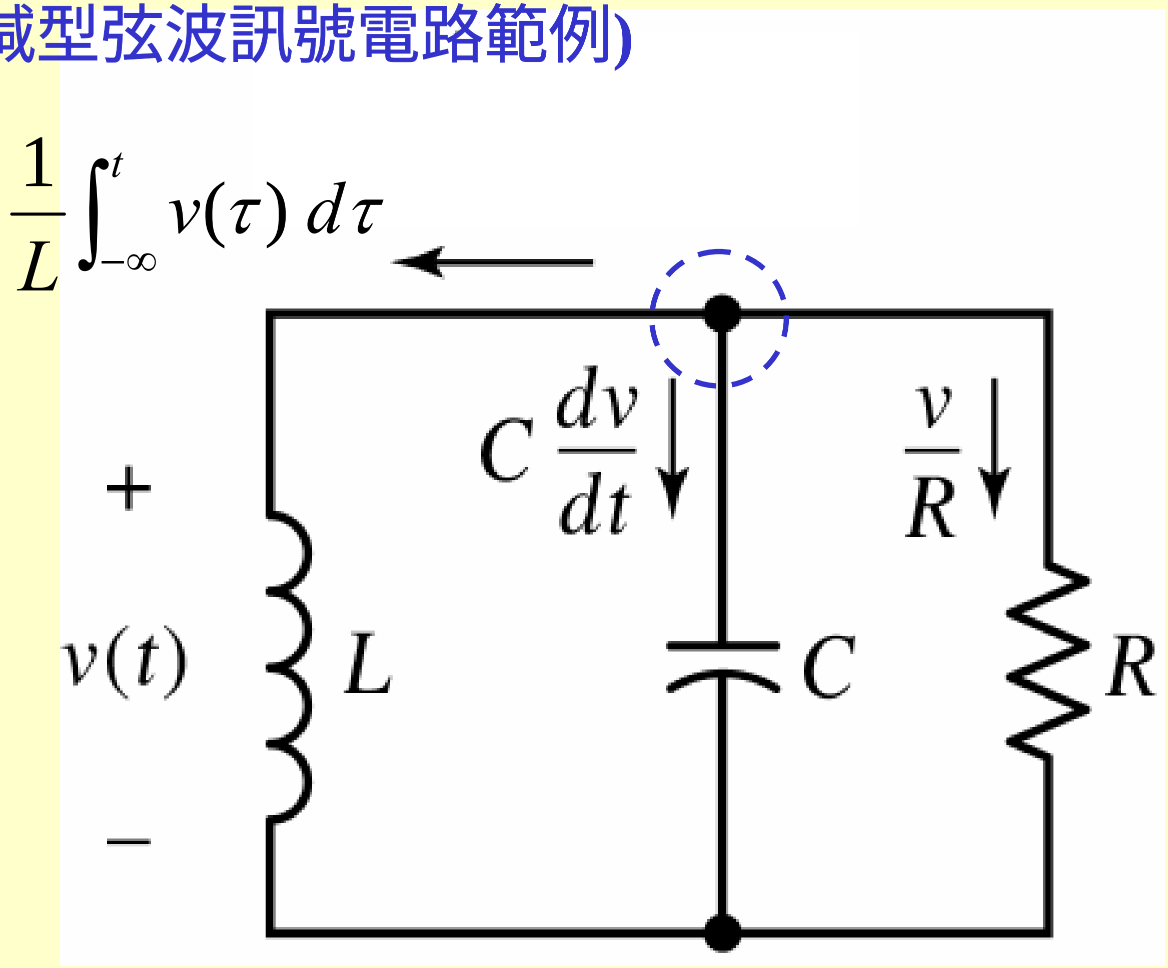
Exponentially damped sinusoidal signal

($Ae^{-\alpha t}\sin(\omega t)$, with A = 60 and α= 6.)



Parallel *LRC* circuit, with ideal *L*,*C and R*

(衰減型弦波訊號電路範例)



$$C\frac{dv(t)}{d}+\frac{v(t)}{R}+\frac{1}{L}\int_{-\infty}^{t}v(\tau)\,d\tau=0$$

$\therefore$

$$C\frac{dv^2(t)}{dt^2}+\frac{1}{R}\frac{dv(t)}{dt}+\frac{1}{L}v(t)\tau=0$$

$Solution:$

$$v(t)=V_0e^{-\frac{t}{2RC}}\cos\left(\omega_0 t\right),\quad t\geq 0,$$

$$\omega_0=\sqrt{\frac{1}{LC}-\frac{1}{4C^2R^2}}$$

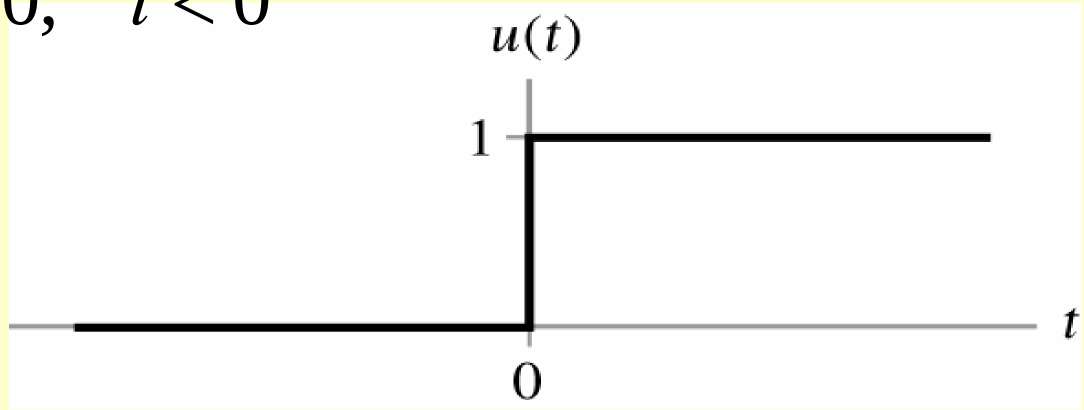
The unit-step function of unit amplitude

Continuous-time Step (function) Signal

(連續性時間步階訊號)

函數定義：

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$



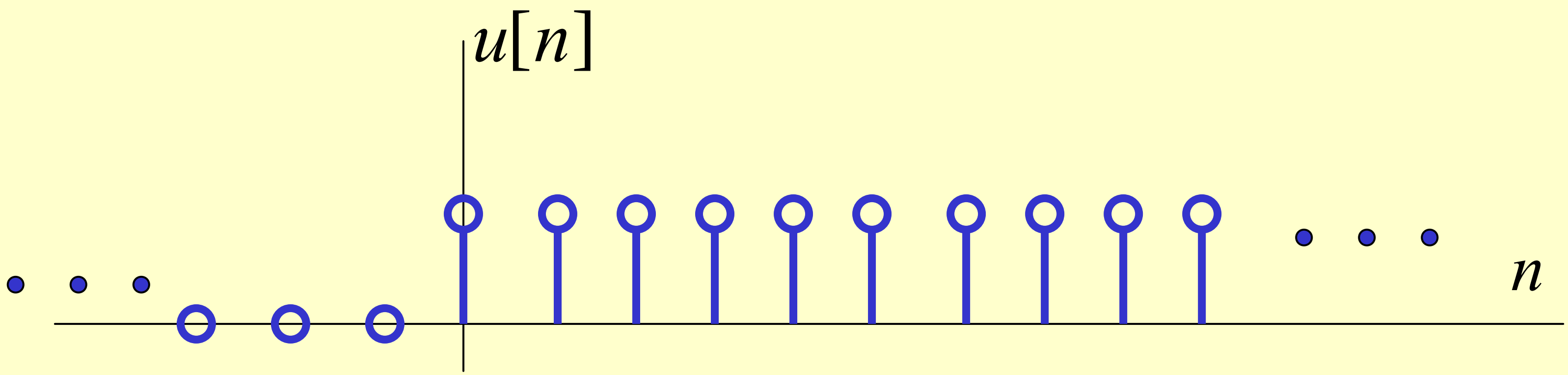
Discrete-Time Step Signal

Discrete-time Step (function) Signal

(離散性時間步階訊號)

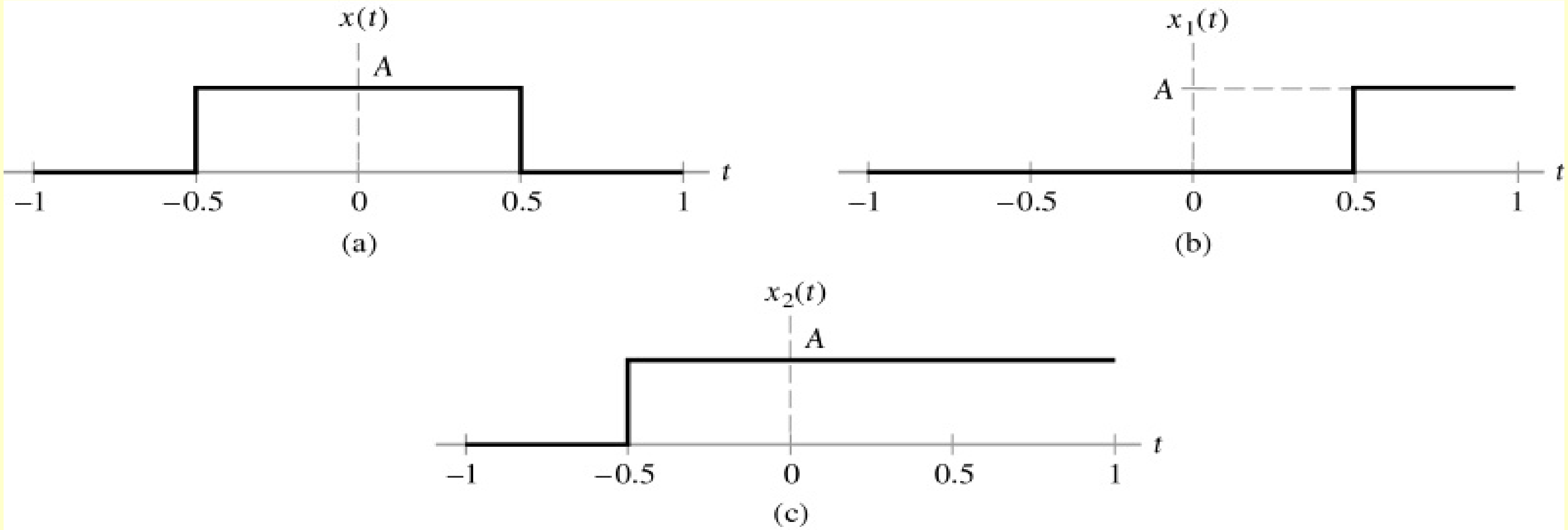
函數定義：

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



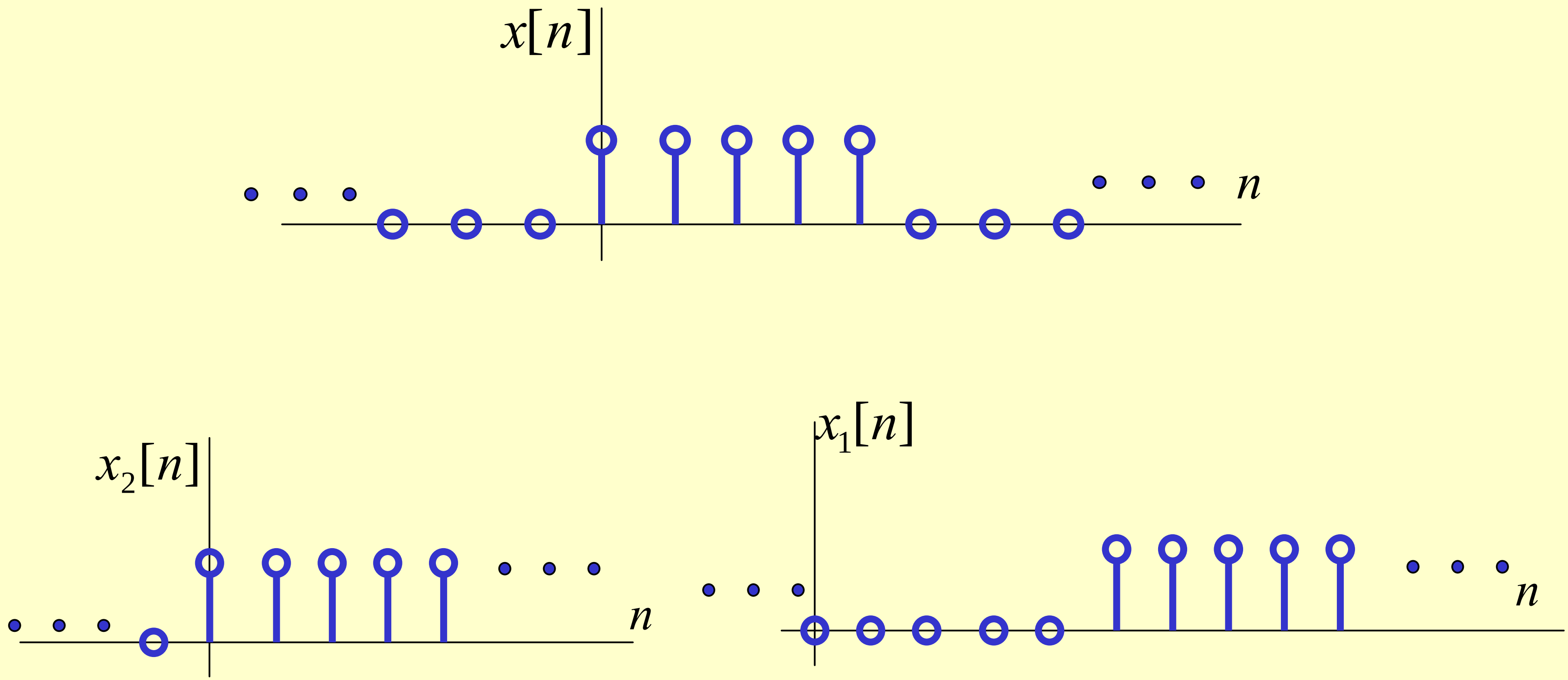
The difference of two step functions:

Note that $x(t) = x_2(t) - x_1(t)$.



The difference of two step functions:

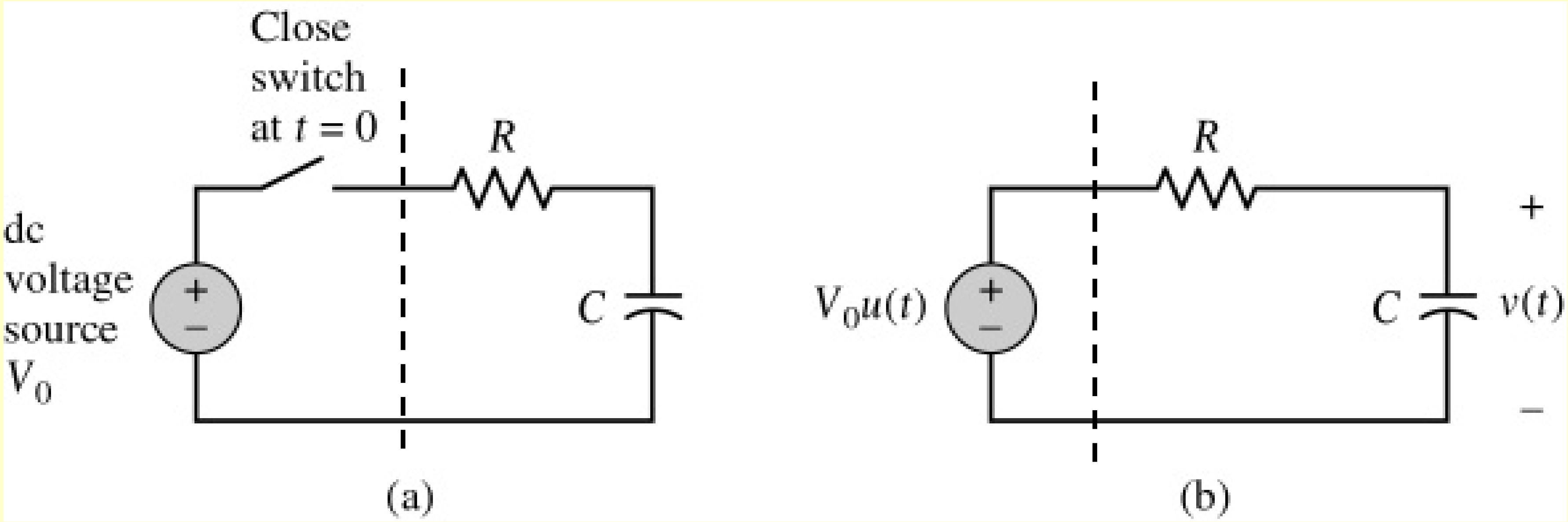
Note that $x[n] = x_2[n] - x_1[n]$.



(連續性時間步階訊號應用範例)

Example:

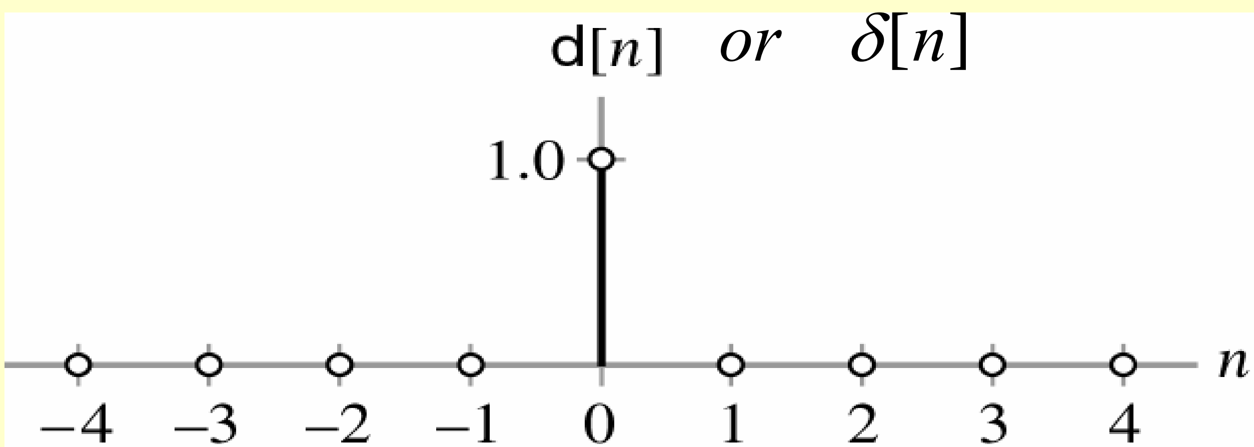
(a) Series RC circuit with a switch that is closed at time t = 0, thereby energizing the voltage source.
(b) Equivalent circuit, using a *step function* to replace the action of the switch.



Discrete-time form of Impulse

離散性時間脈衝訊號定義：

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$



Continuous-time form of Impulse

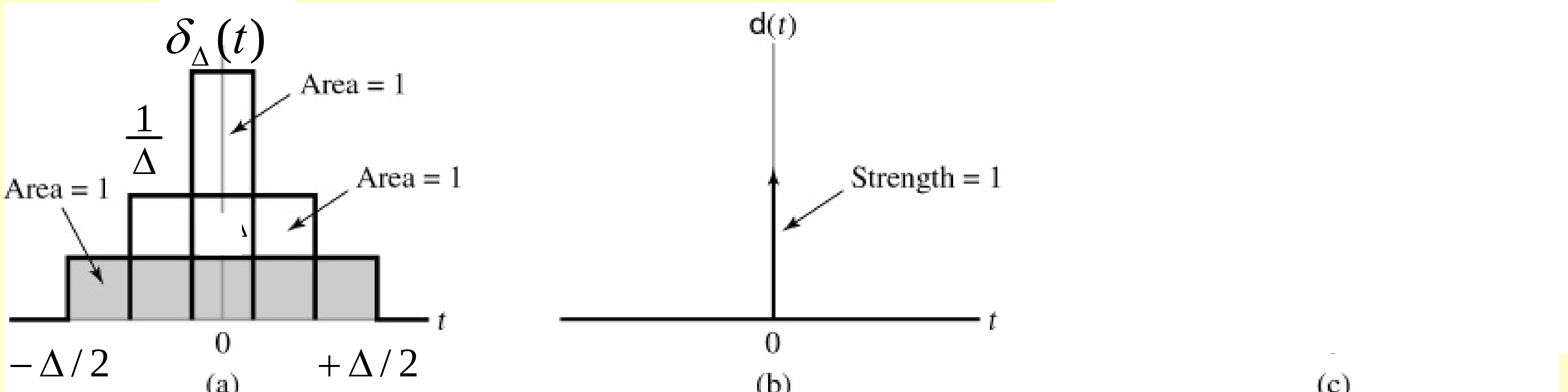
連續性時間脈衝訊號定義：

$$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta}, & -\frac{\Delta}{2} < t < +\frac{\Delta}{2} \\ 0, & others \end{cases}$$

$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$

$$\delta(t): \begin{cases} \int_{-\infty}^{+\infty} \delta(t)\, dt = 1, & t = 0 \\ 0, & t \neq 0 \end{cases}$$

$\delta_\Delta(t)$ 是集中於 t=0 的連續時間極窄脈波(面積= 1)



δ(t) vs. u(t)

Unit-step function 定義：

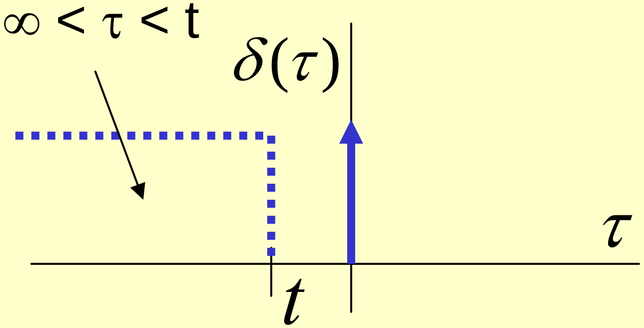
$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Continuous-time Impulse function 積分：

$$\int_{-\infty}^{t} \delta(\tau)\, d\tau = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

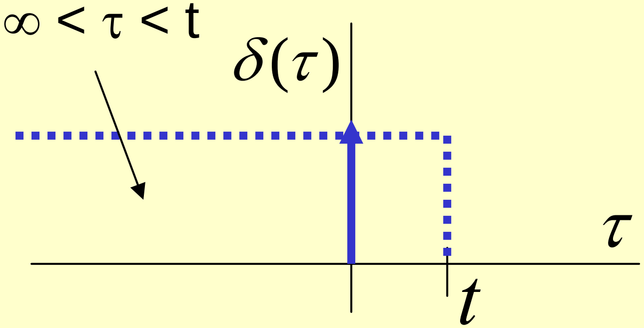
積分區間 t < 0

∞ < τ < t



積分區間 t ≥ 0

∞ < τ < t



$$\because u(t) = \begin{cases} 1, & t \ge 0 \\ 0, & t < 0 \end{cases} \quad \& \quad \int_{-\infty}^{t} \delta(\tau)\, d\tau = \begin{cases} 1, & t \ge 0 \\ 0, & t < 0 \end{cases}$$

$$\therefore$$

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau$$

$$\therefore$$

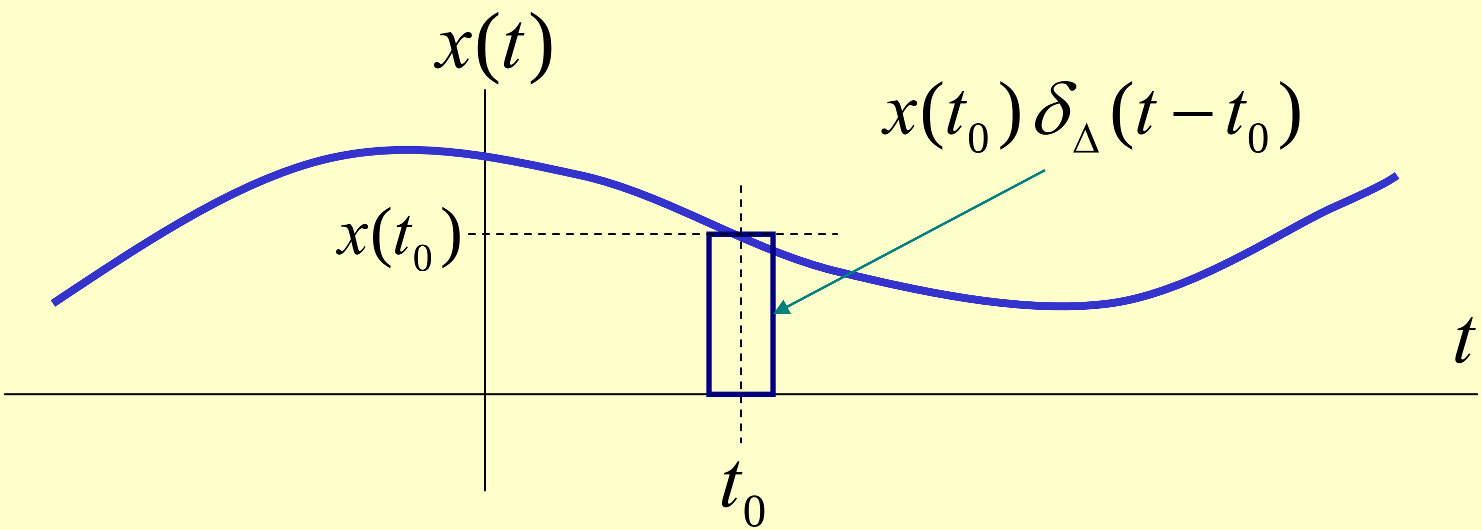
$$\frac{d}{dt} u(t) = \frac{d}{dt} \int_{-\infty}^{t} \delta(\tau)\, d\tau = \delta(t)$$

單位脈衝 δ(t) 時間平移特性

當 $t = t_0$ 時 x(t) 是連續的，也是單位脈衝所在的時間

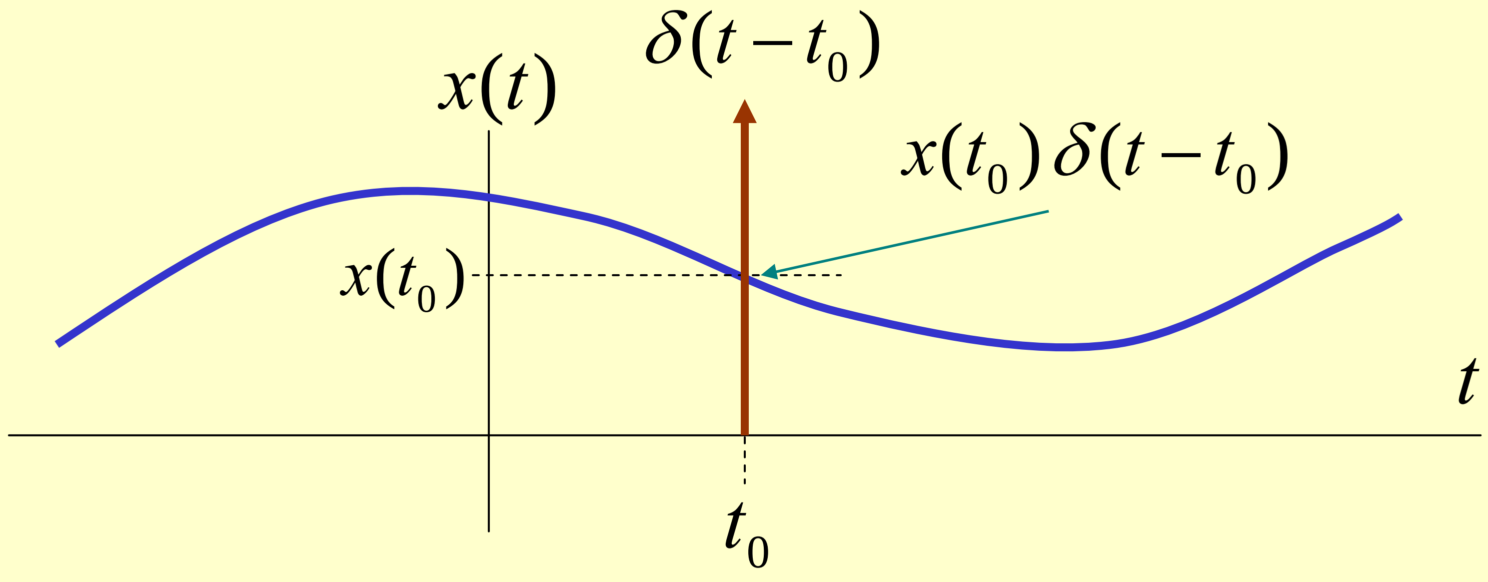
$$\int_{-\infty}^{\infty} x(t)\,\delta_{\Delta}(t-t_0)\,dt = x(t_0)$$

$\delta_{\Delta}(t-t_0)$ 是集中於 $t = t_0$ 的連續時間
極窄脈波(面積= 1)



當 $t = t_0$ 時 $x(t)$ 是連續的，也是單位脈衝所在的時間

$$\int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt = x(t_0)$$



單位脈衝 δ(t) 時間比例條調整特性

$$\boxed{\delta(at)=\frac{1}{a}\delta(t), \qquad a>0}$$

$$\because \delta_\Delta(t)=\begin{cases}\frac{1}{\Delta}, & -\frac{\Delta}{2}<t<+\frac{\Delta}{2}\\ 0, & \end{cases} \quad ; \quad \therefore \delta_\Delta(at)=\begin{cases}\frac{1}{\Delta}, & -\frac{\Delta}{2a}<t<+\frac{\Delta}{2a}\\ 0, & \end{cases}$$

壓縮版本

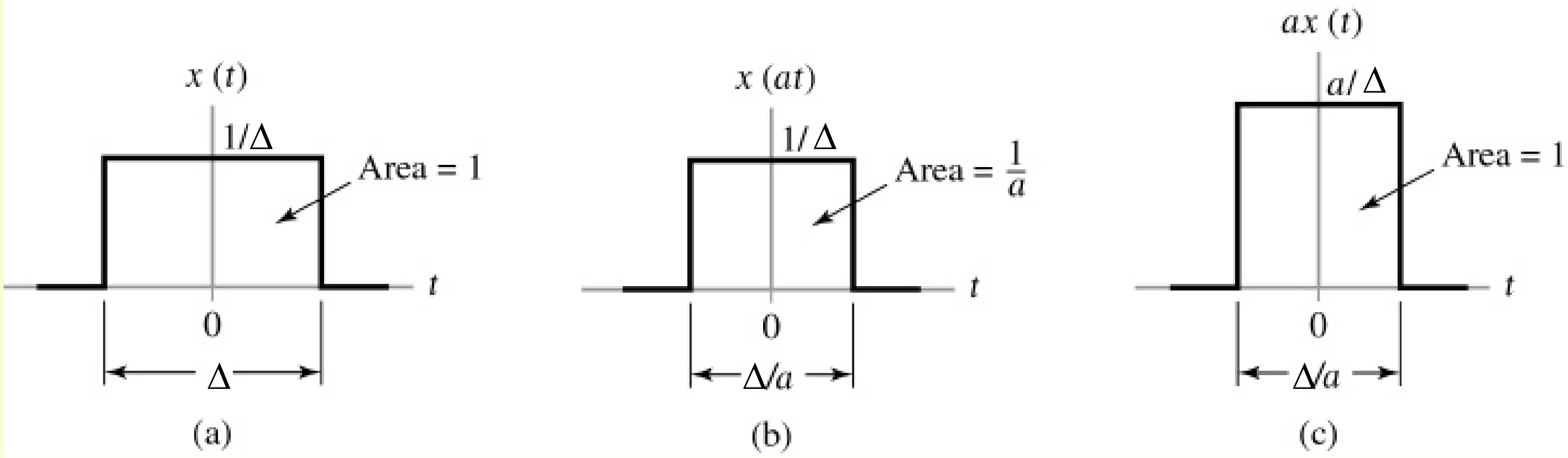
$$\textit{area of } \delta_\Delta(t)=1; \qquad \textit{area of } \delta_\Delta(at)=\frac{1}{a}$$

$$\therefore \delta(t)=\lim_{\Delta\to 0}\delta_\Delta(t)=\lim_{\Delta\to 0}a\cdot\delta_\Delta(at)=a\cdot\lim_{\Delta\to 0}\delta_\Delta(at)=a\cdot\delta(at)$$

$$\therefore \boxed{\delta(at)=\frac{1}{a}\delta(t)}$$

Steps involved in proving the time-scaling property of the unit impulse

(a) Rectangular pulse $x_\Delta(t)$ of amplitude $1/\Delta$ and duration Δ, symmetric about the origin.
(b) Pulse $x_\Delta(t)$ compressed by factor a.
(c) Amplitude scaling of the compressed pulse, restoring it to unit area.



單位脈衝 δ[n] 與 Convolution Sum

任一離散時間系統輸入訊號 x[n] 可以表成離散時間單位脈衝訊號δ[n] 的加權疊加:

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\,\delta[n-k]$$

離散時間系統輸出訊號 y[n] 可以表成離散時間脈衝響應 h[n] 的加權疊加:

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\,h[n-k]$$

即為 Convolution Sum

單位脈衝 δ(t) 與 Convolution Integral

任一連續時間系統輸入訊號 x(t) 可以表成連續時間單位脈衝訊號δ(t) 的加權積分:

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

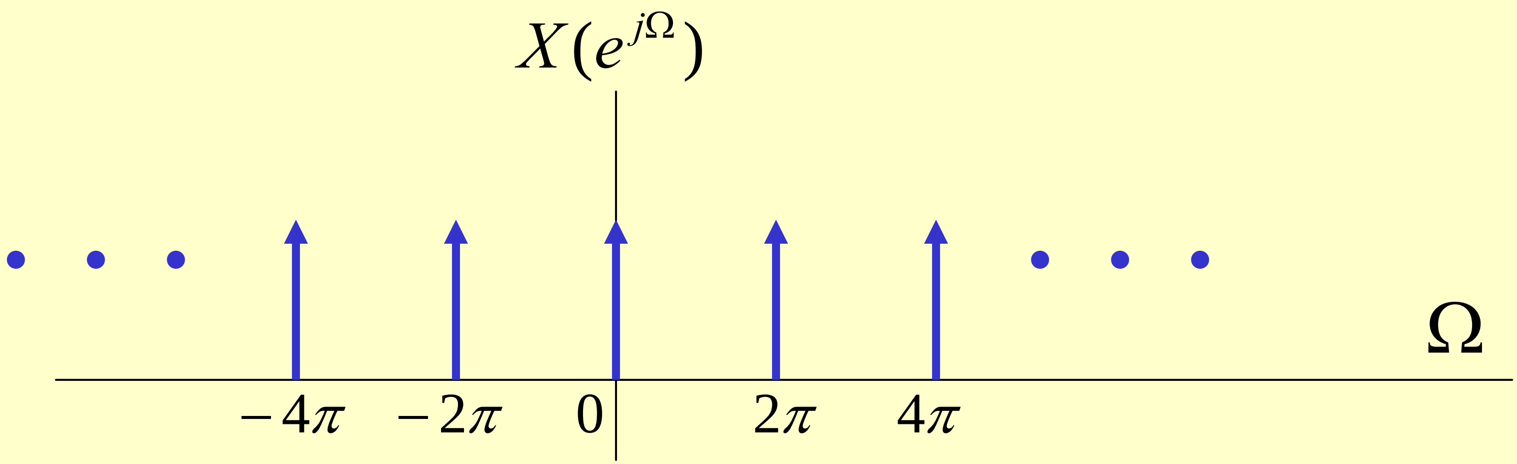
連續時間系統輸出訊號 y(t) 可以表成連續時間脈衝響應 h(t) 的加權積分:

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\,h(t-\tau)\,d\tau$$

即為 Convolution Integral

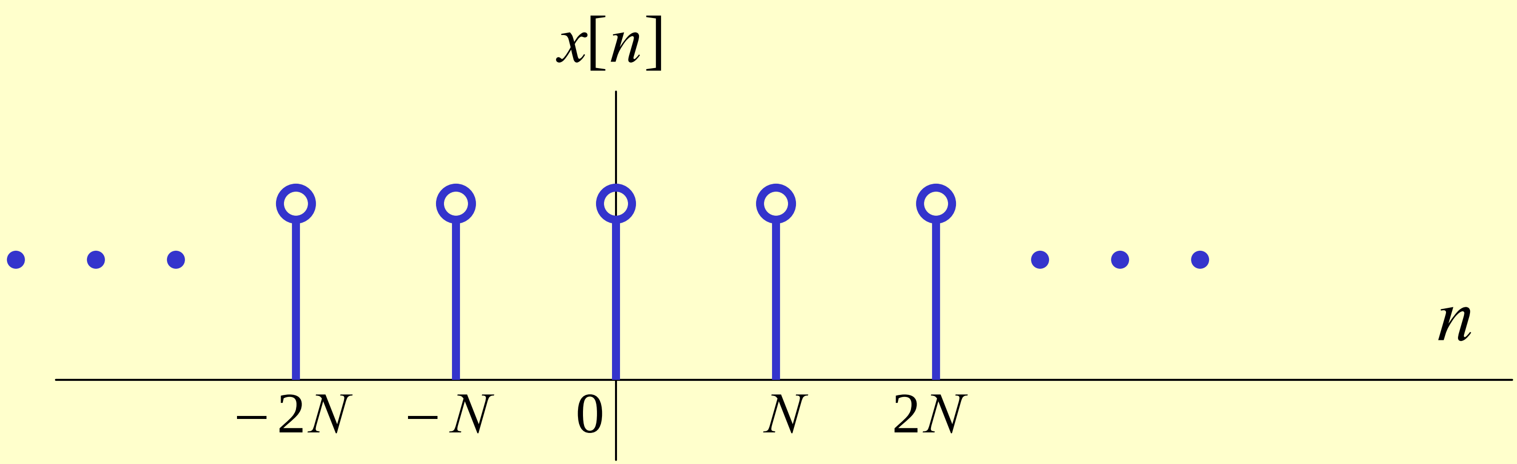
連續單位脈衝 δ(Ω) 序列

$$X(e^{j\Omega}) = \sum_{k=-\infty}^{+\infty} \delta\left(\Omega - 2\pi k\right)$$



離散時間單位脈衝 δ[n] 序列

$$x[n] = \sum_{k=-\infty}^{+\infty} \delta[n - kN]$$

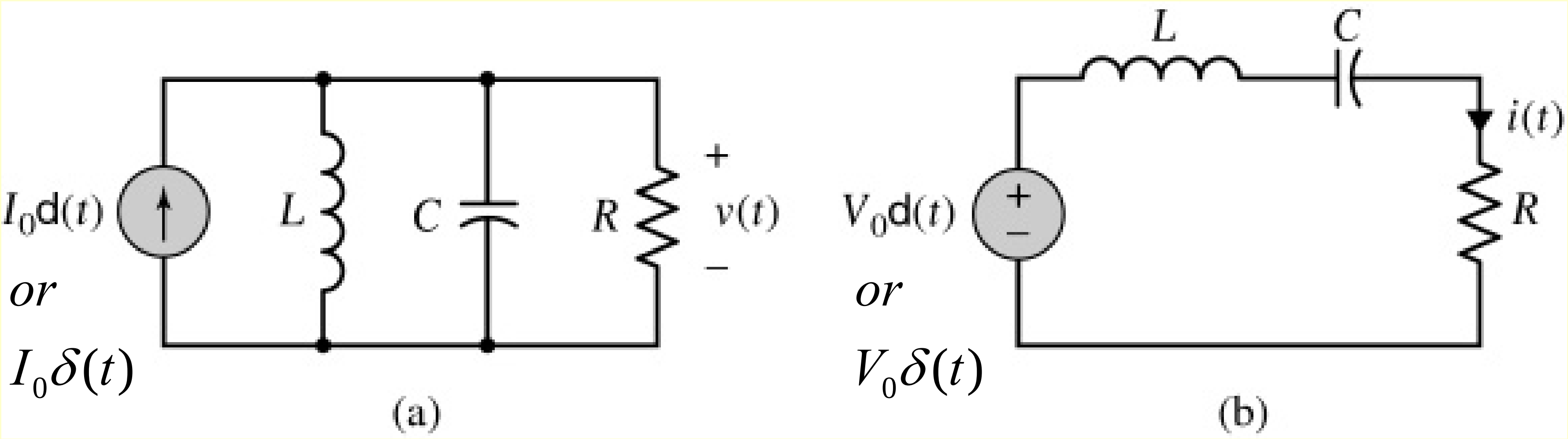


(連續性時間脈衝訊號應用範例)

Example:

(a) Parallel *LRC* circuit driven by an impulsive current signal.

(b) Series *LRC* circuit driven by an impulsive voltage signal.



離散時間單位脈衝 δ[n] 乘積與褶積

乘法性質： 回憶 only for n = k, δ[n-k] = 1.

$$x[n]\cdot\delta[n-k]=x[k]\cdot\delta[n-k]=x[k]$$

離散時間單位脈衝 δ[n] 乘積與摺積

摺積性質：

$$x[n]*\delta[n-n_0]=\sum_{k=-\infty}^{+\infty}\delta[k-n_0]\,x[n-k]$$

$$\because k=n_0\ only,$$

$$\therefore$$

$$x[n]*\delta[n-n_0]=x[n-n_0]$$

連續時間單位脈衝 δ(t) 乘積與摺積

乘法性質：

$$x(t)\cdot\delta(t-\tau)=x(\tau)\cdot\delta(t-\tau)$$

連續時間單位脈衝 δ(t) 乘積與摺積

摺積性質：

$$x(t) * \delta(t - t_0) = \int_{-\infty}^{+\infty} \delta(\tau - t_0)\, x(t - \tau)\, d\tau$$

$$\because \tau = t_0 \ only,$$

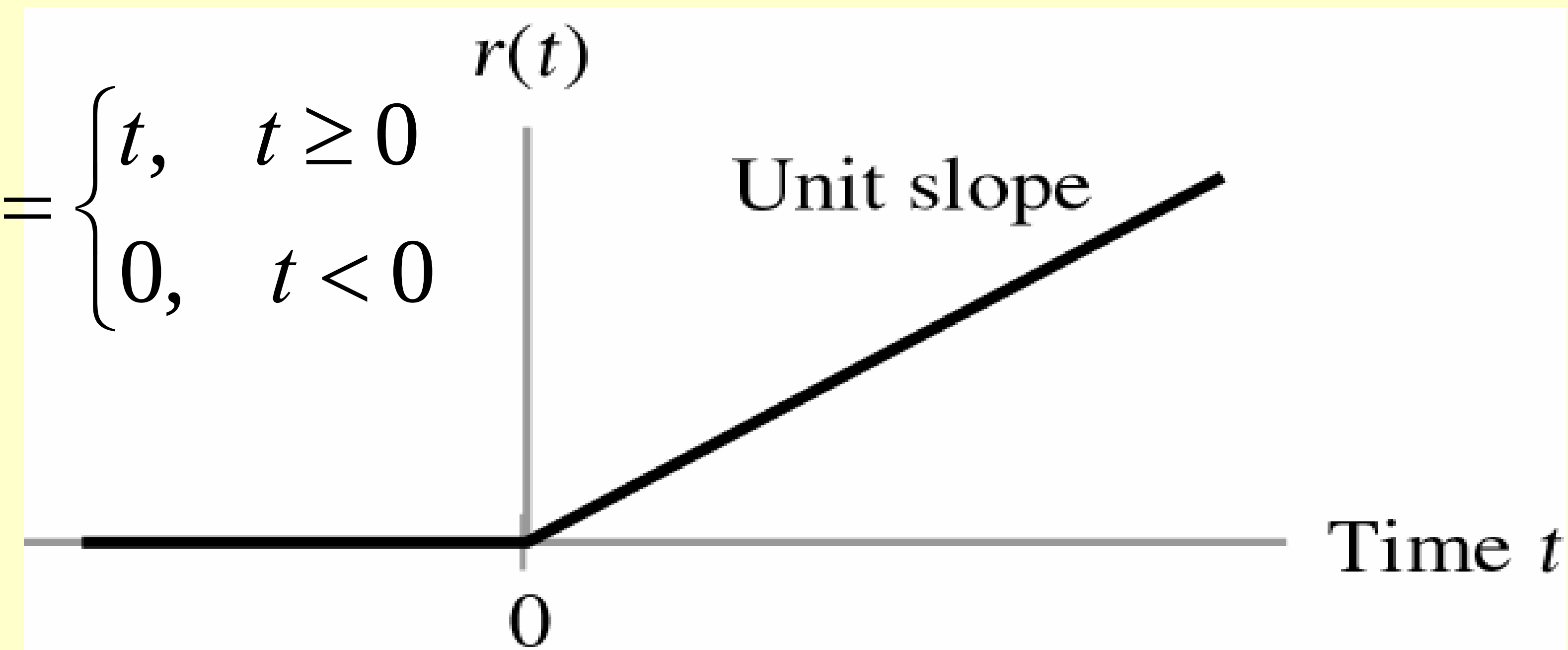
$$\therefore$$

$$x(t) * \delta(t - t_0) = x(t - t_0) \overbrace{\int_{-\infty}^{+\infty} \delta(\tau - t_0)\, d\tau}^{1} = x(t - t_0)$$

Ramp function of unit slope

(連續性時間斜坡訊號)

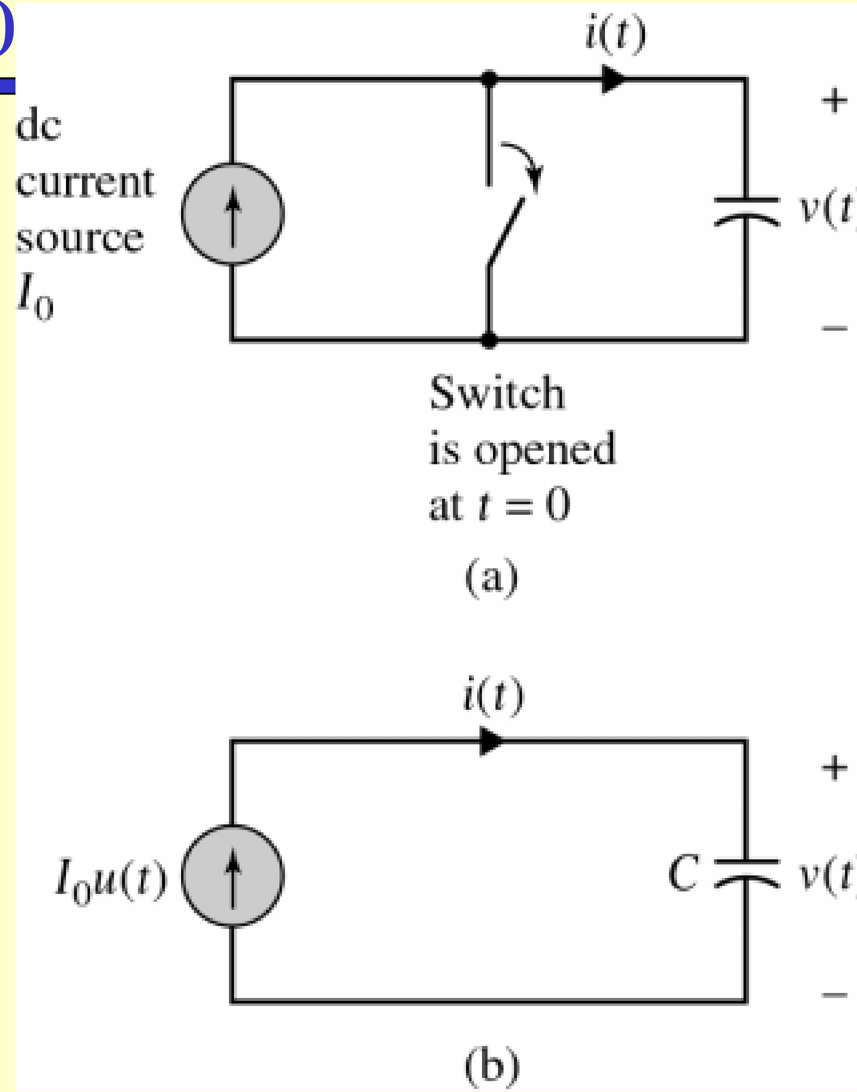
$$r(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$



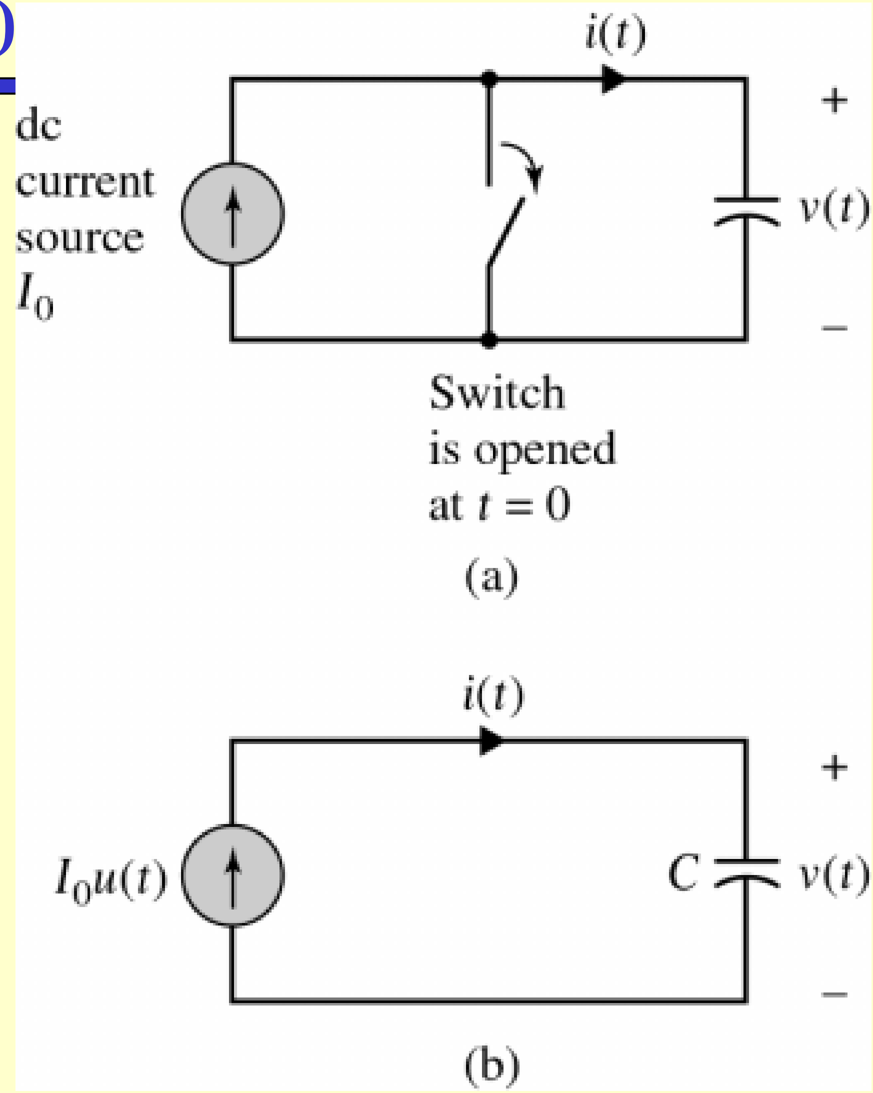
(連續性時間斜坡訊號應用範例)

(a) Parallel circuit consisting of a current source, switch, and capacitor, the capacitor is initially assumed to be *uncharged*, and the switch is opened at time $t = 0$.

(b) Equivalent circuit replacing the action



(連續性時間斜坡訊號應用範例)



(b) Equivalent circuit replacing the action of opening the switch with the step function $u(t)$.

連續性時間斜坡訊號推導

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)\,d\tau = \frac{1}{C}\int_{-\infty}^{t}[I_0\,u(\tau)]\,d\tau$$

$$= \begin{cases} \dfrac{I_0}{C}t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$= \frac{I_0}{C}tu(t) \quad = \frac{I_0}{C}r(t)$$