Lecture 2-1

Linear Time-Invariant System
(LTI System)
線性非時變系統

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Review

Linear Property:

- Scaling
  - $x(t) \rightarrow y(t)$
  - $c\,x(t) \rightarrow c\,y(t)$
- Superposition
  - $x_1(t) \rightarrow y_1(t)$
  - $x_2(t) \rightarrow y_2(t)$
  - $x_1(t)+x_2(t) \rightarrow y_1(t)+y_2(t)$

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Review

Time-Invariant Property:

- $x(t) \rightarrow y(t)$
- $x(t - t_0) \rightarrow y(t - t_0)$

Overview

- 摺積和 (Convolution Sum)
- 摺積和計算程序 (~ Evaluation Procedure)
- 摺積積分 (Convolution Integral)
- 摺積積分計算程序 (~ Evaluation Procedure)
- LTI 串、並聯連結
- LTI 系統特性
- 步階響應(Step Response)
- 微分、差分方程式表示LTI系統
- MATLAB 實作

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- 脈衝響應 (Impulse Response):
  - –系統對於一個*脈衝函數 (訊號)* δ 輸入所做出的輸出反應。
  - –實物上，為系統對於一個持續時間非常短的高振幅輸入所做出的反應。

- 任何系統*輸入*可表成時間平移*脈衝訊號*的加權疊加，系統*輸出*必為時間平移*脈衝響應*的加權疊加。

- 離散時間系統的加權疊加以來“*摺積和*”(convolution sum)表示。

- 連續時間系統的加權疊加以來“*摺積積分*”(convolution Integral)表示。

脈衝訊號時間平移的加權疊加

$$\because \quad \delta[n]=\begin{cases}1, & n=0\\ 0, & otherwise\end{cases} \quad , \quad \therefore x[n]\,\delta[n]=x[0]\,\delta[n]$$

注意：x[n] 為因應不同自變數 n 而獲得的應變數值

注意：x[0] 為自變數 n=0 而獲得的應變數值 = constant

脈衝訊號時間平移的加權疊加

$$\because \quad \delta[n]=\begin{cases}1, & n=0\\ 0, & otherwise\end{cases} \quad , \quad \therefore x[n]\,\delta[n]=x[0]\,\delta[n]$$

$$\therefore x[n]\,\delta[n-k]=x[k]\,\delta[n-k]$$

注意：x[k] 為自變數 n=k

而獲得的應變數值 = constant

時間平移 k 的脈衝訊號

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脈衝訊號時間平移的加權疊加

$$\because \quad \delta[n]=\begin{cases}1, & n=0\\ 0, & otherwise\end{cases} \quad , \quad \therefore x[n]\,\delta[n]=x[0]\,\delta[n]$$

$$\therefore x[n]\,\delta[n-k]=x[k]\,\delta[n-k]$$

$$x[n]=\ldots+x[-1]\delta[n+1]+x[0]\delta[n]+x[1]\delta[n-1]+\ldots$$

$$=\sum_{k=-\infty}^{+\infty}x[k]\delta[n-k]$$

x[n] 可寫成時間平移脈衝的加權疊加

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If the system input is $x[n]$ and the H{} represents the system, the system output $y[n]$ will be,

$$y[n] = H\{x[n]\}$$

$$= H\{\sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]\},$$ 考慮線性特性:疊加性

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If the system input is $x[n]$ and the H{} represents the system, the system output $y[n]$ will be,

$$y[n] = H\{x[n]\}$$

$$= H\{\sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]\}, \quad 考慮線性特性:疊加性$$

$$= \sum_{k=-\infty}^{+\infty} \mathrm{H}\{x[k]\delta[n-k]\}, \quad 考慮線性特性:\mathrm{Scaling}$$

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If the system input is $x[n]$ and the H{} represents the system, the system output $y[n]$ will be,

$$y[n] = H\{x[n]\}$$

$$= H\{\sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]\}, \quad \text{考慮線性特性:疊加性}$$

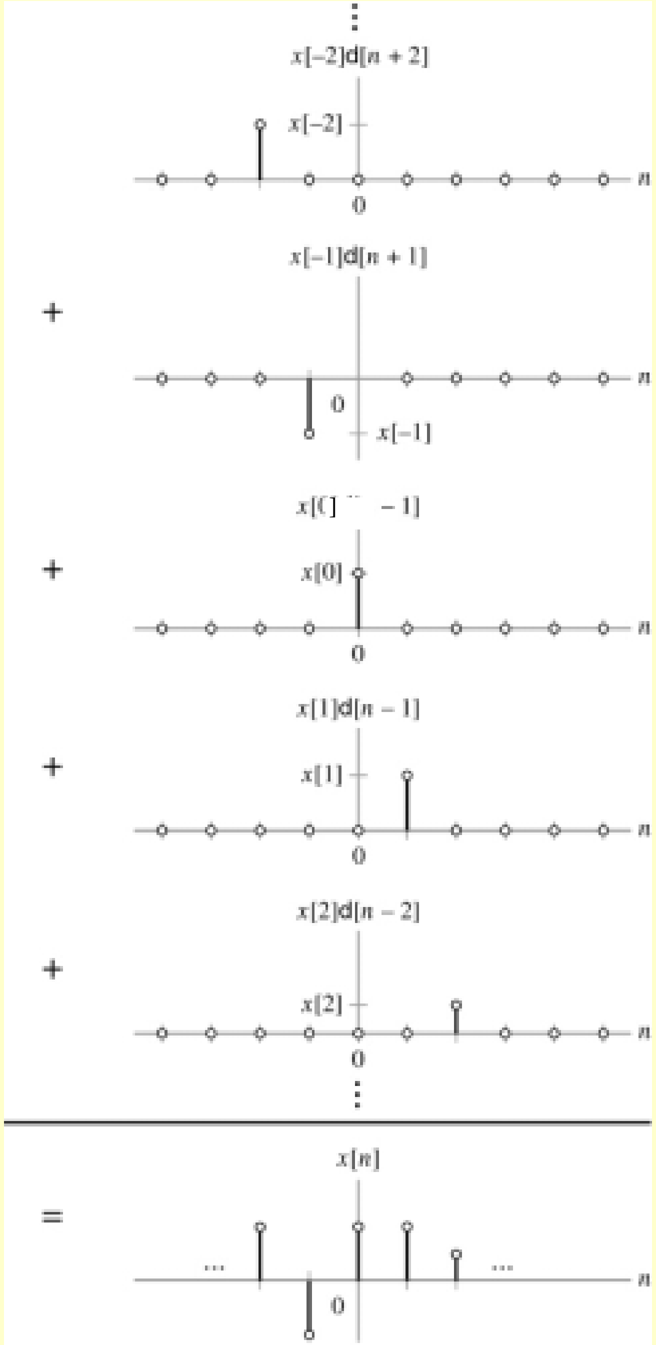
$$= \sum_{k=-\infty}^{+\infty} \mathrm{H}\{x[k]\delta[n-k]\}, \quad \text{考慮線性特性:Scaling}$$

$$= \sum_{k=-\infty}^{+\infty} x[k]H\{\delta[n-k]\}, \quad \because \textit{let } h[n] \textit{ be the impulse response of } \delta[n].$$

$$= \sum_{k=-\infty}^{+\infty} x[k]\, h[n-k].$$

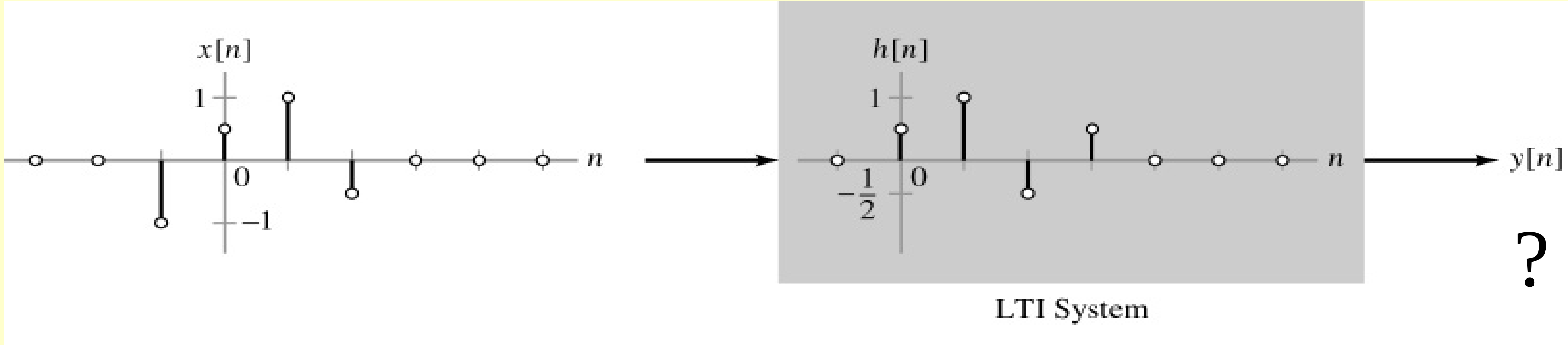
輸入訊號時間平移的加權疊加圖解說明

$$x[n]=\sum_{k=-\infty}^{+\infty}x[k]\delta[n-k]$$

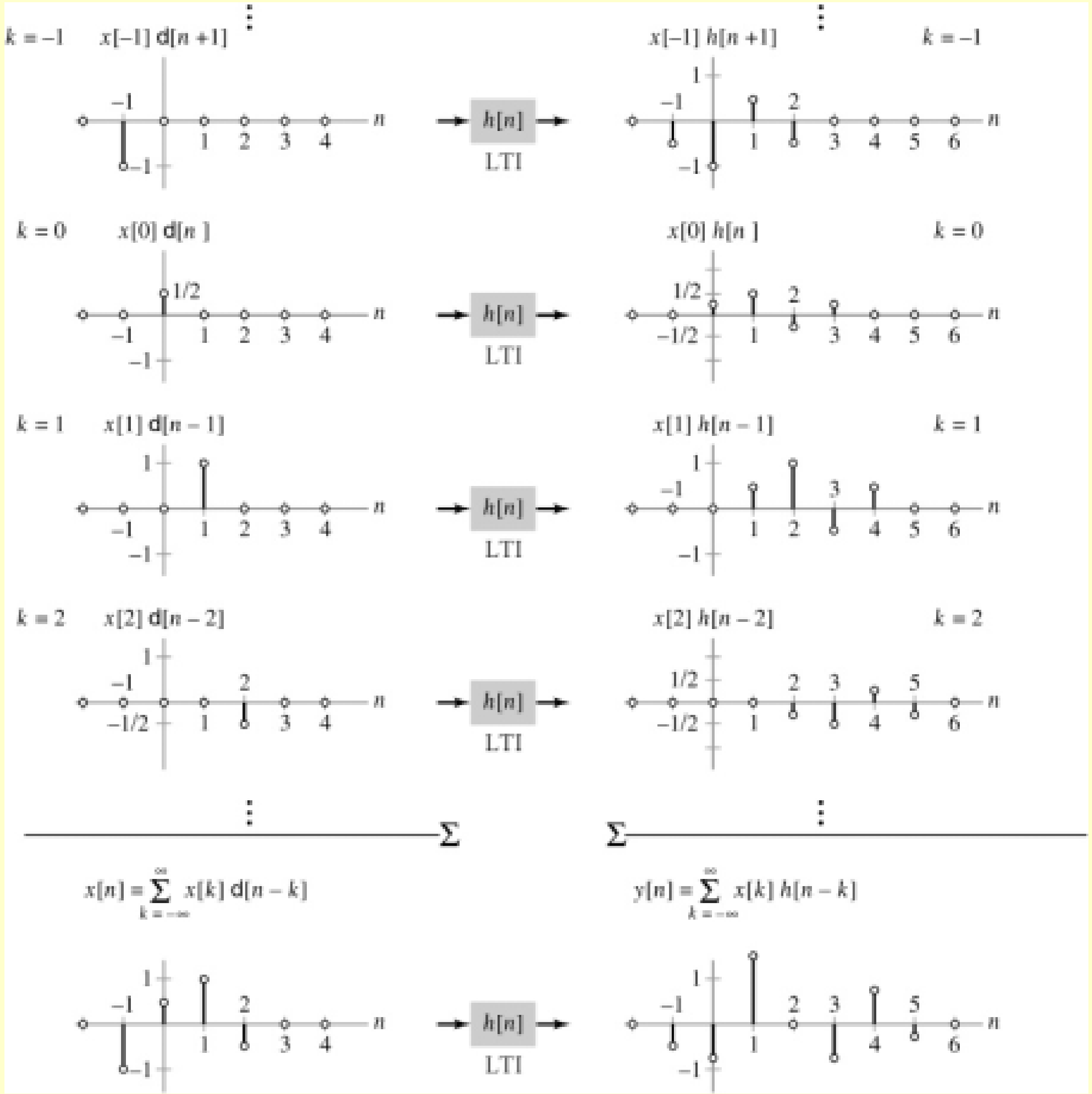


Notation of the Convolution Sum

$$y[n]=\sum_{k=-\infty}^{+\infty}x[k]h[n-k]=x[n]*h[n]$$



(a)


k = −1
x[−1] d[n +1]
x[−1] h[n +1]
k = −1
k = 0
x[0] d[n]
x[0] h[n]
k = 0
k = 1
x[1] d[n − 1]
x[1] h[n − 1]
k = 1
k = 2
x[2] d[n − 2]
x[2] h[n − 2]
k = 2
h[n]
LTI
Σ
Σ
x[n] = Σ_{k=−∞}^{∞} x[k] d[n − k]
y[n] = Σ_{k=−∞}^{∞} x[k] h[n − k]
1/2
−1/2
n

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$$x[n] = x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2]$$

$$h[n] = h[0]\delta[n] + h[1]\delta[n-1] + h[2]\delta[n-2] + h[3]\delta[n-3]$$

$$x[-1]\delta[n+1] \quad \rightarrow \quad x[-1]\delta[n+1] * h[n] = x[-1]h[n+1]$$

$$x[0]\delta[n] \quad \rightarrow \quad x[0]\delta[n] * h[n] = x[0]h[n]$$

$$x[1]\delta[n-1] \quad \rightarrow \quad x[1]\delta[n-1] * h[n] = x[1]h[n-1]$$

$$x[2]\delta[n-2] \quad \rightarrow \quad x[2]\delta[n-2] * h[n] = x[2]h[n-2]$$

$$y[n] = x[-1]h[n+1] + x[0]h[n] + x[1]h[n-1] + x[2]h[n-2]$$

$$= \sum_{k=-1}^{2} x[k]\, h[n-k]$$

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Example 2.2　摺積和 計算練習

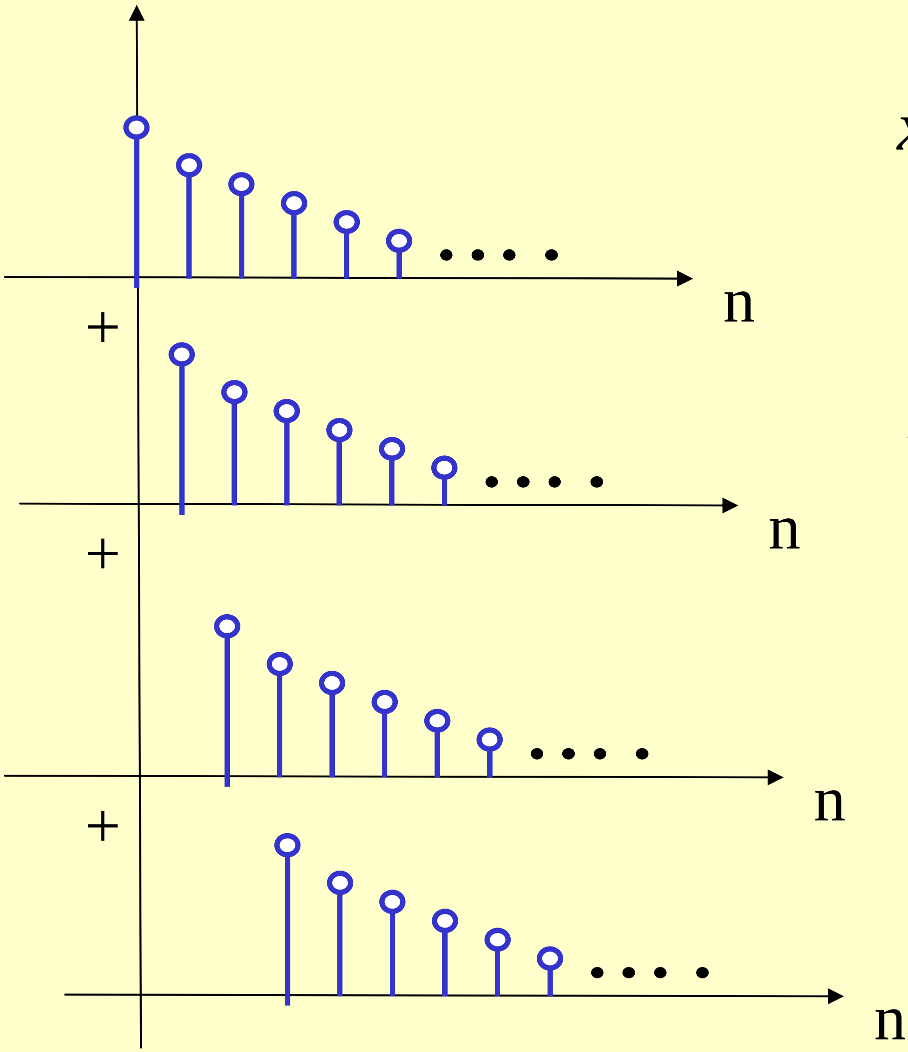
系統脈衝響應，輸入如下:

$$h[n] = \left(\frac{3}{4}\right)^n u[n], \quad x[n] = u[n],$$

$$y[n] = ? \quad y[-5] = ?, \ y[5] = ?, \ y[10] = ?$$

系統輸出為何？ 當 n = -5, +5, +10 時

對應輸入，系統脈衝響應: y[n] = ?

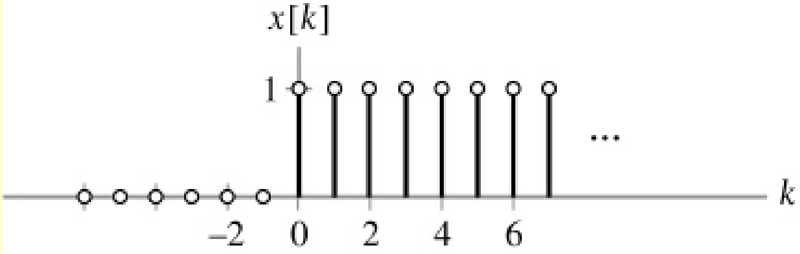


$$x[0]\delta[n]*h[n]=x[0]h[n]$$

$$x[1]\delta[n-1]*h[n]=x[1]h[n-1]$$

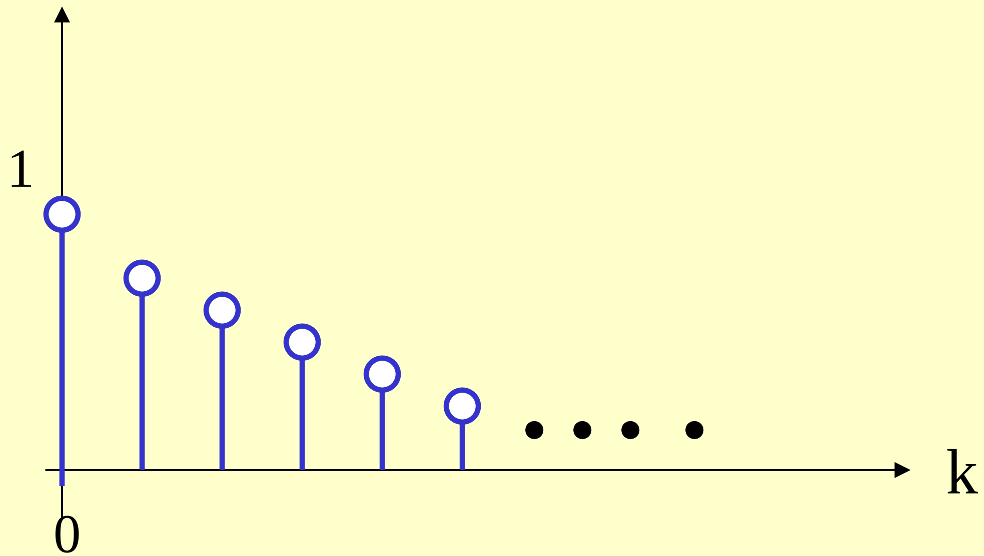
$$x[2]\delta[n-2]*h[n]=x[2]h[n-2]$$

⋮



$$x[k] = u[k]$$

$$h[k] = \left(\frac{3}{4}\right)^k u[k],$$



先找出 $x[k]$ 和 $h[k]$ = ?

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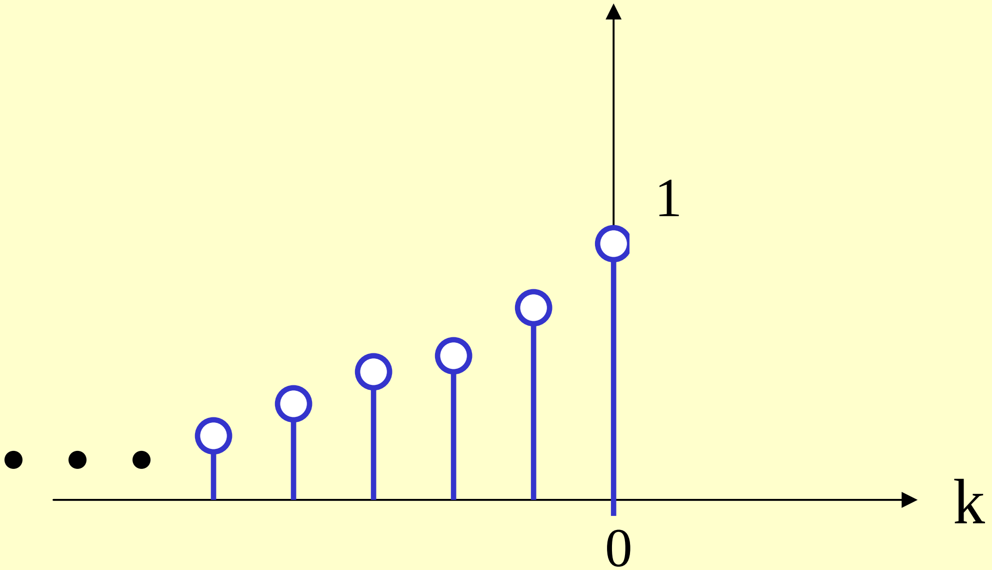
$$h[-k] = \left(\frac{3}{4}\right)^{-k} u[-k]$$



再找出反射運算後 $h[-k]$ = ?

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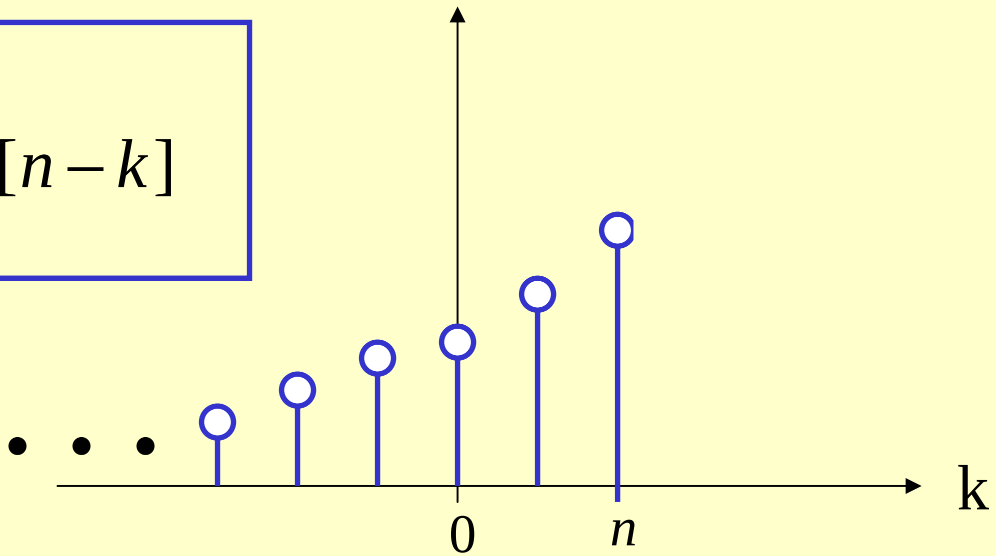
$$h[n-k]=\left(\frac{3}{4}\right)^{n-k}u[n-k]$$



再找出反射運算以及時間平移後 $h[-k+n]$ = ?

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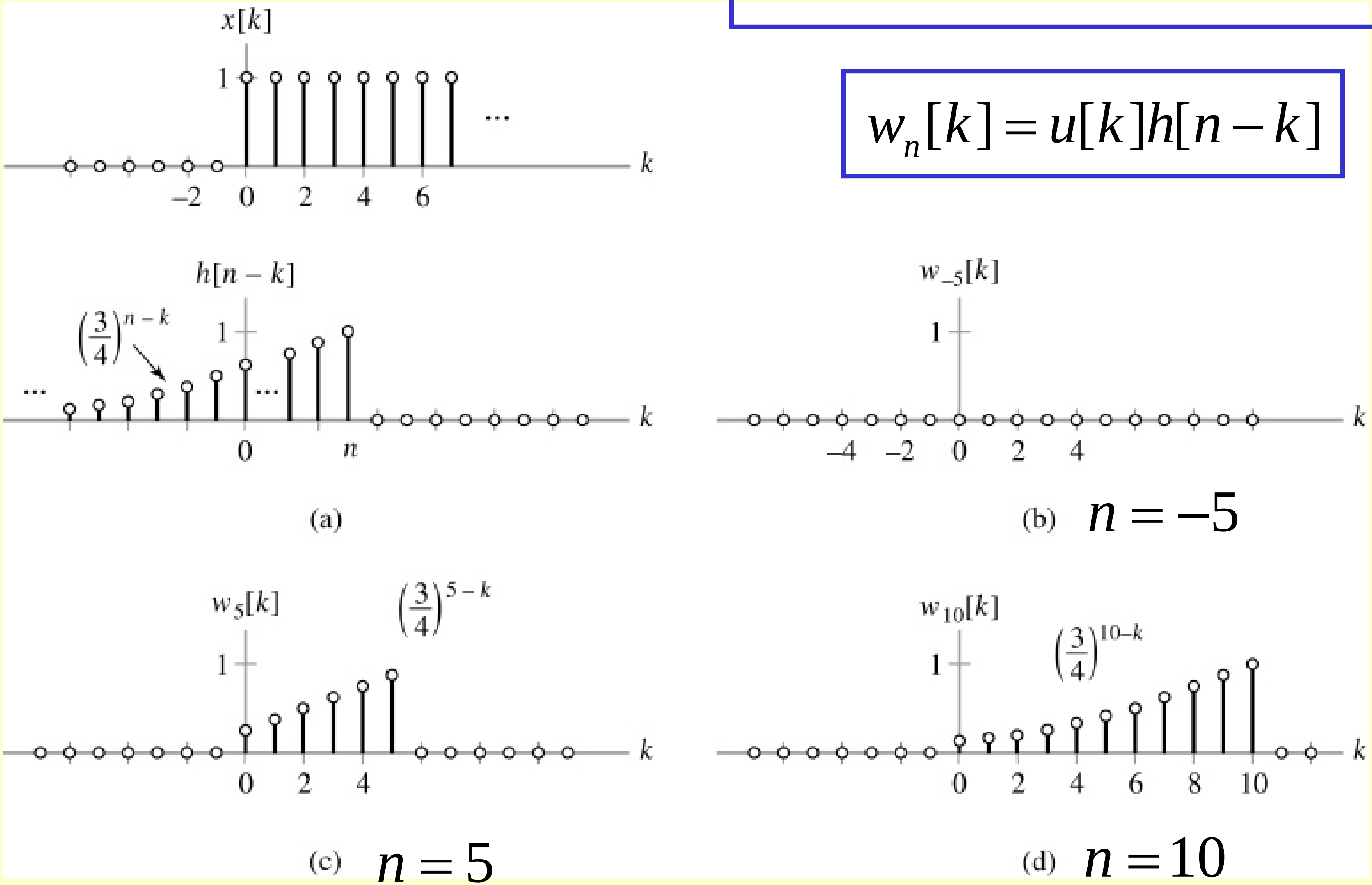
$$x[k] = u[k]$$

$$h[n-k] = \left(\frac{3}{4}\right)^{n-k} u[n-k]$$

$$w_n[k] = u[k]h[n-k]$$



(b) $n = -5$

(c) $n = 5$

(d) $n = 10$

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$$y[n]=\sum_{k=-\infty}^{+\infty}x[k]h[n-k]=\sum_{k=-\infty}^{+\infty}u[k]\left(\frac{3}{4}\right)^{n-k}u[n-k]$$

$$=\begin{cases}\sum_{k=0}^{n}\left(\frac{3}{4}\right)^{n-k}, & n\geq 0\\ 0, & otherwise\end{cases}$$

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$$y[n]=\sum_{k=-\infty}^{+\infty}x[k]h[n-k]=\sum_{k=-\infty}^{+\infty}u[k]\left(\frac{3}{4}\right)^{n-k}u[n-k]$$

$$=\begin{cases}\sum_{k=0}^{n}\left(\frac{3}{4}\right)^{n-k}, & n\geq 0\\ 0, & otherwise\end{cases}$$

$\therefore$

(1) $y[-5]=0$

(2) $y[5]=\sum_{k=0}^{5}\left(\frac{3}{4}\right)^{5-k}=\left(\frac{3}{4}\right)^{5}\sum_{k=0}^{5}\left(\frac{3}{4}\right)^{-k}=\left(\frac{3}{4}\right)^{5}\sum_{k=0}^{5}\left(\frac{4}{3}\right)^{k}$

$$=\left(\frac{3}{4}\right)^{5}\frac{1-\left(\frac{4}{3}\right)^{6}}{1-\frac{4}{3}}=3.288$$ 有限幾何級數

(3) $y[10]=?$

$$y[n]=\sum_{k=-\infty}^{+\infty}x[k]h[n-k]=\sum_{k=-\infty}^{+\infty}u[k]\left(\frac{3}{4}\right)^{n-k}u[n-k]$$

$$=\begin{cases}\sum_{k=0}^{n}\left(\frac{3}{4}\right)^{n-k}, & n\geq 0\\ 0, & otherwise\end{cases}$$

$\therefore$

(1) $y[-5]=0$

(2) $y[5]=\sum_{k=0}^{5}\left(\frac{3}{4}\right)^{5-k}=\left(\frac{3}{4}\right)^{5}\sum_{k=0}^{5}\left(\frac{3}{4}\right)^{-k}=\left(\frac{3}{4}\right)^{5}\sum_{k=0}^{5}\left(\frac{4}{3}\right)^{k}$

$$=\left(\frac{3}{4}\right)^{5}\frac{1-\left(\frac{4}{3}\right)^{6}}{1-\frac{4}{3}}=3.288$$ 有限幾何級數

(3) $y[10]=?$ 學生試一試

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Example 2.3 系統 *脈衝響應* = ?

系統輸出入關係如右:

$$y[n] = \frac{1}{4}\sum_{k=0}^{3} x[n-k]$$

令輸入 $x[n] = \delta[n]$，輸出即為系統 *脈衝響應*:

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Example 2.3 (cont.)

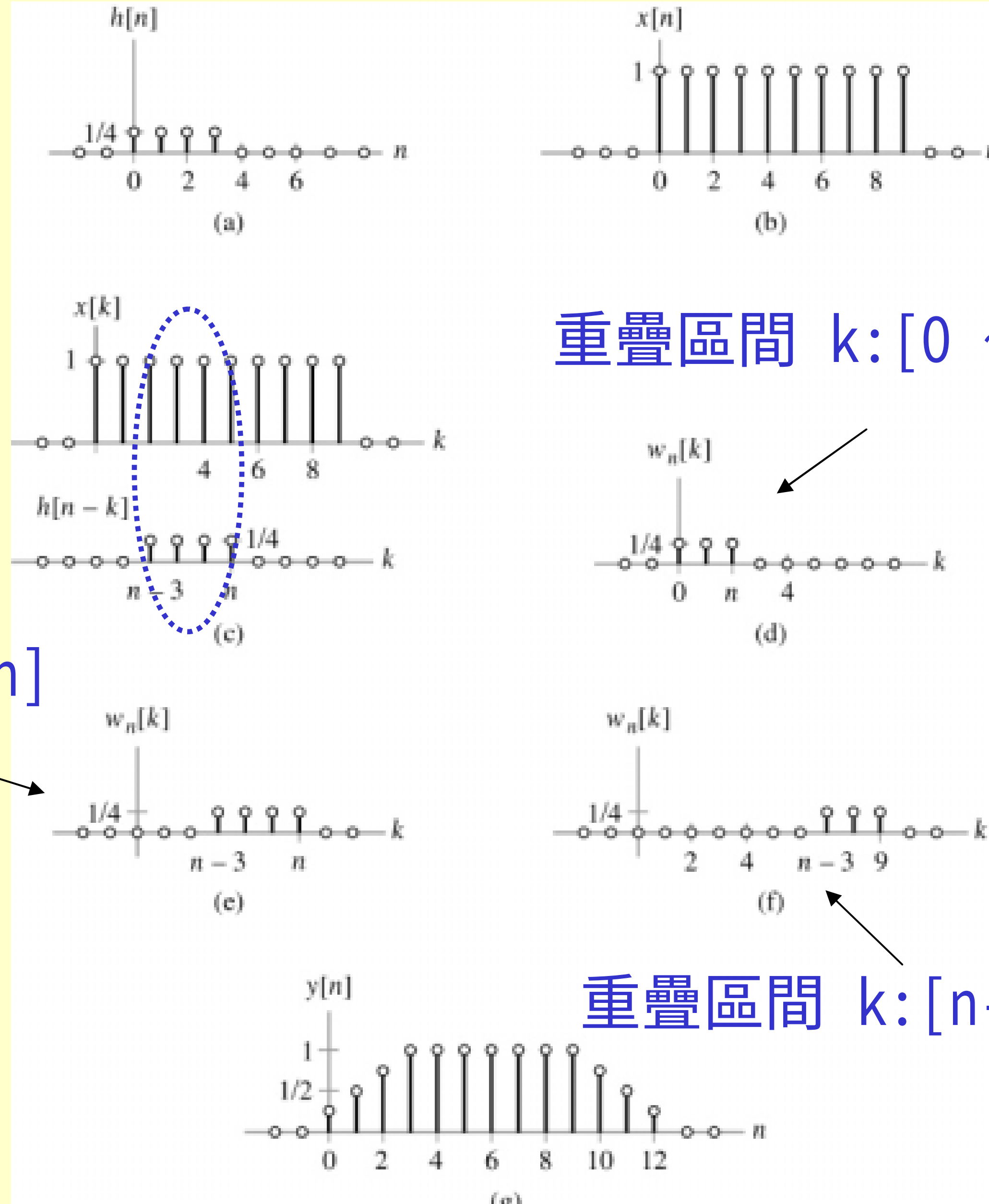
系統輸出入關係如右: $y[n]=\frac{1}{4}\sum_{k=0}^{3}x[n-k]$

令輸入 $x[n]=\delta[n]$，輸出即為系統*脈衝響應*:

$$h[n]=\frac{1}{4}\sum_{k=0}^{3}\delta[n-k]=\frac{1}{4}\left(\delta[n]+\delta[n-1]+\ldots+\delta[n-3]\right)$$

$$=\frac{1}{4}\left(u[n]-u[n-4]\right)$$

今若輸入： $x[n]=u[n]-u[n-10]$ ，輸出 = ?



重疊區間 k:[0 ~ n]

重疊區間 k:[n-3 ~ n]

重疊區間 k:[n-3 ~ 9]

Solution: $$y[n]=\sum_{k=-\infty}^{+\infty}x[k]h[n-k]$$

(1) n < 0, y[n] = 0, *no overlay*

(2) $0 \le n \le 3$, $y[n] = \sum_{k=0}^{n}\frac{1}{4}$ *overlay k : 0 ~ n*

(3) $4 \le n \le 9$, $y[n] = \sum_{k=n-3}^{n}\frac{1}{4}$ *overlay k : (n-3) ~ n*

(4) $10 \le n \le 12$, $y[n] = \sum_{k=n-3}^{9}\frac{1}{4}$ *overlay k : (n-3) ~ 9*

(5) n > 12, y[n] = 0, *no overlay*

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Example 2.4 迴歸性系統

系統輸出入關係如右: $y[n]-\rho\, y[n-1]=x[n]$

若輸入 $x[n]=b^{n}u[n+4]$ ，輸出 = ?

(系統為已知 <u>因果性系統</u>，且 b ≠ ρ)

Hint: for a causal system, h[n] = 0 for n < 0.

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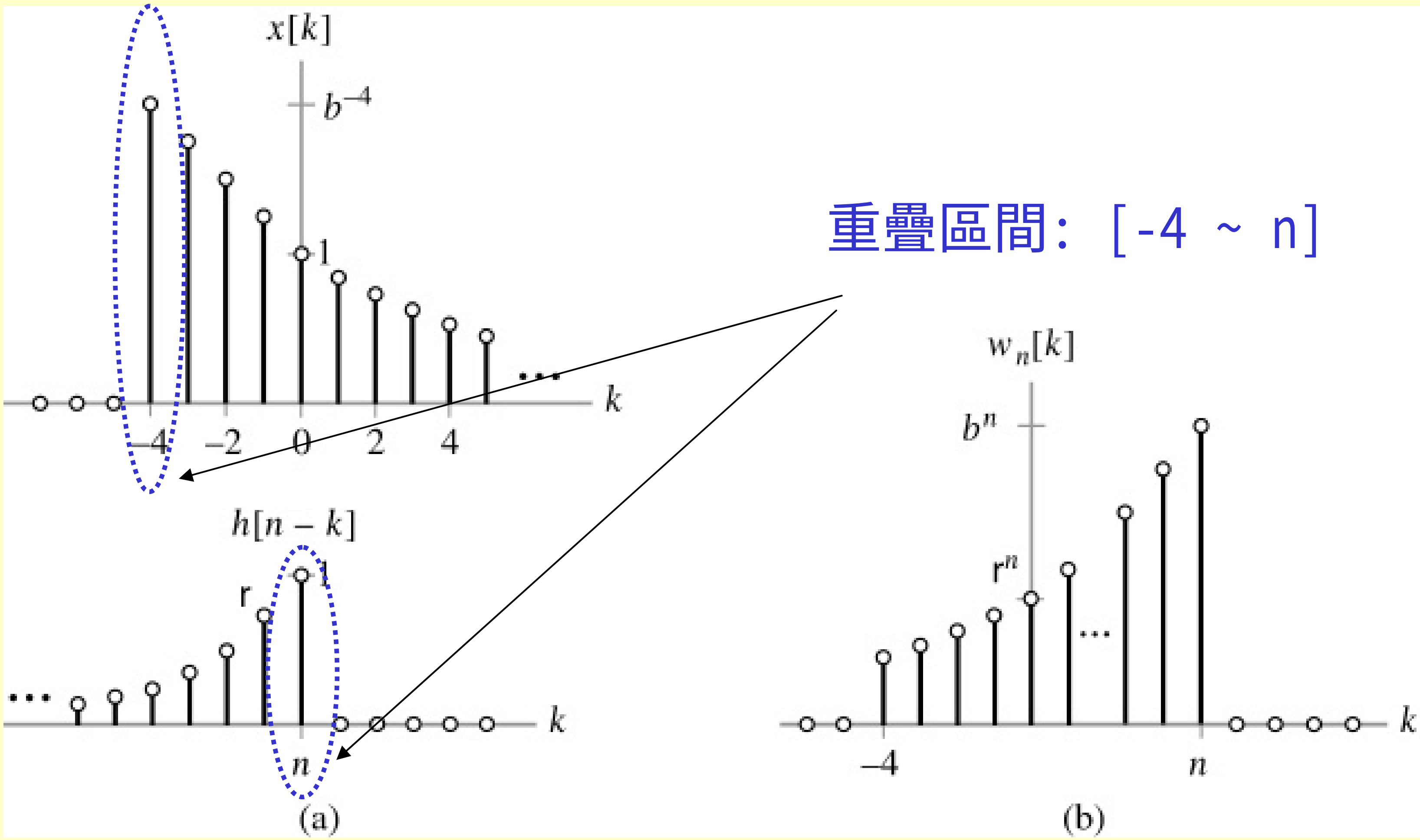


Solution: impulse response h[n]: $h[n] = \rho\, h[n-1] + \delta[n]$

$$\therefore \quad causal\ system: \quad h[n] = 0, \quad for\ n < 0$$

$$h[0] = \delta[0] = 1$$

$$h[1] = \rho\, h[0] + \delta[1] = \rho$$

$$h[2] = \rho\, h[1] + \delta[2] = \rho^2$$

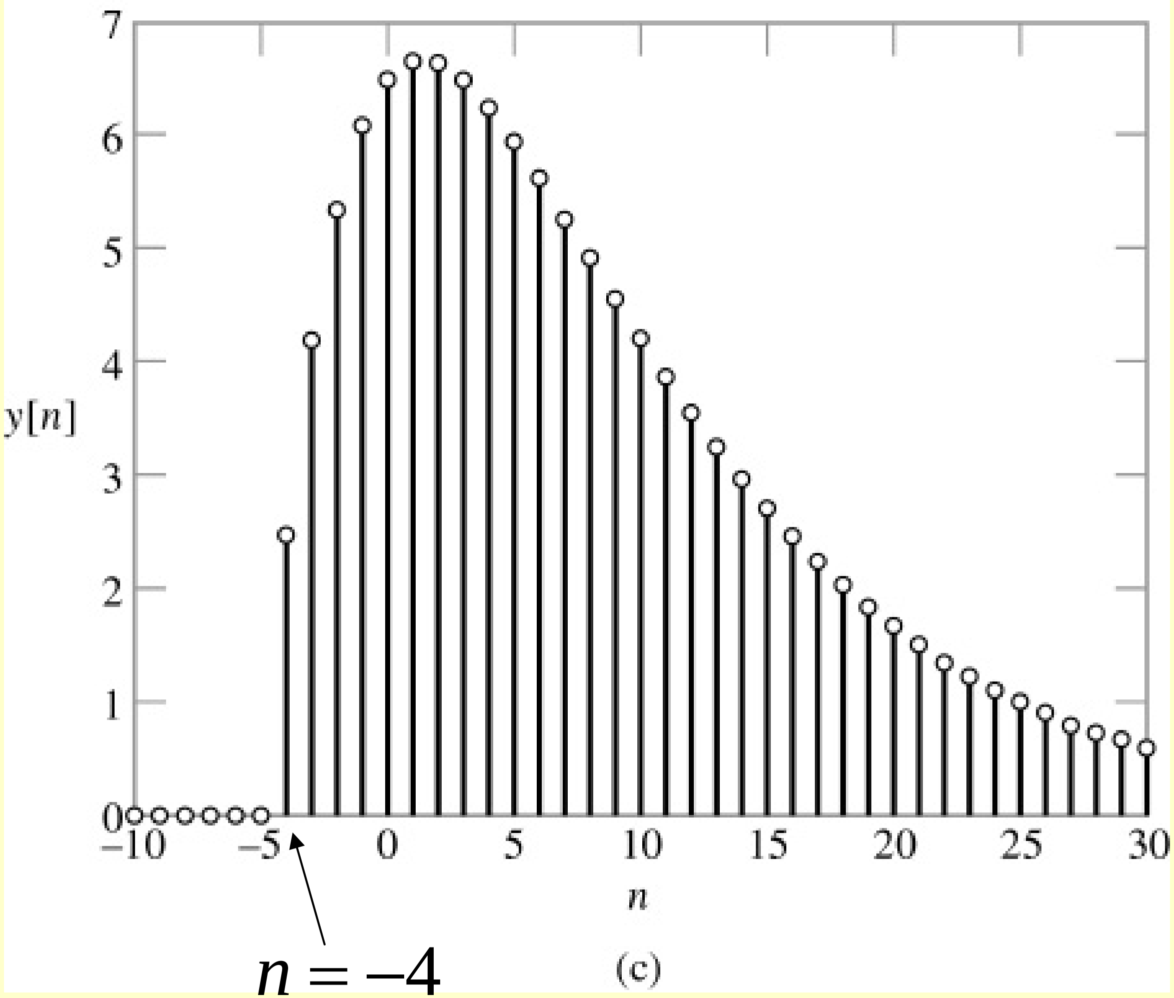
$$\vdots$$

$$h[n] = \rho^n\, u[n]$$

若輸入 $x[n] = b^n u[n+4]$ 重疊區間：[- 4, n]

$$y[n] = x[n] * h[n] = \{\rho^n u[n]\} * \{b^n u[n+4]\} = \sum_{k=-4}^{n} b^k\ \rho^{n-k}$$

Plot $y[n]$, $\rho = 0.9$ and $b = 0.8$.

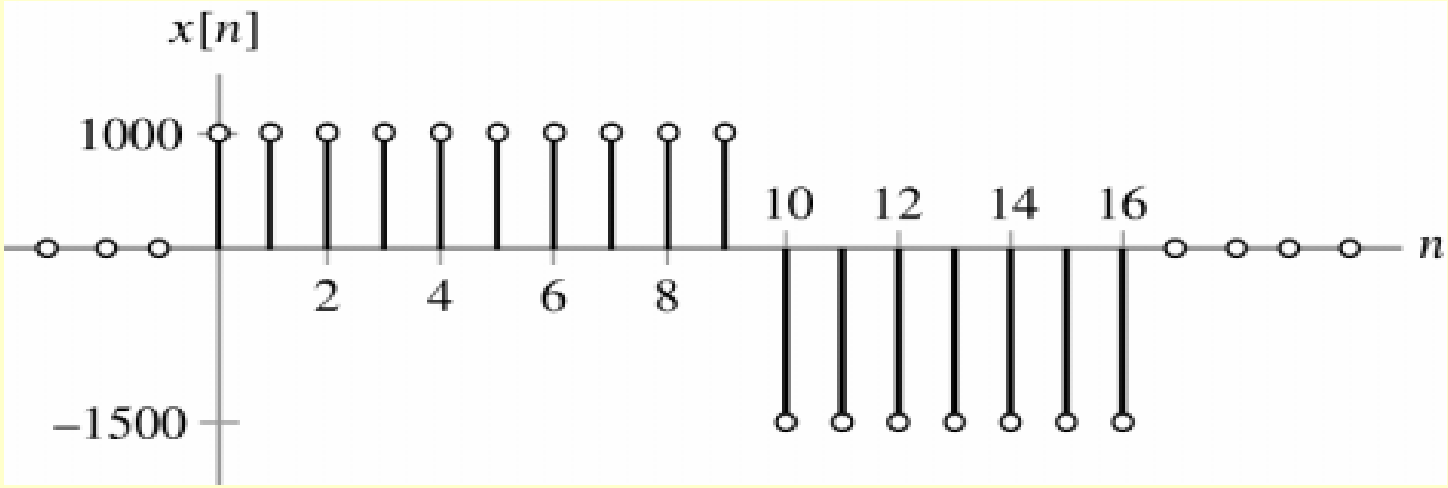


(c)

Example 2.5 (Applies Ex2.4 solution)

在一投資帳戶中前10年皆存入1000元，後7年則皆提出1500元，若年利率是8%，請問餘額 = ？

帳戶存取可用下圖表示：



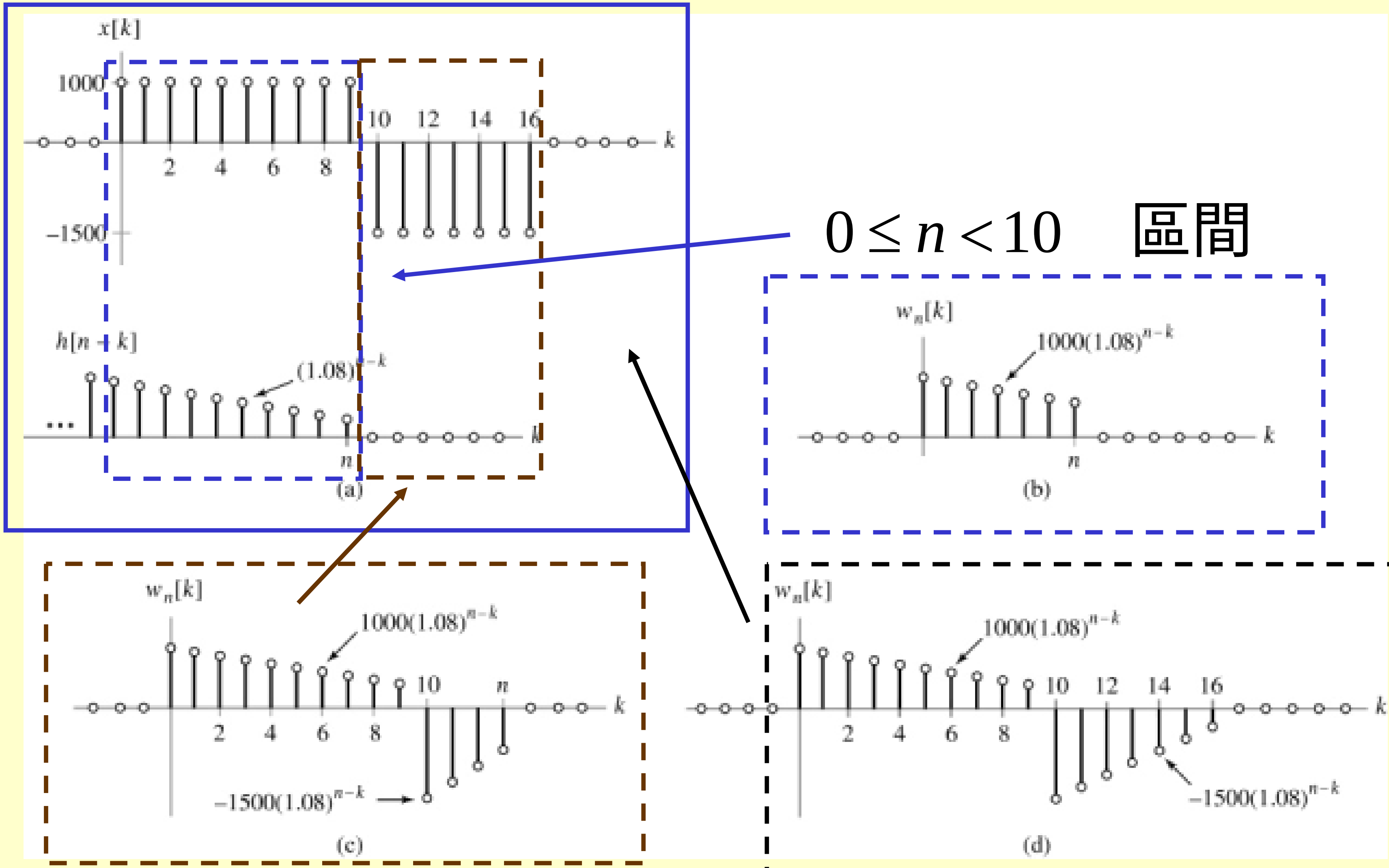
在投資帳戶中，餘額計算可用一階遞迴方程式表示：

$$y[n]-\rho\, y[n-1]=x[n]$$

若存款利率：8%， ρ 為 1.08

在 Ex2.4 中得知系統脈衝響應： $h[n]=\rho^n u[n]=(1.08)^n u[n]$

餘額可由 convolution sum 求得 y[n] = x[n] * h[n] = ?



$10 \le n \le 16$ 區間

$16 < n$ 區間

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(1) $n < 0$, $y[n] = 0$, no overlay

(2) $0 \le n \le 9$, $$y[n] = \sum_{k=0}^{n} 1000(1.08)^{n-k} = 12500\left((1.08)^{n+1} - 1\right)$$

(3) $10 \le n \le 16$, $$y[n] = \sum_{k=0}^{9} 1000(1.08)^{n-k} - \sum_{k=10}^{n} 1500(1.08)^{n-k}$$
$$= 7246.89(1.08)^{n} - 18750\left((1.08)^{n-9} - 1\right)$$

(4) $n \ge 17$ $$y[n] = \sum_{k=0}^{9} 1000(1.08)^{n-k} - \sum_{k=10}^{16} 1500(1.08)^{n-k}$$
$$= 3340.17(1.08)^{n}$$

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計算式參考：

$$\sum_{k=0}^{9} 1000(1.08)^{n-k}$$

$$= 1000(1.08)^n \sum_{k=0}^{9} (1.08)^{-k} = 1000(1.08)^n \sum_{k=0}^{9} (1.08^{-1})^k$$

$$= 1000(1.08)^n \frac{1-\left(1.08^{-1}\right)^{10}}{1-1.08^{-1}}; \quad \textit{finite geometry series}$$

$$= 1000(1.08)^n \frac{0.5368065}{1-0.9259259} = 1000(1.08)^n (7.2468852)$$

$$\approx 7246.89\left(1.08\right)^n$$

計算式參考：

$$\sum_{k=10}^{n} 1500(1.08)^{n-k}, \quad let\ m = k - 10,$$

$$= 1500(1.08)^{n} \sum_{k=10}^{n} (1.08)^{-k} = 1500(1.08)^{n} \sum_{m=0}^{n-10} (1.08^{-1})^{m+10},$$

$$= 1500(1.08)^{n-10} \sum_{m=0}^{n-10} (1.08^{-1})^{m} = 1500(1.08)^{n-10} \frac{1-\left(1.08^{-1}\right)^{n-9}}{1-1.08^{-1}},$$

$$= 1500(1.08)^{n-9}(1.08^{-1}) \frac{1-\left(1.08^{-1}\right)^{n-9}}{1-1.08^{-1}},$$

$$= \frac{1500(1.08)^{n-9}(1.08^{-1})}{1-1.08^{-1}} - \frac{1500(1.08^{-1})}{1-1.08^{-1}},$$

$$= \frac{1500(1.08^{-1})}{1-1.08^{-1}} \left\{(1.08)^{n-9} - 1\right\} = 18750 \ \left\{(1.08)^{n-9} - 1\right\}$$

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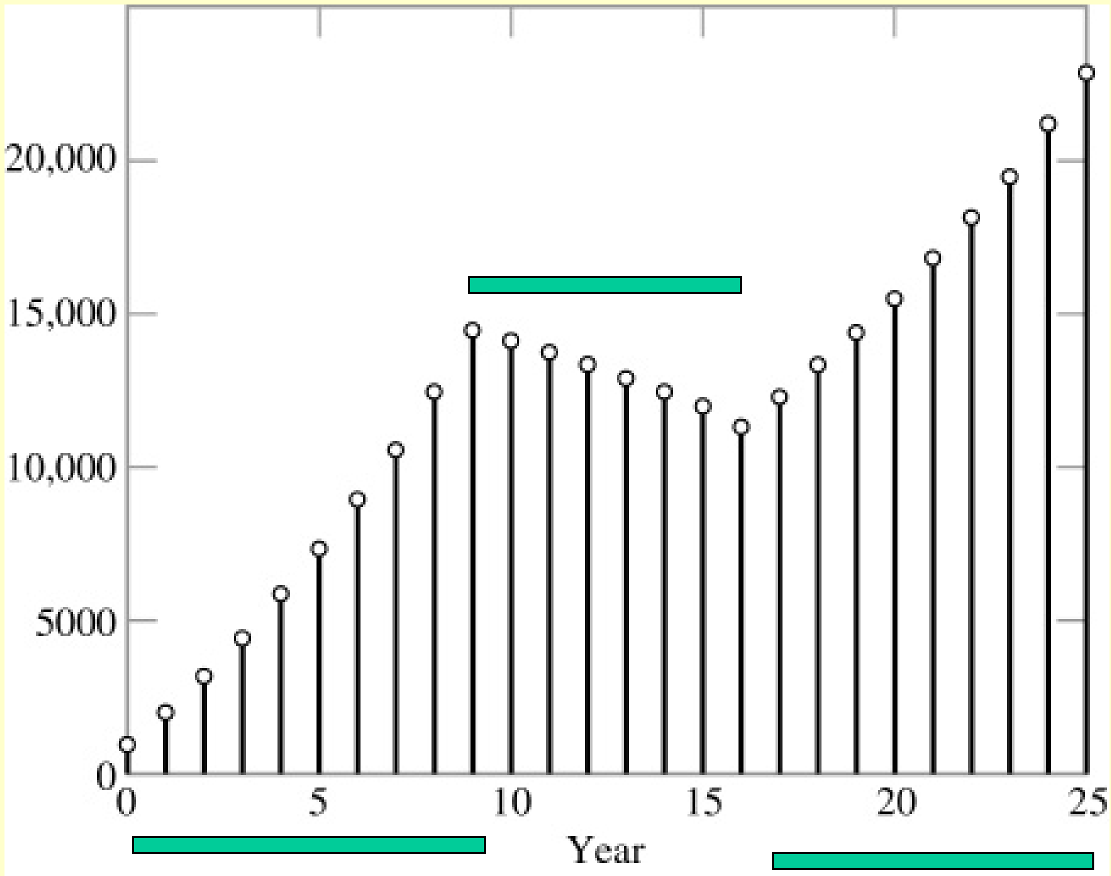
The $y[n]$ represents the value of the investment immediately after the deposit or withdrawal at the start of year n.



(e)

EX: Y[n] = x[n]*h[n] = ?