Audio & DSP Lab.

# Lecture 2-2

## Linear Time-Invariant System
## (LTI System)
## 線性非時變系統

# Convolution Integral

連續時間訊號也可表示為時間平移脈衝的加權疊加 (*積分*)：

$$x(t)=\int_{-\infty}^{+\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

連續時間訊號的系統輸出：

$$y(t)=H\{x(t)\}=H\left\{\int_{-\infty}^{+\infty} x(\tau)\,\delta(t-\tau)\,d\tau\right\}$$

由於線性特性，連續時間訊號的系統輸出：

$$y(t) = H\{x(t)\} = \int_{-\infty}^{+\infty} x(\tau)\, H\{\delta(t-\tau)\}\, d\tau$$

定義系統對單一脈衝輸入的響應： $H\{\delta(t)\} = h(t)$

定義系統滿足非時變性： $H\{\delta(t-\tau)\} = h(t-\tau)$

因此在 LTI 系統，連續時間訊號的系統輸出：

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau = x(t) * h(t)$$

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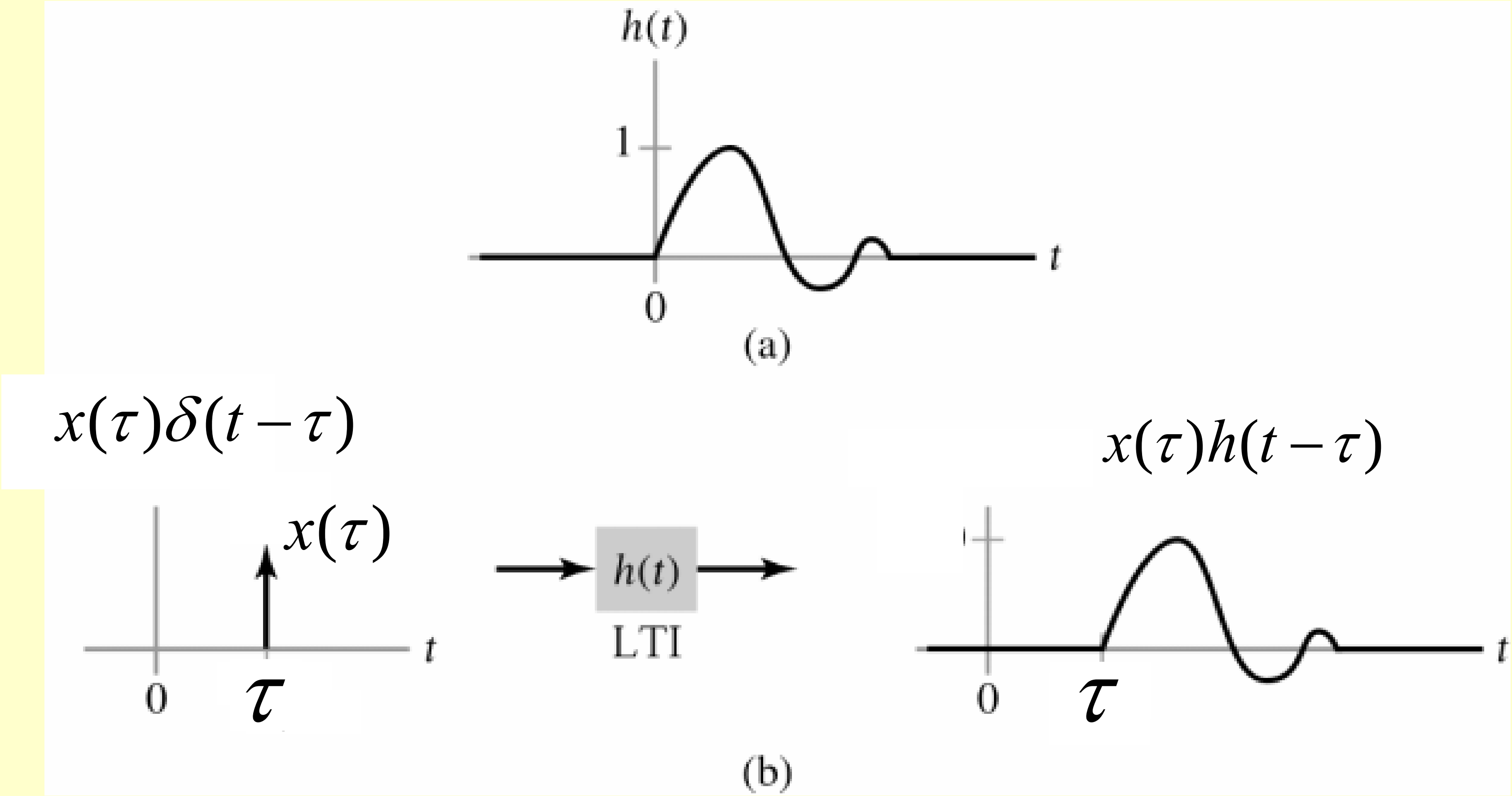
# Convolution Integral 圖解說明：



# Convolution Integral 計算程序 ：$y(t)=\int_{-\infty}^{+\infty} x(\tau)\,h(t-\tau)\,d\tau$

(1)畫出 x(τ) 和 h(τ) 圖形

(2)應用反射特性獲得 h(-τ) 圖形

(3)將h(-τ)圖形向 t 平移獲得 h(t-τ) 圖形

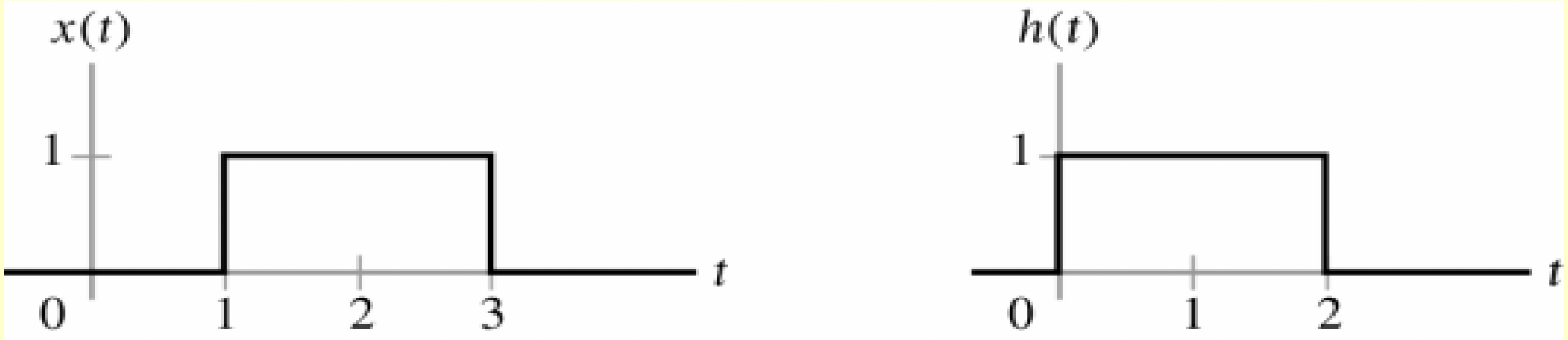
(4)若t > 0，向右平移，若t < 0，向左平移

(5)寫出中繼式： $w_i(\tau)=x(\tau)h(t-\tau)$

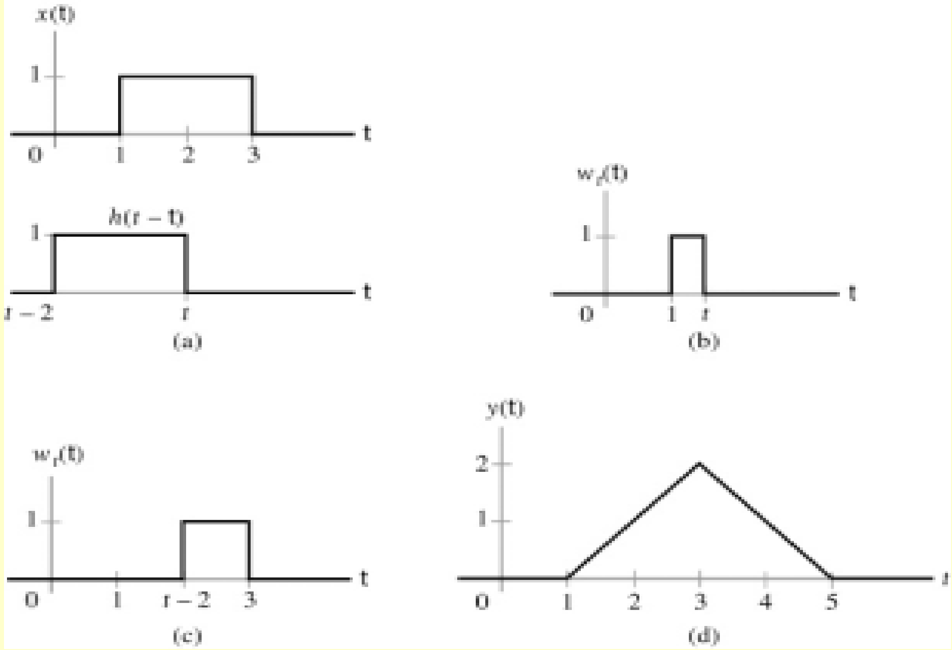
(6)將對每個平移 t 區間，做積分運算： $\int_{-\infty}^{+\infty} w_i(\tau)\,d\tau$

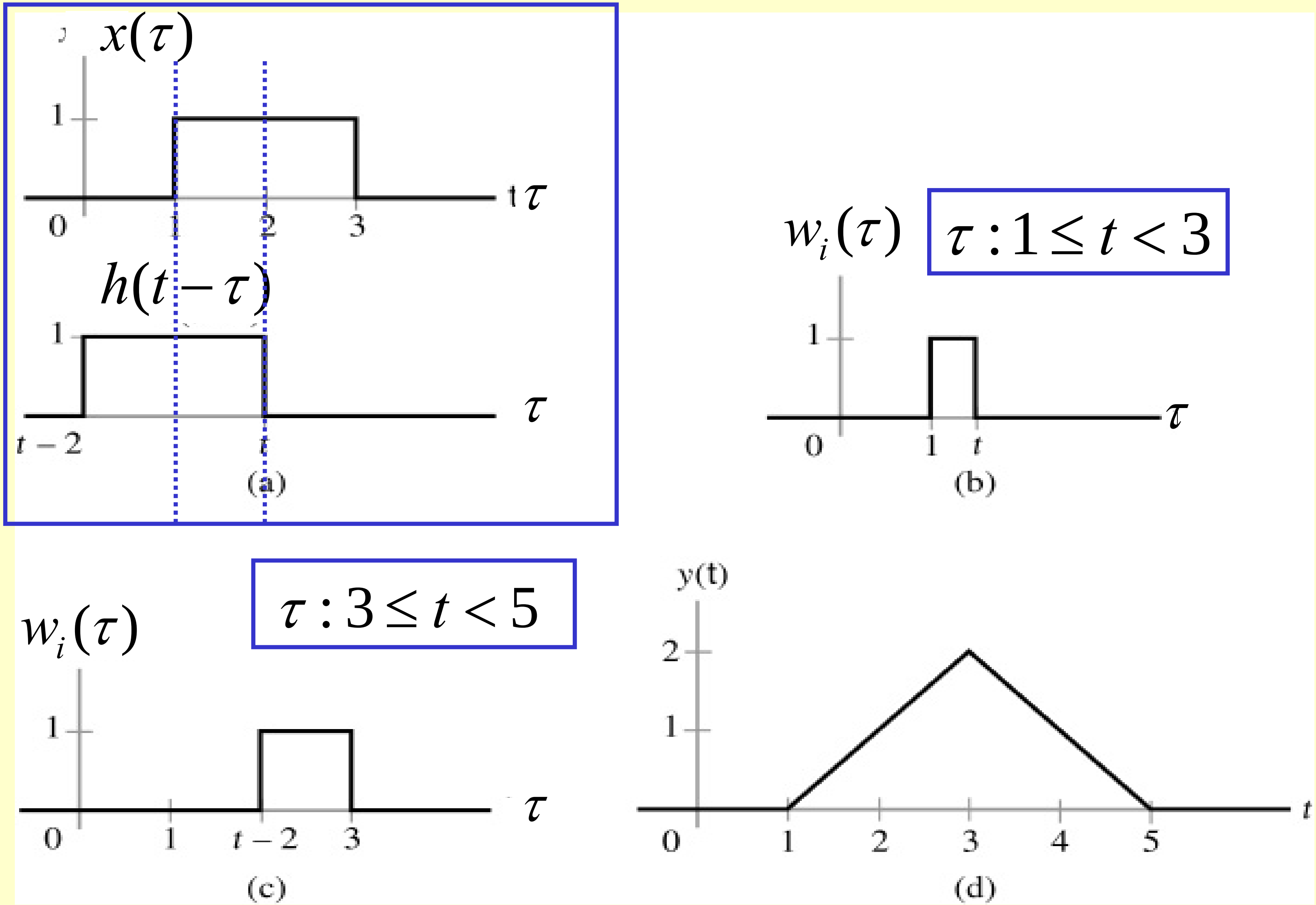
(7)積分區域：τ = -∞ ~ +∞

# EX2.6: $y(t) = x(t) * h(t) = ?$



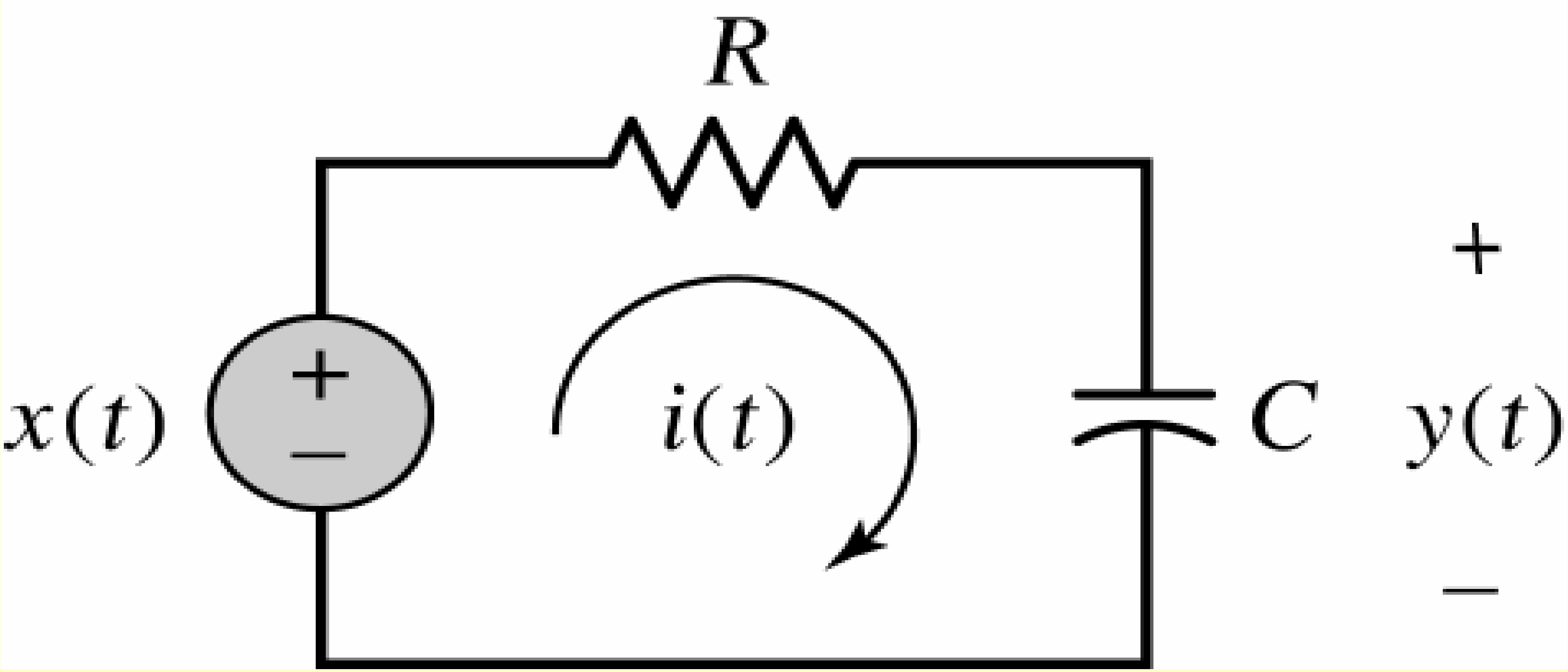
Find and $x(\tau)$ and the reflected, time-shifted impulse, h(t-$\tau$).
Find the product signal $w_t(\tau)$ for $1 \leq t < 3$.
Find the product signal $w_t(\tau)$ for $3 \leq t < 5$.
Find the system output $y(t)$.





$x(\tau)$
$h(t-\tau)$
$\tau$
$t-2$
$t$
(a)
$w_i(\tau)$
$\tau : 1 \le t < 3$
(b)
$w_i(\tau)$
$\tau : 3 \le t < 5$
(c)
y(t)
(d)

*EX: A RC* circuit system with the voltage source $x(t)$ as input and the voltage measured across the capacitor $y(t)$, as output.



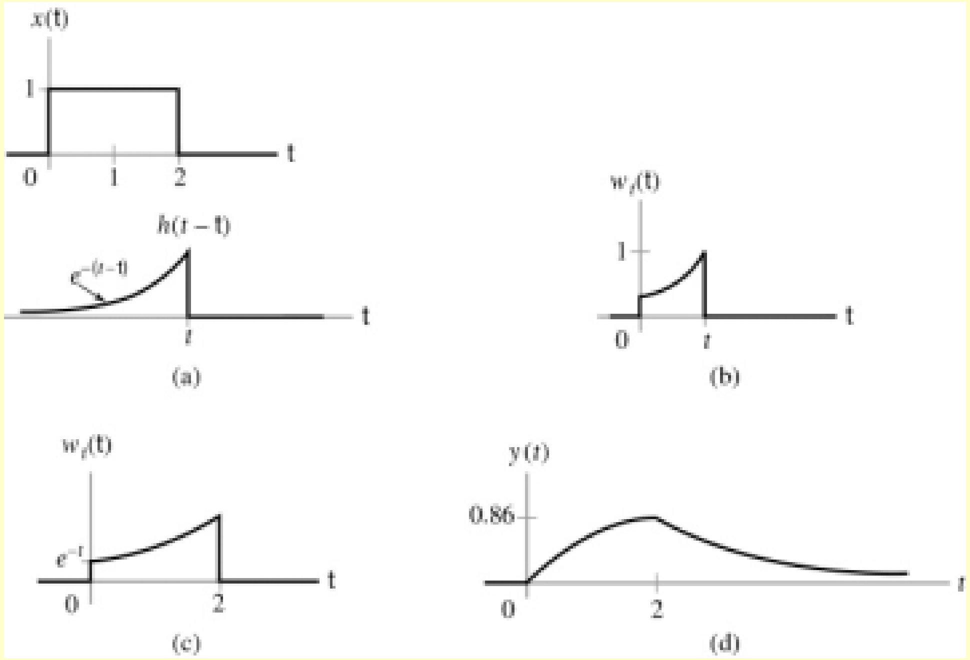
Ex 2.7:

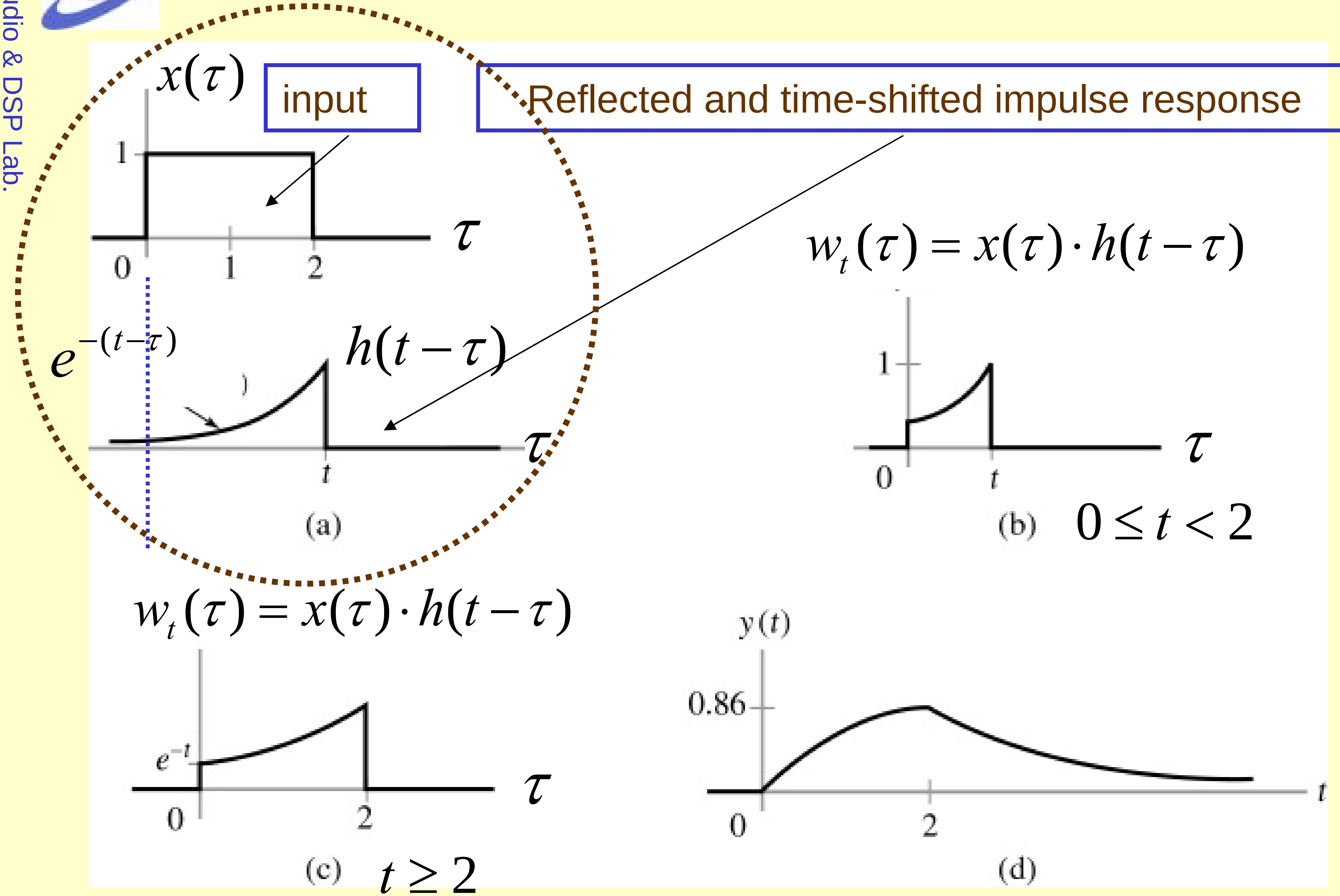
(a) The input $x(\tau)$ superimposed over the reflected and time-shifted impulse response $h(t-\tau)$, depicted as a function of $\tau$.

(b) The product signal $w_t(\tau)$ for $0 \le t < 2$.

(c) The product signal $w_t(\tau)$ for $t \ge 2$.

(d) The system output $y(t)$.




$x(\tau)$
input
Reflected and time-shifted impulse response
1
$\tau$
0
1
2
$w_t(\tau) = x(\tau) \cdot h(t-\tau)$
$e^{-(t-\tau)}$
$h(t-\tau)$
$\tau$
$t$
(a)
1
$\tau$
0
$t$
(b)
$0 \le t < 2$
$w_t(\tau) = x(\tau) \cdot h(t-\tau)$
$e^{-t}$
$\tau$
0
2
(c)
$t \ge 2$
$y(t)$
0.86
$t$
0
2
(d)

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Ex 2.8:

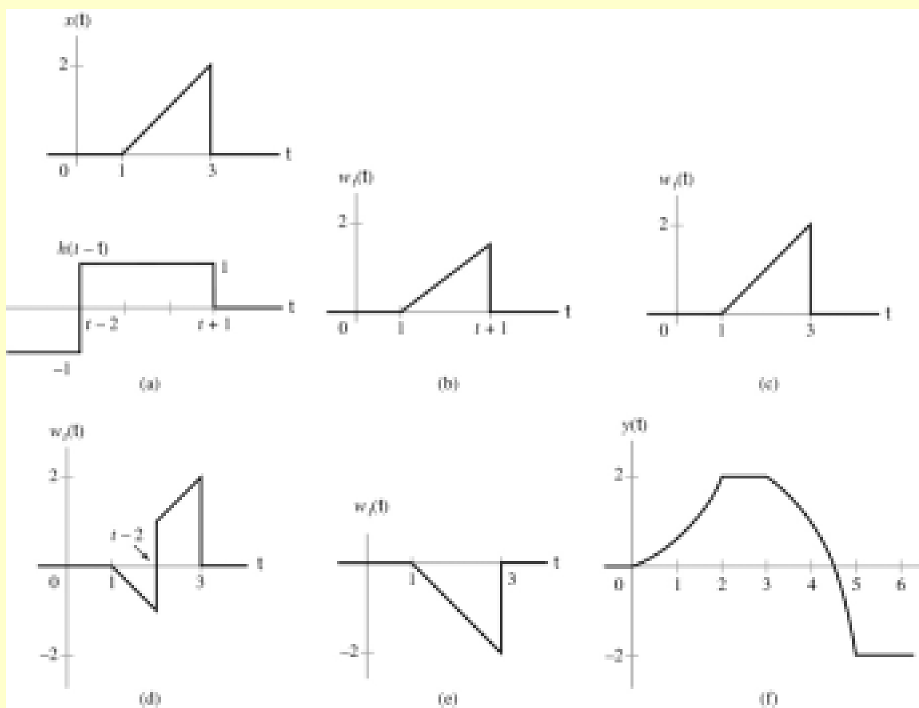
(a) The input $x(\tau)$ superimposed on the reflected and time-shifted impulse response $h(t-\tau)$, depicted as a function of $\tau$.

(b) The product signal $w_t(\tau)$ for $0 \le t < 2$.

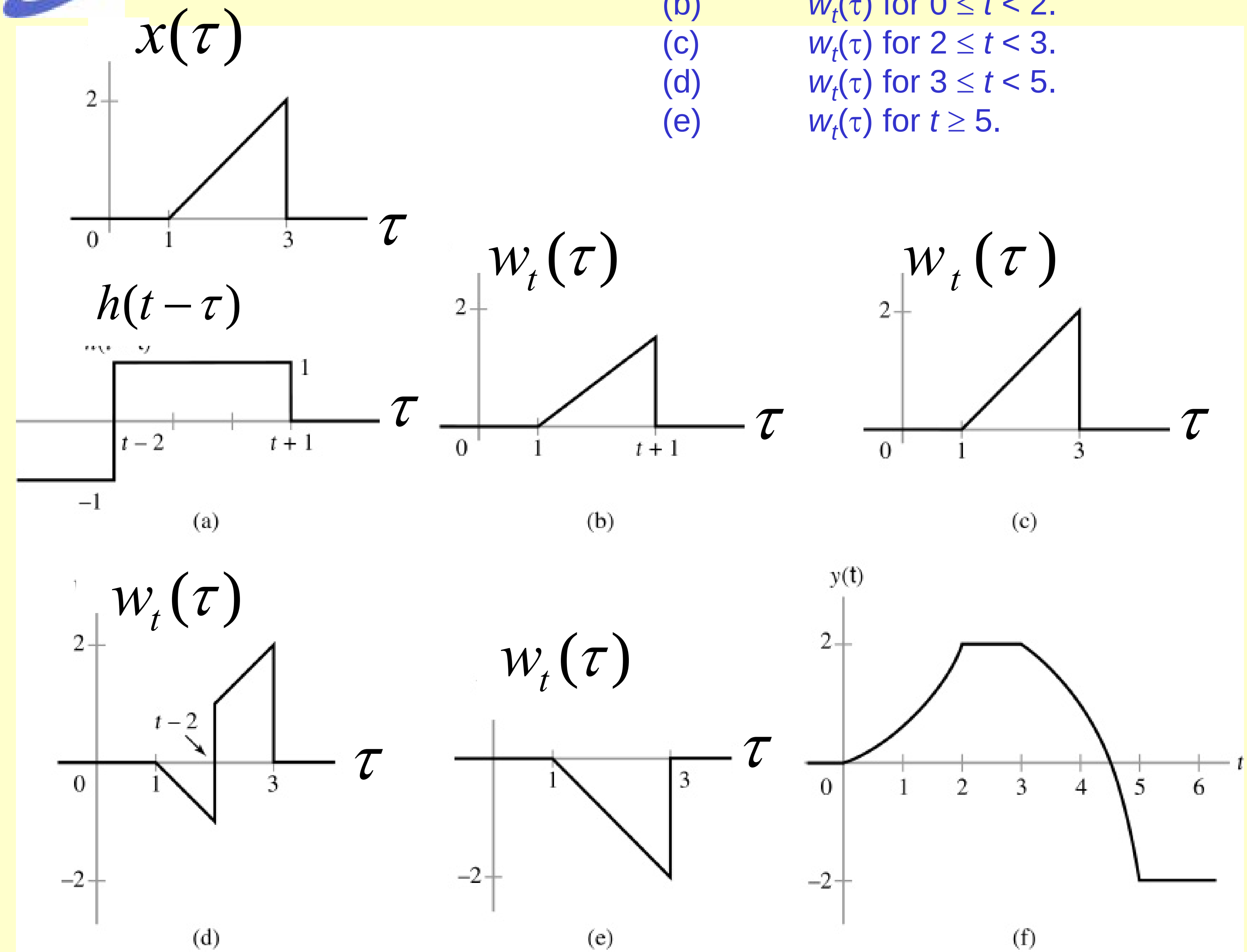
(c) The product signal $w_t(\tau)$ for $2 \le t < 3$.

(d) The product signal $w_t(\tau)$ for $3 \le t < 5$.

(e) The product signal $w_t(\tau)$ for $t \ge 5$. The system output $y(t)$.



(b) $w_t(\tau)$ for $0 \le t < 2$.
(c) $w_t(\tau)$ for $2 \le t < 3$.
(d) $w_t(\tau)$ for $3 \le t < 5$.
(e) $w_t(\tau)$ for $t \ge 5$.



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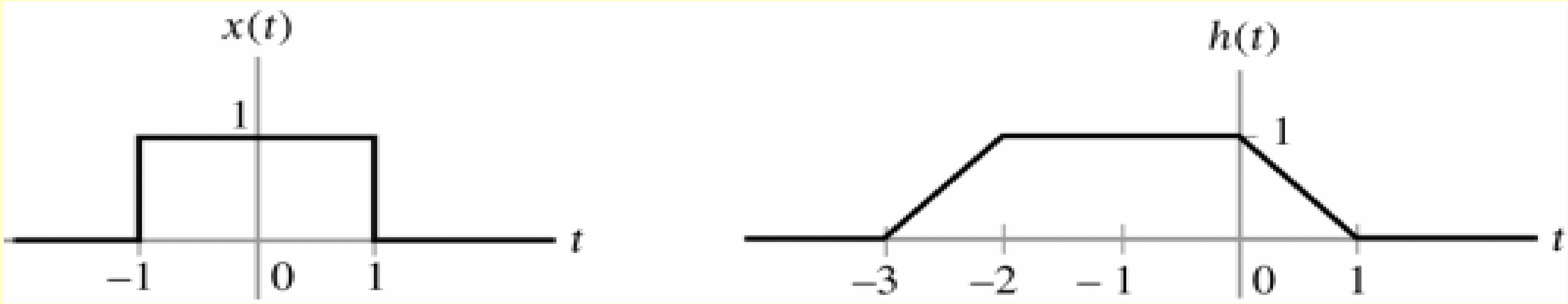
# P2.5: 試求系統輸出： $y(t) = x(t) * h(t)$



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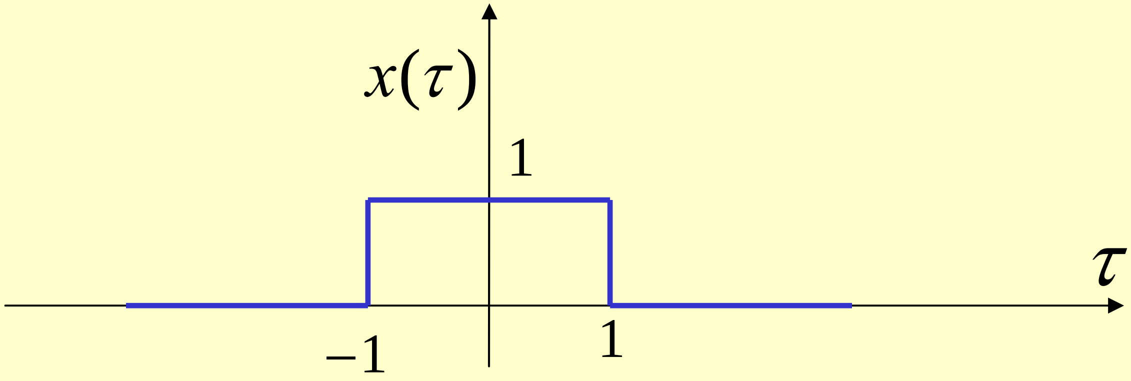
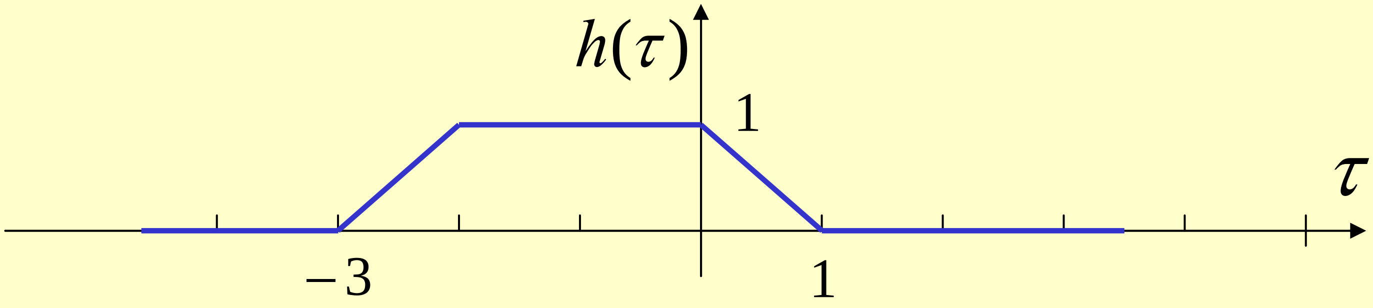
# Step 1:

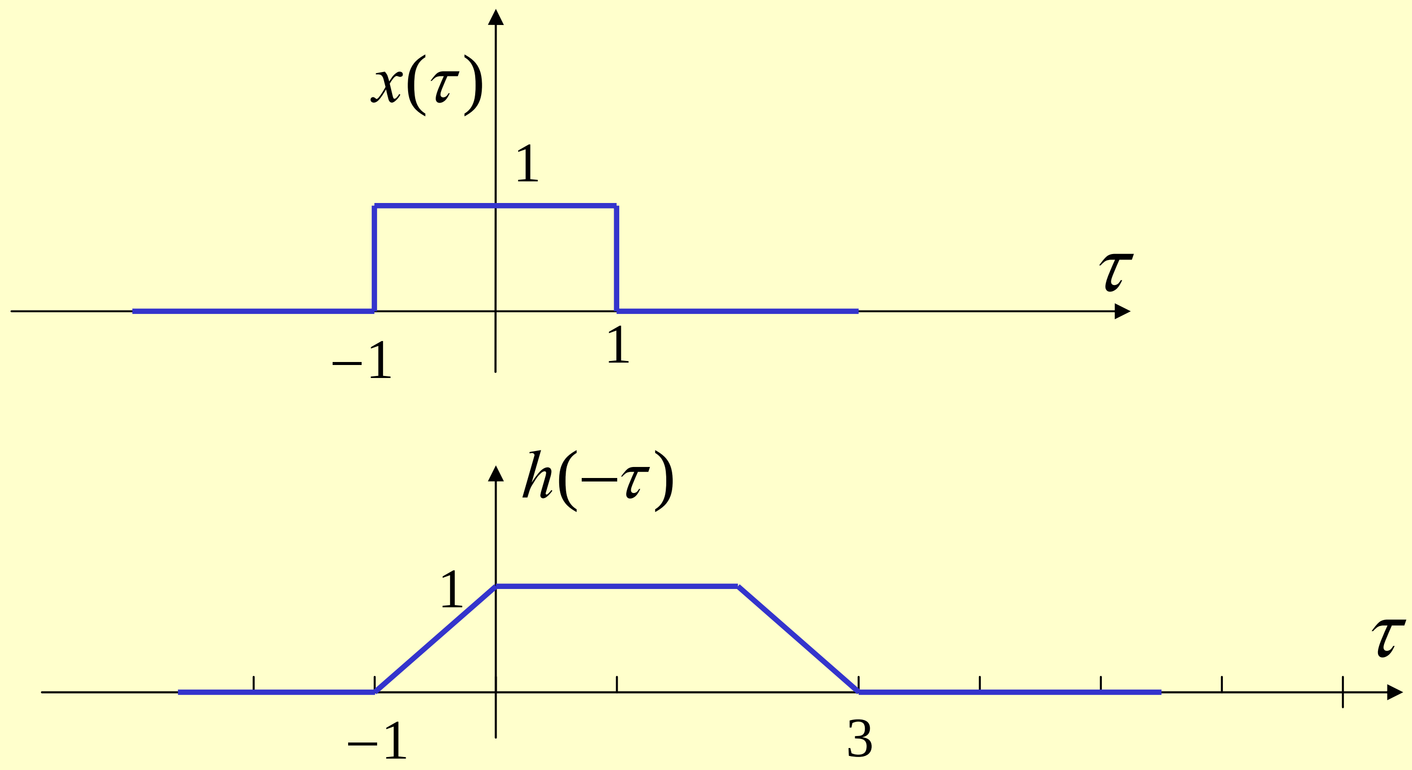
Produce new set of dependent variables, x(τ) and h(τ) with replace t by τ.



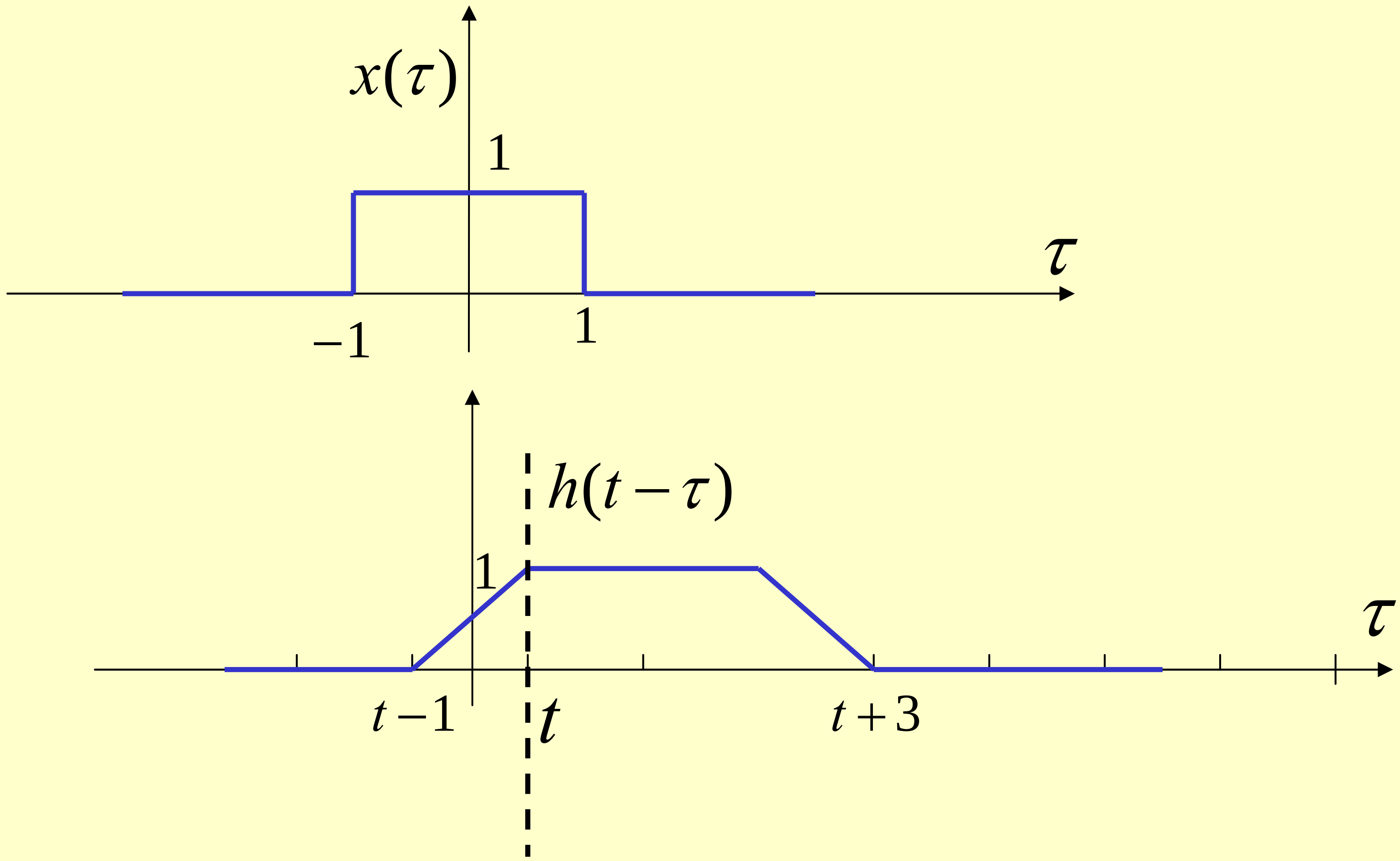
# Step 2:

Obtain the h(-τ) from h(τ) by reflection operation.



# Step 3:

Obtain the h(t-τ) from h(-τ) by time-shifting operation.



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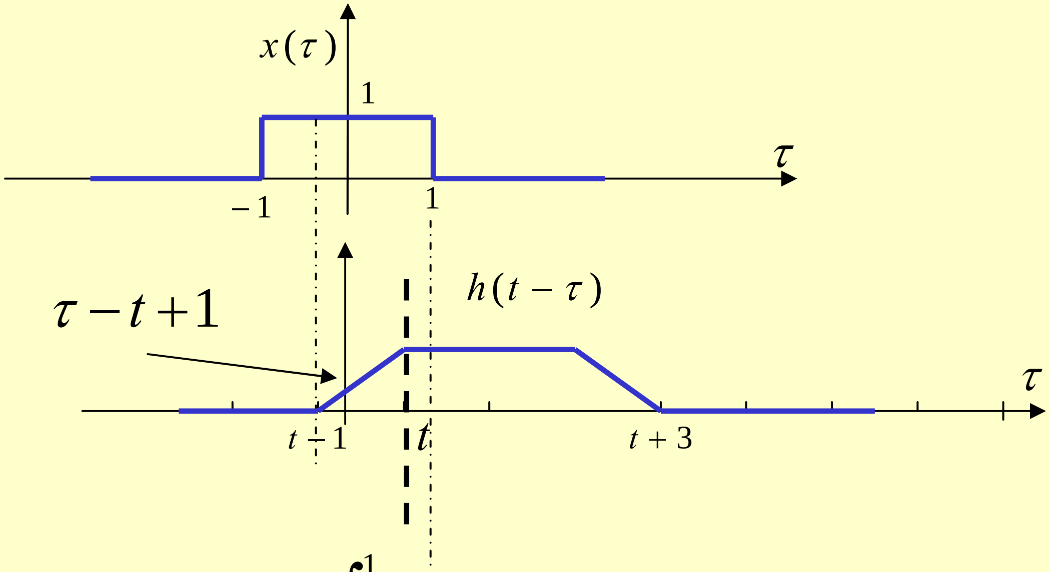
# 詳細推導




e.g.,

*if* $0 \le t < 1,$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)\, d\tau = \int_{t-1}^{t} (\tau - t + 1)\ d\tau + \int_{t}^{1} 1\ d\tau$$

$$= \left(\frac{\tau^2}{2} - \tau(t-1)\Big|_{t-1}^{t}\right) + \left(\tau\Big|_{t}^{1}\right)$$

$$= \frac{t^2}{2} - t(t-1) - \frac{(t-1)^2}{2} + (t-1)^2 + (1-t)$$

$$= \frac{t^2}{2} - t^2 + t - \frac{t^2}{2} + t - \frac{1}{2} + t^2 - 2t + 1 + 1 - t$$

$$= \frac{3}{2} - t$$

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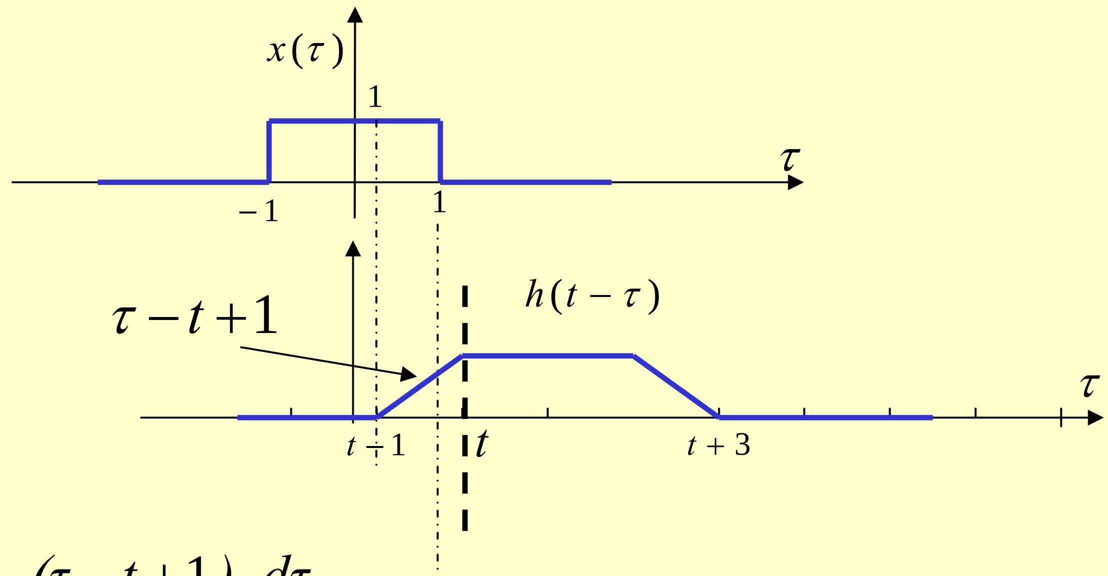
# 詳細推導




e.g.,

$if\ 1 \le t < 2,$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)\,d\tau = \int_{t-1}^{1} (\tau - t + 1)\ \ d\tau$$

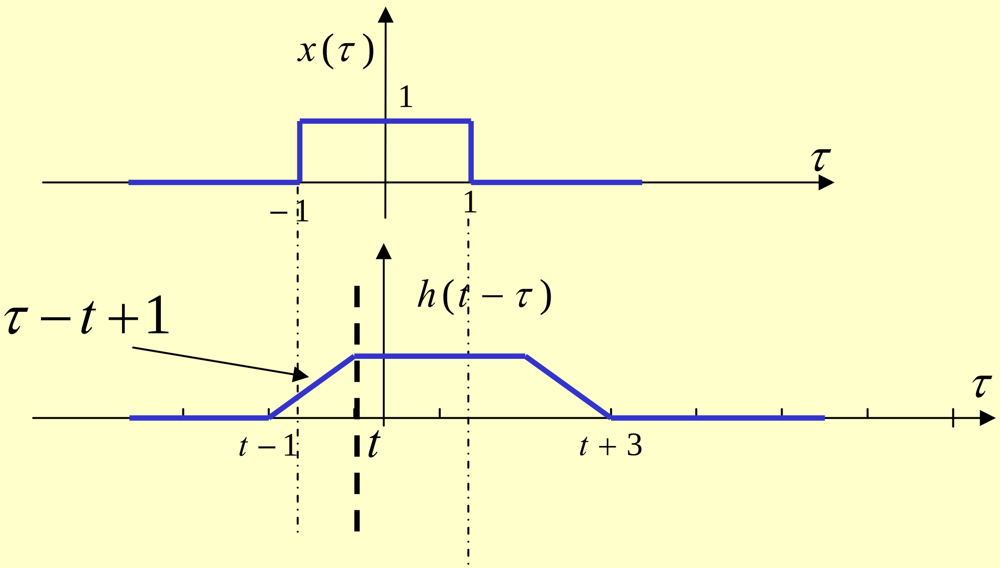
$$= \left(\frac{\tau^2}{2} - \tau(t-1)\Big|_{t-1}^{1}\right)$$

$$= \frac{1}{2} - (t-1) - \frac{(t-1)^2}{2} + (t-1)^2$$

$$= \frac{1}{2} - t + 1 - \frac{t^2}{2} + t - \frac{1}{2} + t^2 - 2t + 1$$

$$= \frac{t^2}{2} - 2t + 2$$

# 詳細推導




e.g.,

$if\ -1 \le t < 0,$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)\,d\tau = \int_{-1}^{t} (\tau - t + 1)\ d\tau + \int_{t}^{1} 1\ d\tau$$

$-t^2-5t-6$

$$= \left(\frac{\tau^2}{2} - \tau(t-1)\Big|_{-1}^{t}\right) + \left(\tau\Big|_{t}^{1}\right)$$

$$= \frac{t^2}{2} - t(t-1) - \frac{(-1)^2}{2} - (t-1) + (1-t)$$

$$= \frac{t^2}{2} - t^2 + t - \frac{1}{2} - t + 1 + 1 - t$$

$$= -\frac{t^2}{2} - t + \frac{3}{2}$$

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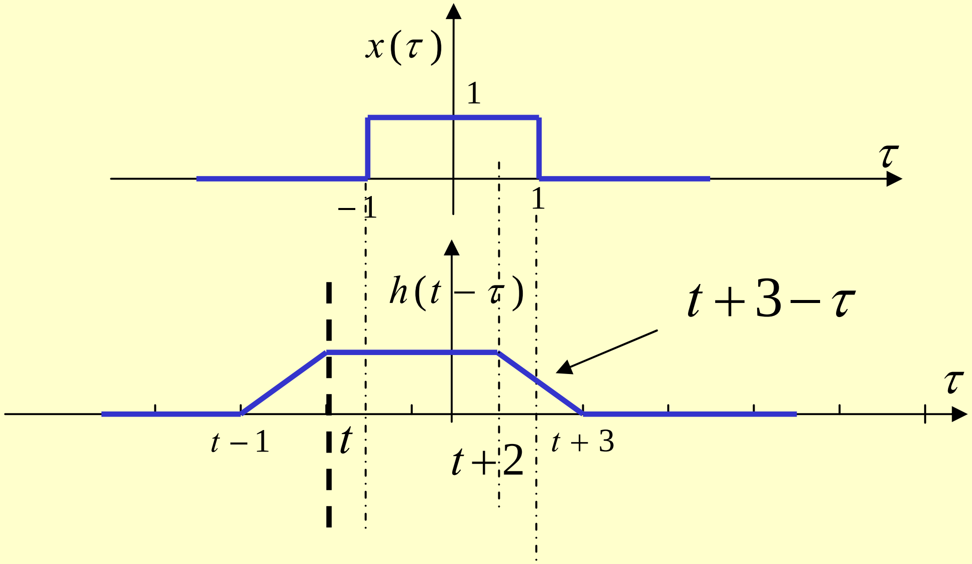



# 詳細推導

e.g.,

$if\ -2 \le t < -1,$

$$y(t)=\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)\,d\tau=\int_{t+2}^{1}(t+3-\tau)\ d\tau+\int_{-1}^{t+2} 1\ d\tau$$

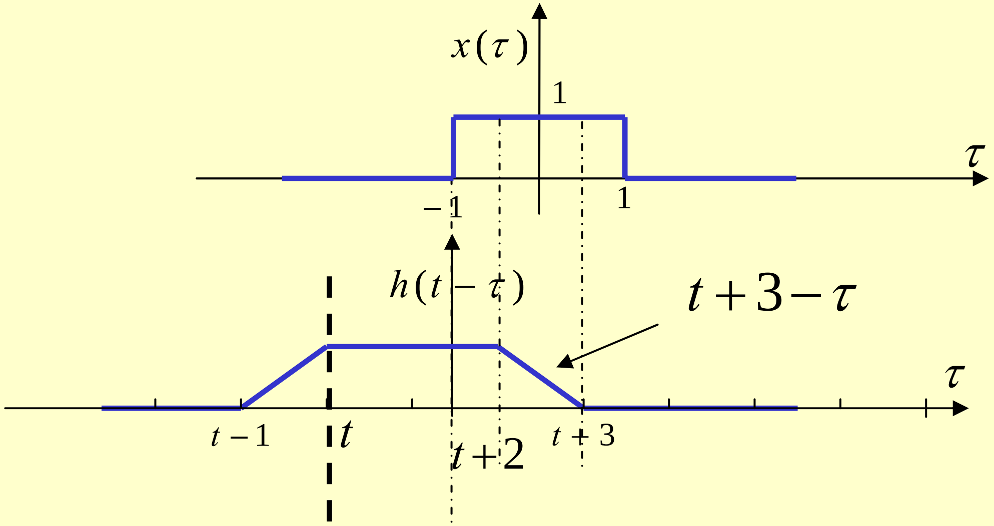
$$=(\tau(3+t)-\frac{\tau^2}{2}\Big|_{t+2}^{1})+(\tau\Big|_{-1}^{t+2})$$

$$=(3+t)-\frac{1}{2}-(t+2)(3+t)+\frac{(t+2)^2}{2}+(t+2+1)$$

$$=3+t-\frac{1}{2}-t^2-5t-6+\frac{t^2}{2}+2t+2+t+3$$

$$=-\frac{t^2}{2}-t+\frac{3}{2}$$

# 詳細推導




e.g.,

$if\ -3 \le t < -2,$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)\,d\tau = \int_{t+2}^{t+3} (t+3-\tau)\ d\tau + \int_{-1}^{t+2} 1\ d\tau$$

$$= (\tau(3+t) - \frac{\tau^2}{2} \Big|_{t+2}^{t+3}) + (\tau \Big|_{-1}^{t+2})$$

$$= (3+t)^2 - \frac{(t+3)^2}{2} - (t+2)(3+t) + \frac{(t+2)^2}{2} + (t+2+1)$$

$$= 9 + 6t + t^2 - \frac{t^2}{2} - 3t - \frac{9}{2} - t^2 - 5t - 6 + \frac{t^2}{2} + 2t + 2 + t + 3$$

$$= t + \frac{7}{2}$$

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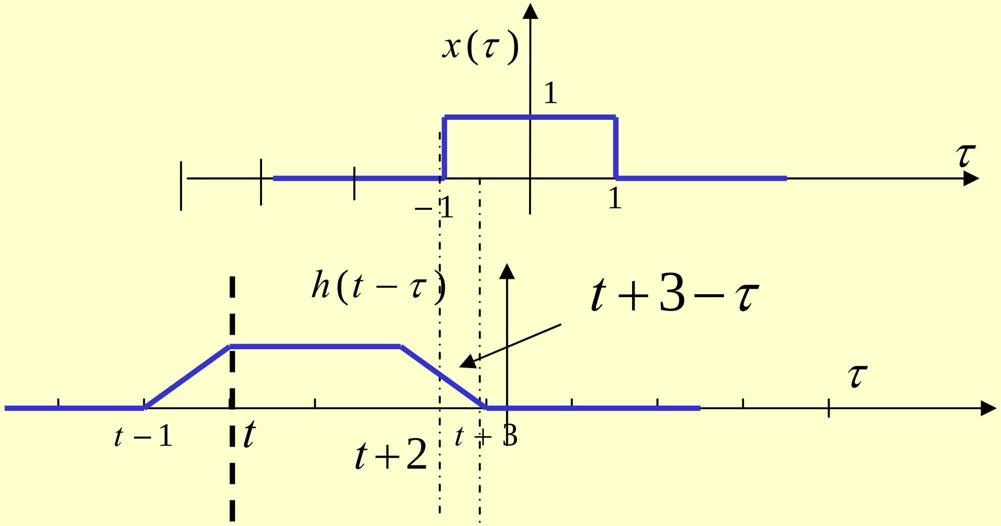



# 詳細推導

e.g.,

$$if\ -4 \le t < -3,$$

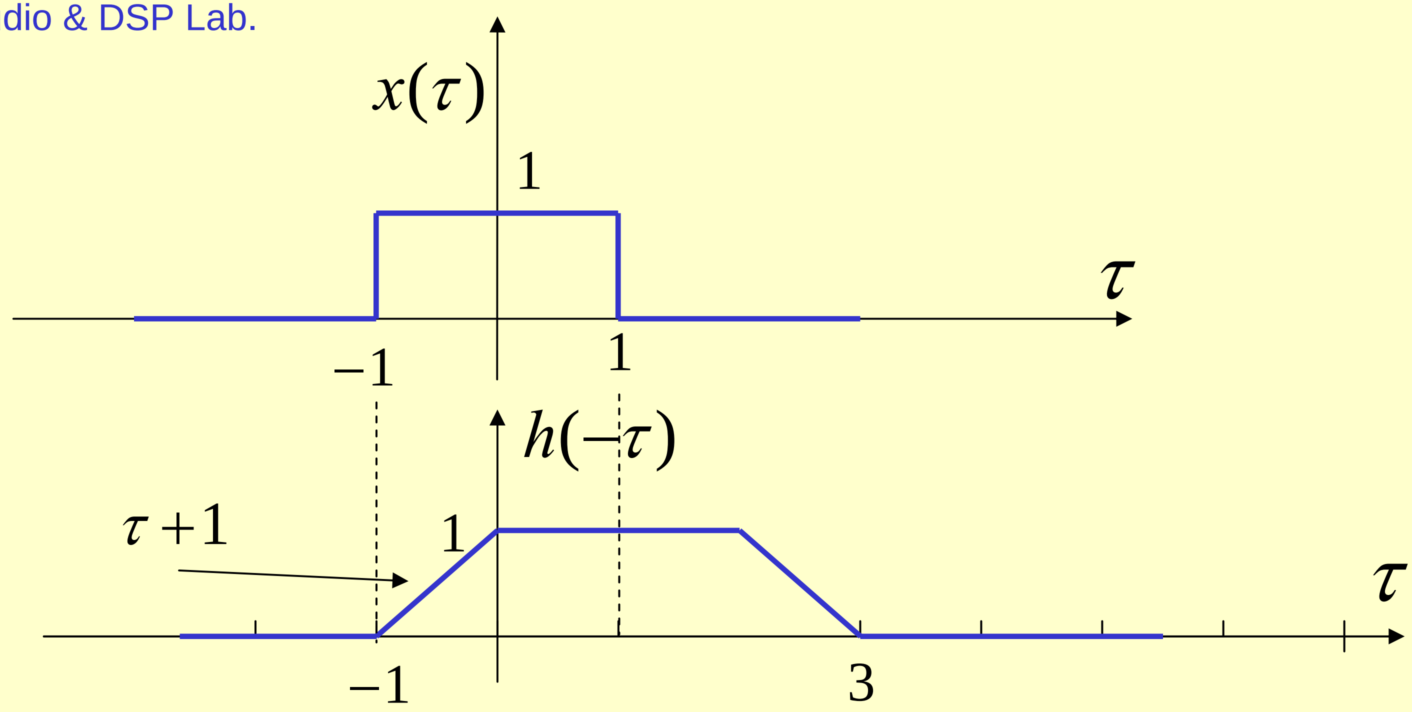
$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)\, d\tau = \int_{-1}^{t+3} (t+3-\tau)\ d\tau$$

$$= (\tau(3+t) - \frac{\tau^2}{2}\Big|_{-1}^{t+3})$$

$$= (3+t)^2 - \frac{(t+3)^2}{2} + (3+t) + \frac{1}{2}$$

$$= 9 + 6t + t^2 - \frac{t^2}{2} - 3t - \frac{9}{2} + 3 + t + \frac{1}{2}$$

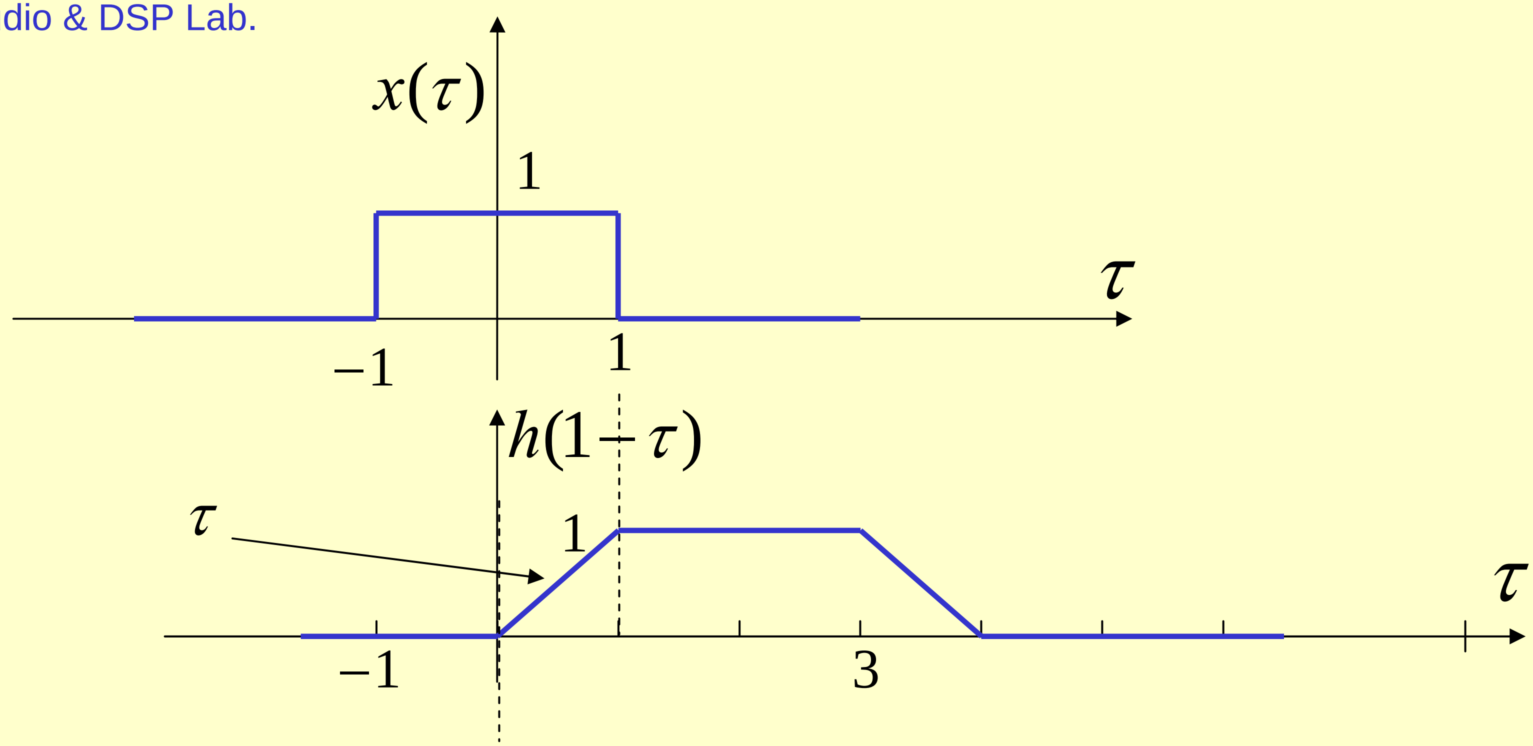
$$= \frac{t^2}{2} + 4t + 8$$




簡易代入：

$$y(0)=\int_{-\infty}^{+\infty} x(\tau)\,h(0-\tau)\,d\tau=\int_{-1}^{0}(\tau+1)\;d\tau+\int_{0}^{1}1\;d\tau$$

$$=\frac{\tau^2}{2}+\tau\Big|_{-1}^{0}+\tau\Big|_{0}^{1}=\frac{1}{2}+1=\frac{3}{2}$$




簡易代入

$$y(1)=\int_{-\infty}^{+\infty} x(\tau)\,h(1-\tau)\,d\tau=\int_{0}^{1}\tau\ \ d\tau$$

$$=\left.\frac{\tau^2}{2}\right|_0^1=\frac{1}{2}$$



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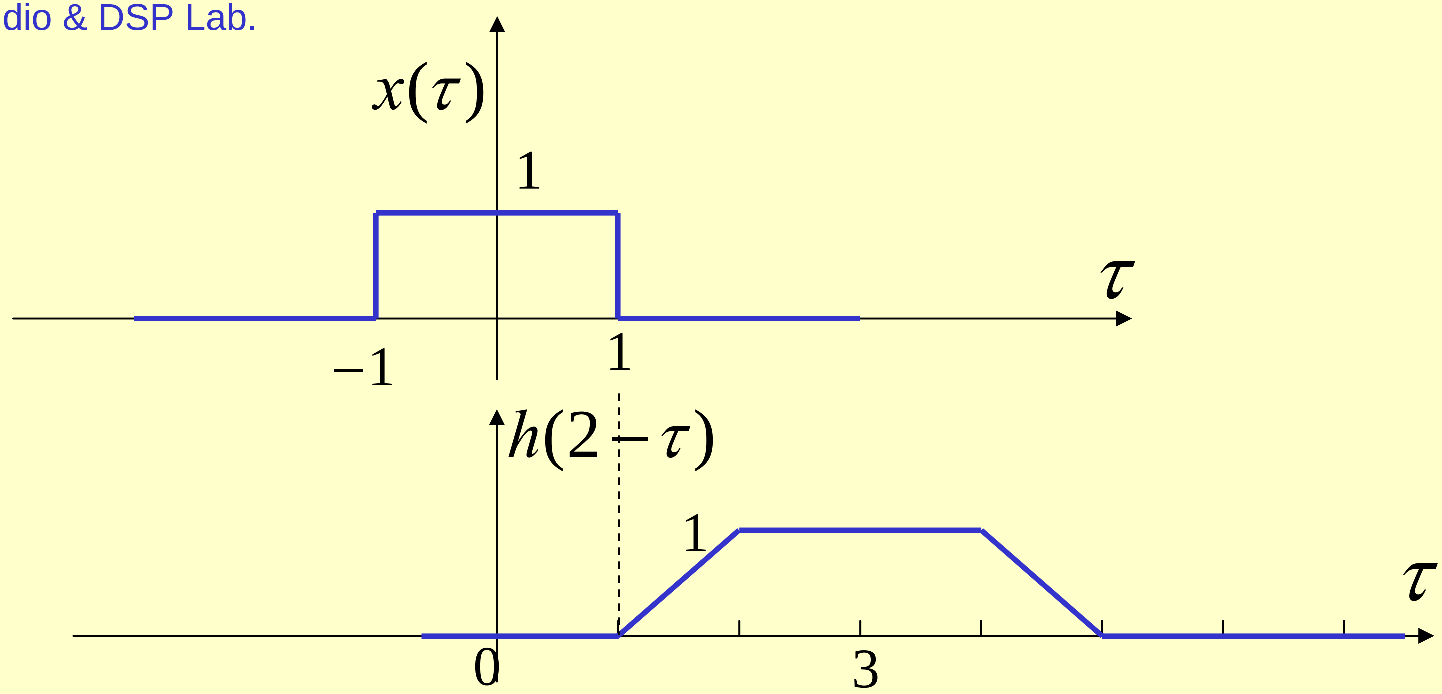


簡易代入

$$y(2)=\int_{-\infty}^{+\infty} x(\tau)\, h(2-\tau)\, d\tau = 0$$

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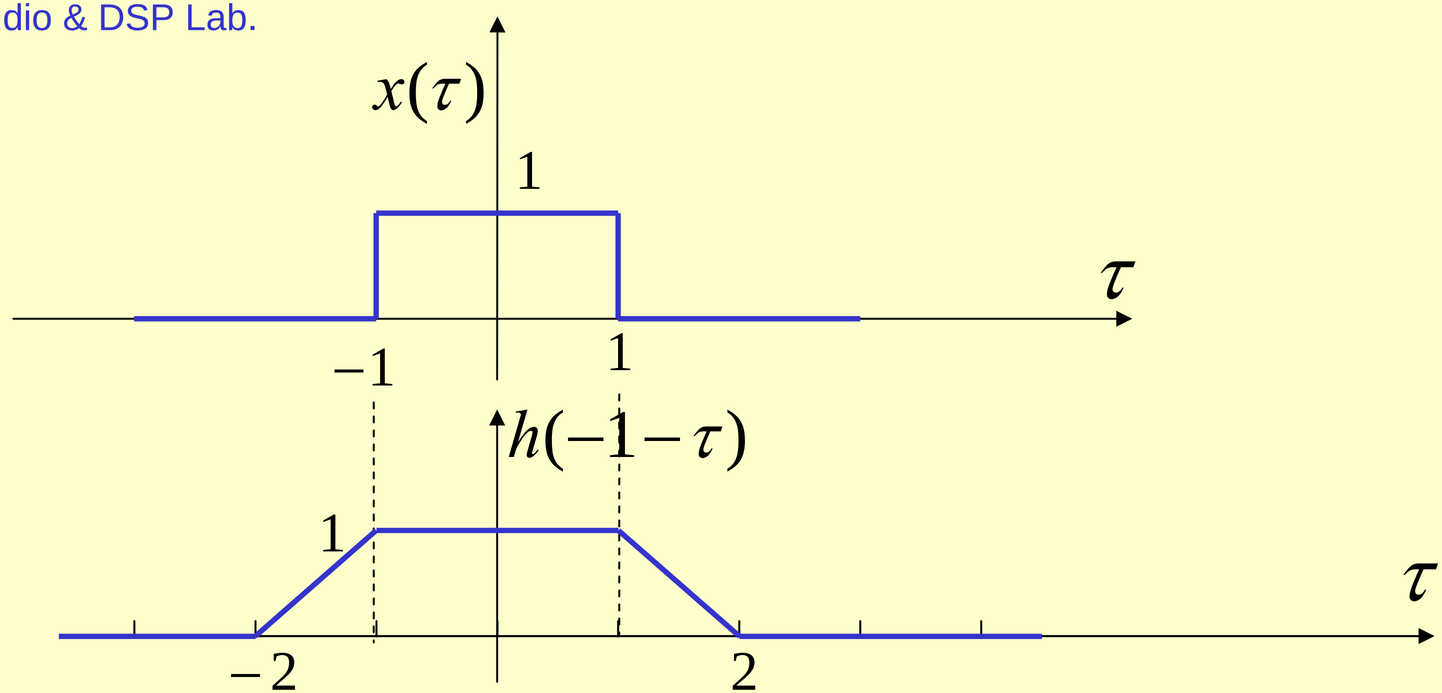
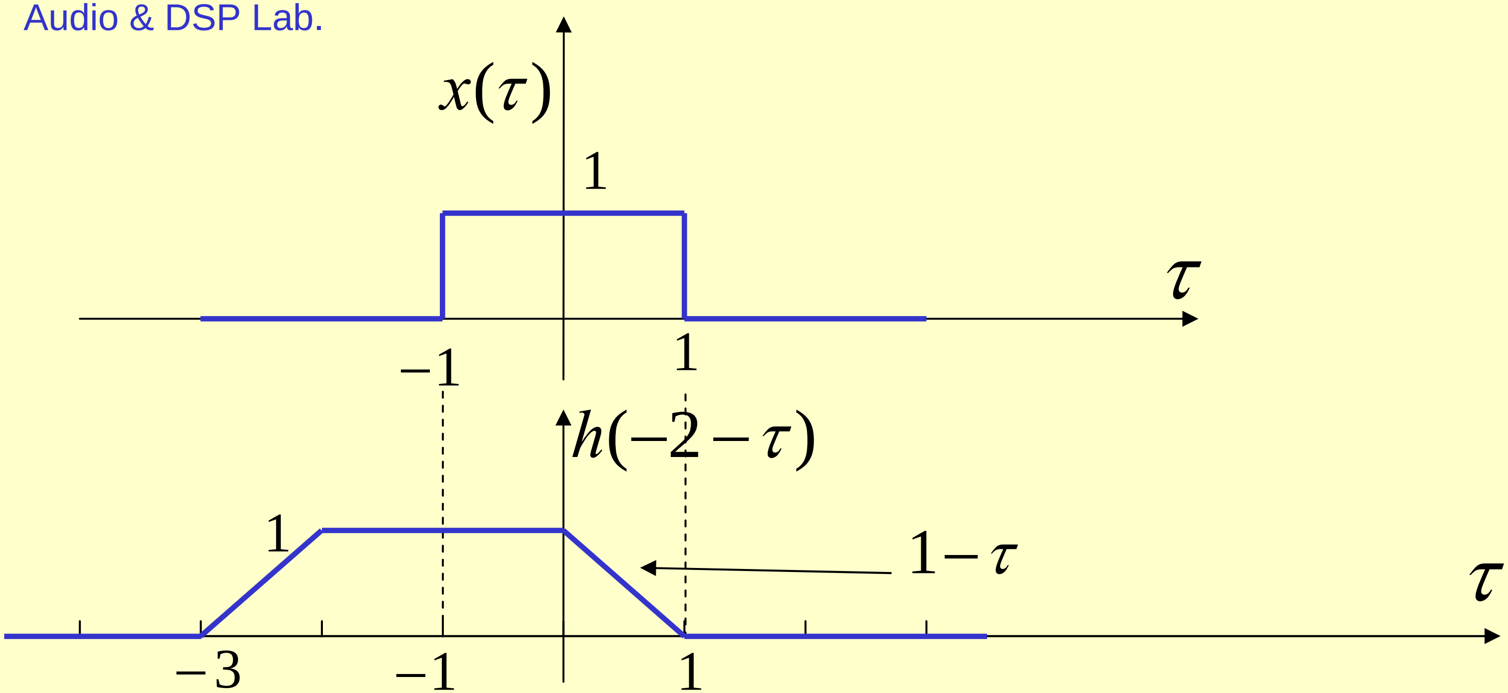



## 簡易代入

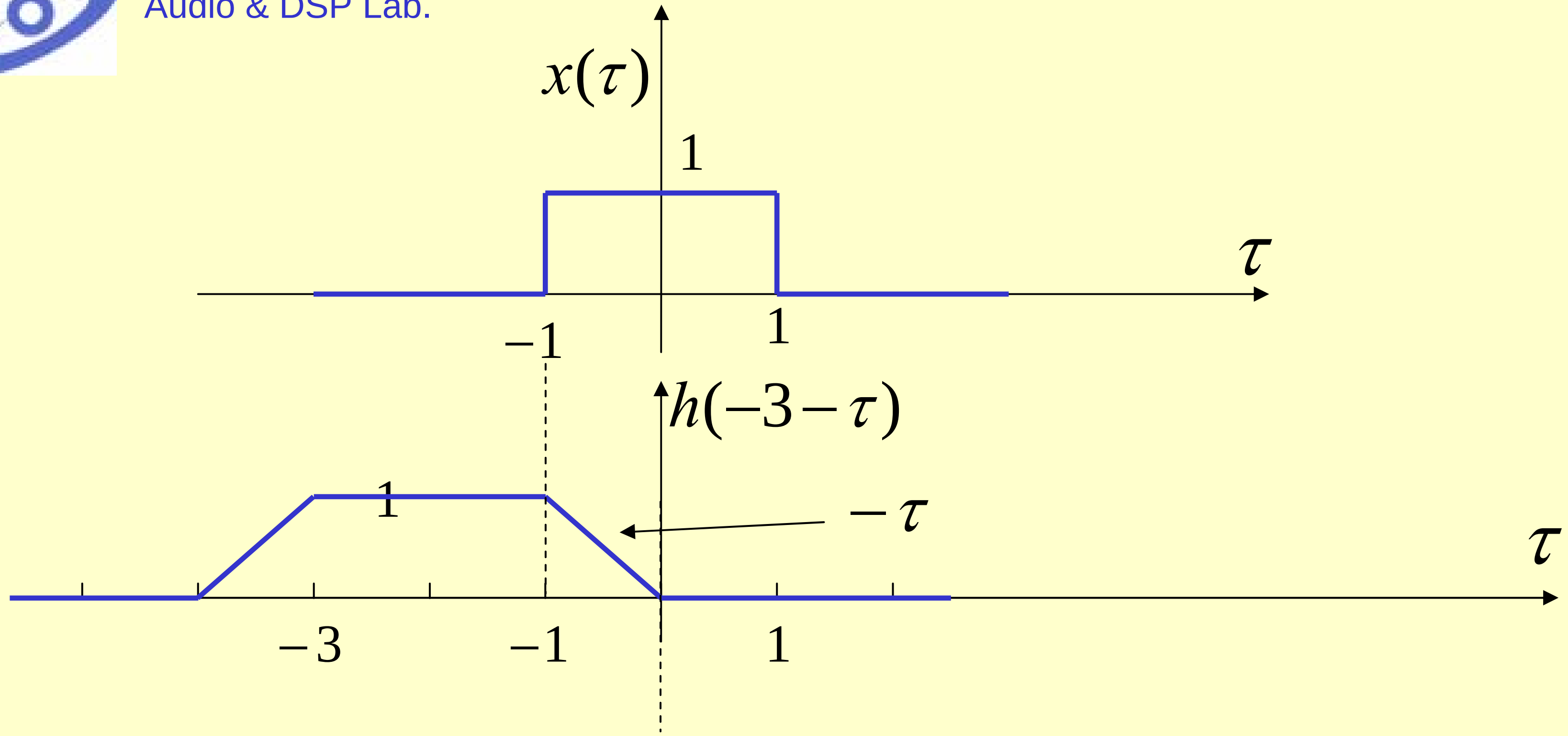
$$y(-1)=\int_{-\infty}^{+\infty} x(\tau)\,h(-1-\tau)\,d\tau=\int_{-1}^{+1} 1\,d\tau=\tau\Big|_{-1}^{+1}=2$$




簡易代入

$$y(-2)=\int_{-\infty}^{+\infty} x(\tau)\,h(-2-\tau)\,d\tau=\int_{-1}^{0} 1\,d\tau+\int_{0}^{1}(1-\tau)\,d\tau$$

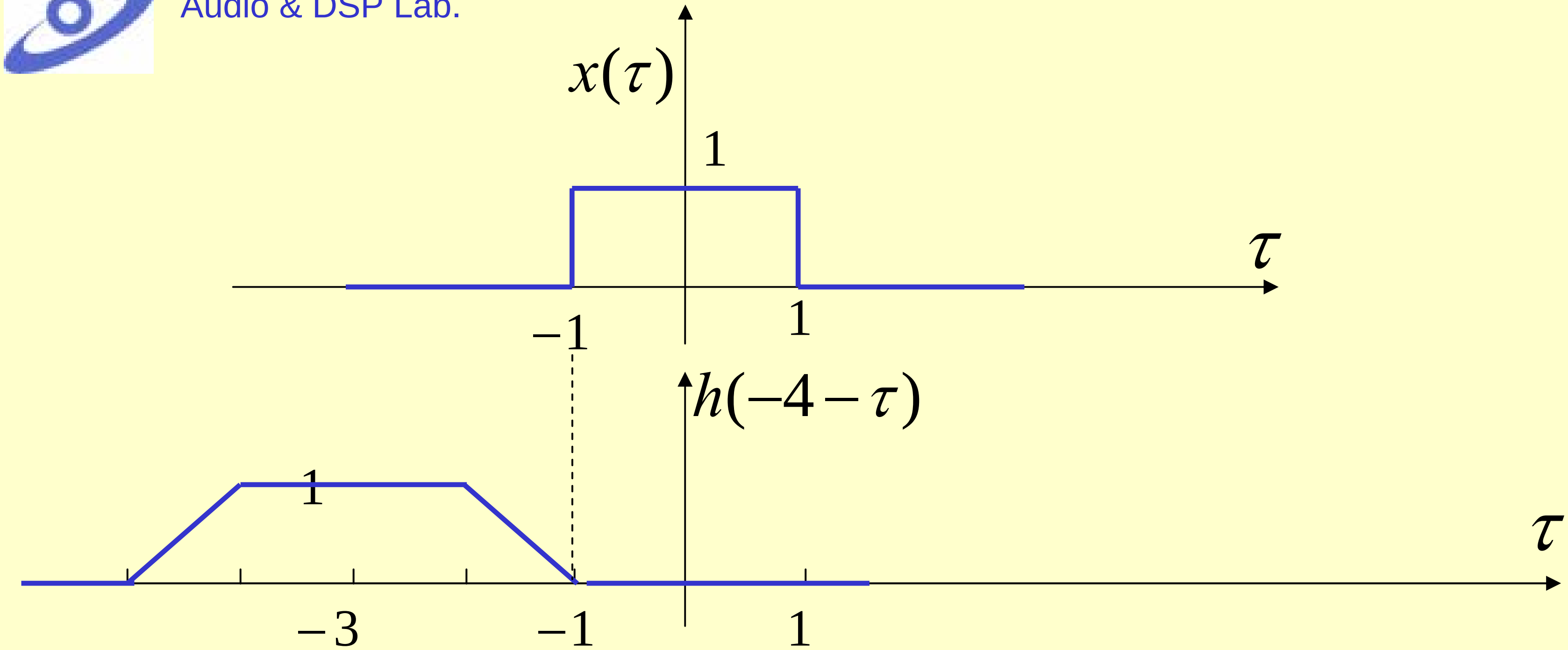
$$=\tau\Big|_{-1}^{0}+(\tau-\frac{\tau^2}{2}\Big|_{0}^{1})=1+\frac{1}{2}=\frac{3}{2}$$




簡易代入

$$y(-3)=\int_{-\infty}^{+\infty} x(\tau)\, h(-3-\tau)\, d\tau = -\int_{-1}^{0} \tau \, d\tau$$

$$= -\frac{\tau^2}{2}\bigg|_{-1}^{0} = \frac{1}{2}$$




簡易代入

$$y(-4) = \int_{-\infty}^{+\infty} x(\tau)\, h(-4-\tau)\, d\tau = 0$$



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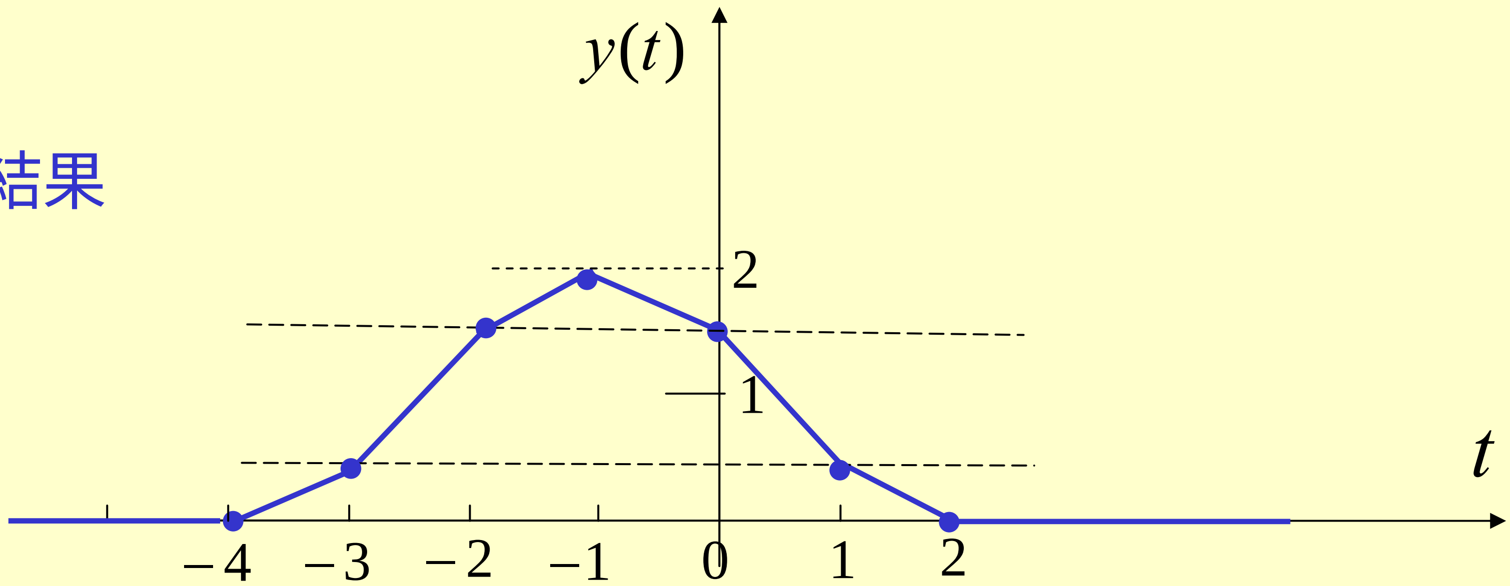
結果



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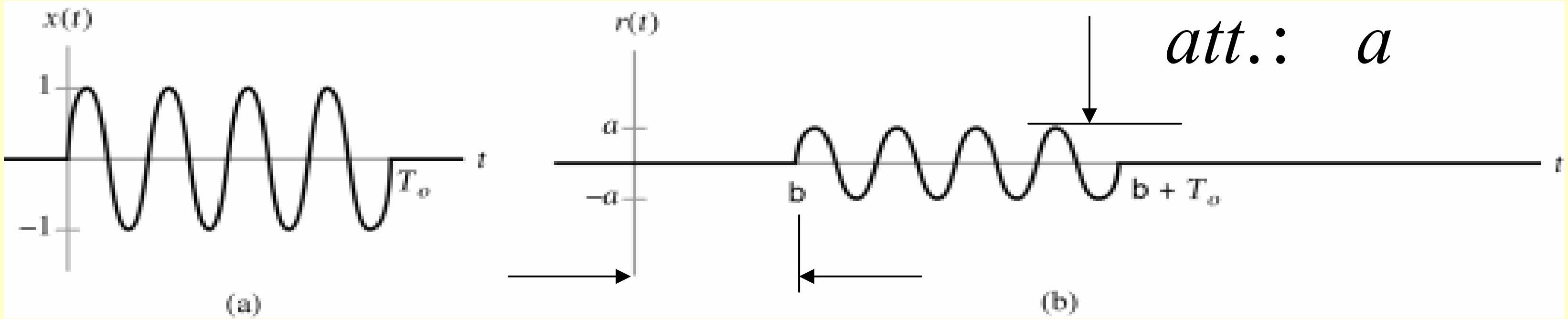
# Radar range measurement

(a) Transmitted RF pulse.

(b) The received echo is an *attenuated* and *delayed* version of the transmitted pulse.

$$r(t) = a \cdot x(t - \beta)$$




$$delay: \quad \beta$$

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Radar sends an impulse “$\delta(t)$” to target, and the impulse response is an attenuated “$a$” and delayed “$\beta$”:

$$h(t) = a\delta(t-\beta)$$

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$$\because h(t) = a\delta(t-\beta), \qquad \therefore h(\tau) = a\delta(\tau-\beta)$$

$$h(-\tau) = a\delta(-\tau-\beta) = a\delta(-(\tau+\beta));$$

$$r(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

$$= \int_{-\infty}^{+\infty} x(\tau)\left(a\delta\left(t-(\tau+\beta)\right)\right)d\tau$$

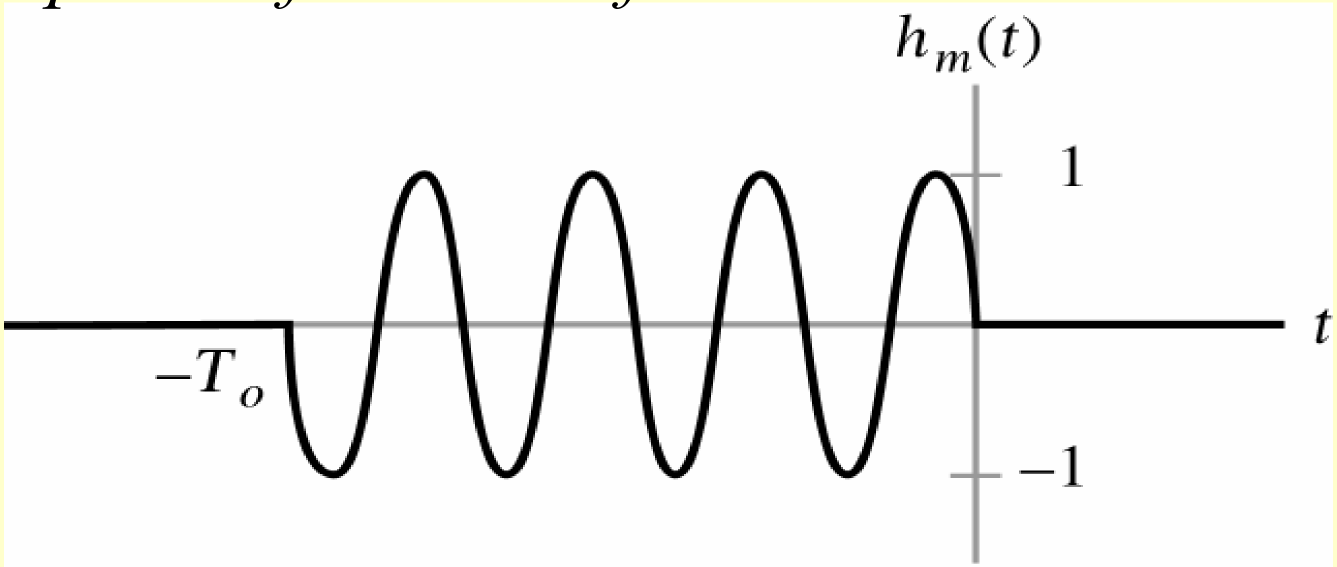
$$= a\int_{-\infty}^{+\infty} x(\tau)\left[\delta\left(-\tau+(t-\beta)\right)\right]d\tau = a\,x(t-\beta)$$

# Match Filter for Radar Range Measurement

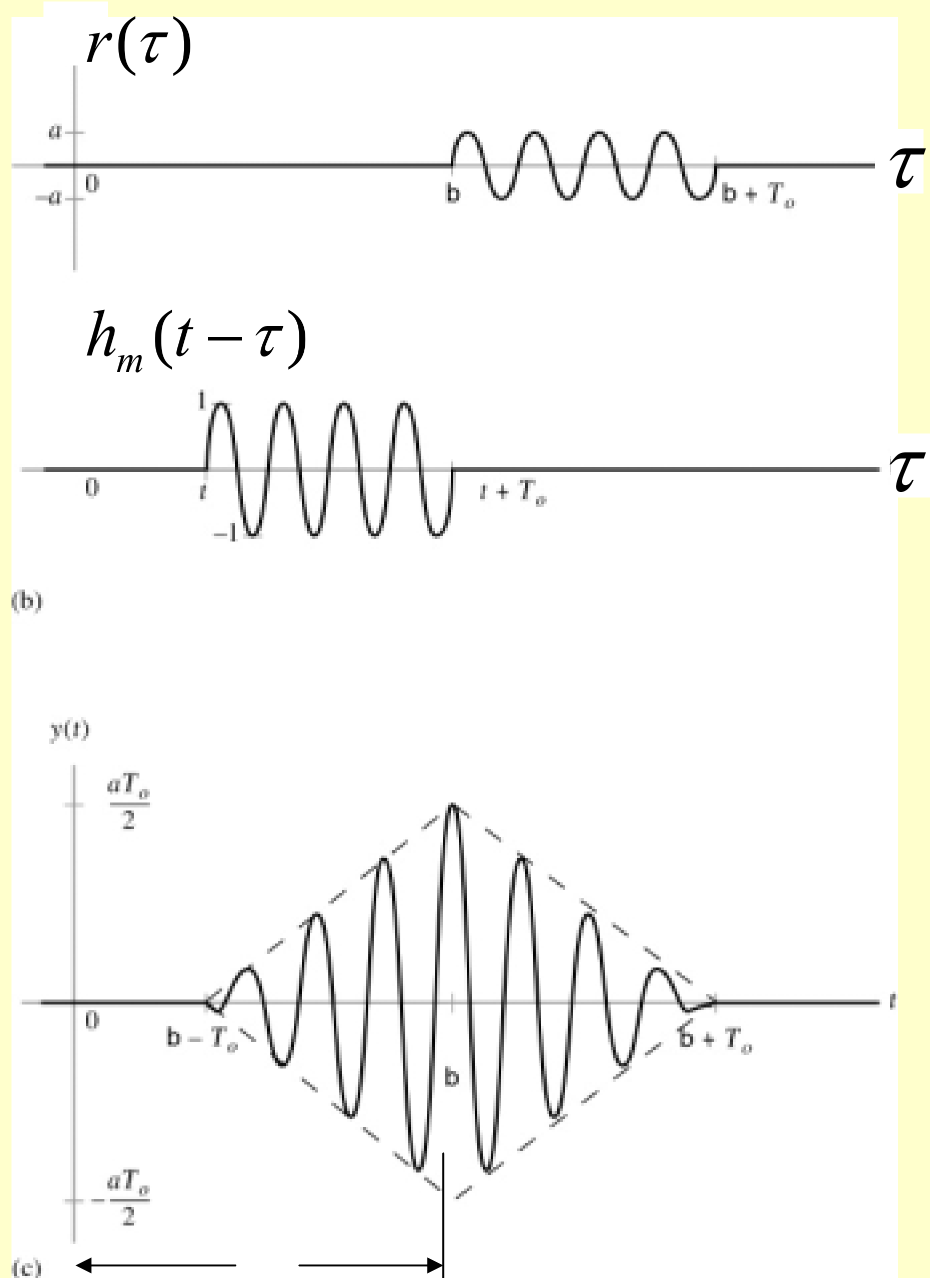
(a) Impulse response of the matched filter for processing the received signal.

*The impulse response of a match filter :*

$$h_m(t) = x(-t)$$




(a)




偵測出 β 值

(b)
The received signal $r(\tau)$ time-shifted matched filter impulse response $h_m(t - \tau)$

(c)
Matched filter output y($t$).

$$y(t) = r(t) * h_m(t)$$

$$= \int_{-\infty}^{+\infty} r(\tau) h_m(t-\tau)\, d\tau$$