Fourier Transform of Signals

訊號的傅立葉轉換

Lecture 3-2

Introduction

- 訊號表示為一組複數弦波的加權疊加
- 把複雜的訊號看成頻率的函數
- 傅立葉表示法
  - 連續時間傅立葉級數 (週期性) : FS
  - 連續時間傅立葉轉換 (非週期性) : FT
  - 離散時間傅立葉級數 (週期性) : DTFS
  - 離散時間傅立葉轉換 (非週期性) : DTFT

Discrete-Time Periodic Signals (DTPS):

The Discrete-Time Fourier Series (DTFS)

基本週期 N 基本頻率 $\Omega_0 = 2\pi/N$ 的週期訊號 x[n] :

X[k] is discrete spectrum & X[k] is periodic function.

$$x[n] = \sum_{k=0}^{N-1} X[k] e^{jk\Omega_0 n} \qquad (1)$$

x[n] is discrete-time signal and x[n] is periodic signal.

$$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk\Omega_0 n} \qquad (2)$$

Orthogonal Property

$$\sum_{n=0}^{N-1} e^{-jk\Omega n} \cdot \sum_{m=0}^{N-1} e^{jm\Omega n} = \sum_{n=0}^{N-1} \cdot \sum_{m=0}^{N-1} e^{-jk\Omega n} \cdot e^{jm\Omega n}$$

$$= \sum_{n=0}^{N-1} \cdot \sum_{m=0}^{N-1} e^{j(m-k)\Omega n}$$

$$if \;\; m = k,$$

$$= \sum_{n=0}^{N-1} \cdot \sum_{m=k}^{k} e^{j(0)\Omega n} = \sum_{n=0}^{N-1} 1 = N$$

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$$\sum_{n=0}^{N-1} e^{-jk\frac{2\pi}{N}n} \cdot \sum_{m=0}^{N-1} e^{jm\frac{2\pi}{N}n} = \sum_{n=0}^{N-1} \cdot \sum_{m=0}^{N-1} e^{-jk\frac{2\pi}{N}n} \cdot e^{jm\frac{2\pi}{N}n}$$

$$= \sum_{n=0}^{N-1} \cdot \sum_{m=0}^{N-1} e^{j(m-k)\frac{2\pi}{N}n}$$

$if \;\; m \neq k, \quad let \quad l = m - k; \quad The \;\; l \;\; is \; an$ integer.

$$= \sum_{n=0}^{N-1} \cdot \sum_{m=0}^{N-1} e^{j\frac{2\pi l}{N}n} = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \left(e^{j\frac{2\pi l}{N}} \right)^{n} = \sum_{m=0}^{N-1} \frac{1 - e^{j\frac{2\pi l}{N}N}}{1 - e^{j\frac{2\pi l}{N}}}$$

$$= \sum_{m=0}^{N-1} \frac{1 - e^{j2\pi l}}{1 - e^{j\frac{2\pi l}{N}}} = \sum_{m=0}^{N-1} \frac{1-1}{1 - e^{j\frac{2\pi l}{N}}} = 0$$

$$\because \quad given \quad x[n]=\sum_{m=0}^{N-1} X[m]e^{jm\Omega_0 n}$$

$$\frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\Omega_0 n} = \frac{1}{N}\sum_{n=0}^{N-1}\left\{\sum_{m=0}^{N-1} X[m]e^{jm\Omega_0 n}\right\}e^{-jk\Omega_0 n}$$

$$=\frac{1}{N}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} X[m]\,e^{j(m-k)\Omega_0 n}$$

$$=\begin{cases}\dfrac{1}{N}\displaystyle\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} X[m]\,e^{j(m-k)\Omega_0 n}=0, & \forall k\neq m\\ \dfrac{1}{N}X[k]\displaystyle\sum_{n=0}^{N-1} 1 = X[k], & \forall k=m\end{cases}$$

$$x[n] \xleftrightarrow{DTFS;\quad \Omega_0} X[k]$$

- X[k] 被稱為 x[n] 的頻域表示法，
- X[k] 和 x[n] 皆以有限 N 個數表示週期，
- DTFS是 唯 一可用電腦運算的傅立葉表示法，
- 複數弦波 $e^{jk\Omega_0 n}$ 只需有 N 個。 (N 週期函數)

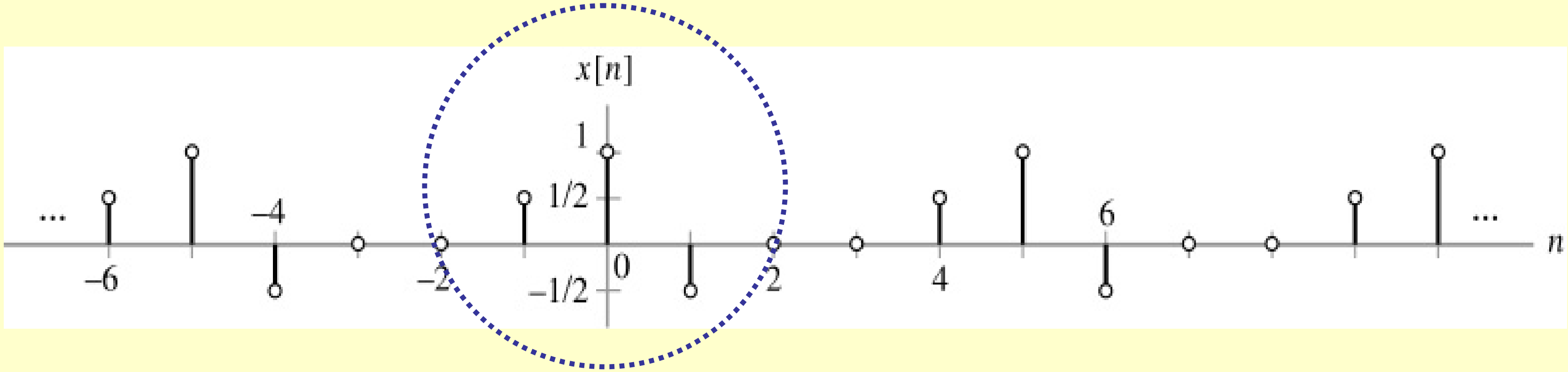
$$i.e.,\ e^{j(k+N)\Omega_0 n} = e^{jN\Omega_0 n} e^{jk\Omega_0 n} = e^{jN\frac{2\pi}{N}n} e^{jk\Omega_0 n}$$

$$= e^{j2\pi n} e^{jk\Omega_0 n} = e^{jk\Omega_0 n}$$

Example 3.2

試以下圖求出訊號 DTFS 係數 X[k] = ?

週期 N = ? 如何選取 那一個週期區間？ 優、缺點？



Time-domain signal for Example 3.2.

solution:

$$\because \quad N=5, \quad \therefore \quad \Omega_0 = {2\pi}/{5},$$

$$\because \quad x[n]\ is\ odd\ symmetric\ signal,\ \underline{select\ n=-2\ to\ +2}:$$

$$X[k]=\frac{1}{5}\sum_{n=-2}^{+2} x[n]e^{-jk\frac{2\pi}{5}n}$$

$$=\frac{1}{5}\left[0\cdot e^{-jk\frac{2\pi}{5}(-2)}+\frac{1}{2}\cdot e^{-jk\frac{2\pi}{5}(-1)}+1\cdot e^{-jk\frac{2\pi}{5}(0)}-\frac{1}{2}\cdot e^{-jk\frac{2\pi}{5}(1)}+0\cdot e^{-jk\frac{2\pi}{5}(2)}\right]$$

$$=\frac{1}{5}\left[\frac{1}{2}\cdot e^{jk\frac{2\pi}{5}}+1-\frac{1}{2}\cdot e^{-jk\frac{2\pi}{5}}\right]=\frac{1}{5}\left[1+\frac{1}{2}\cdot e^{jk\frac{2\pi}{5}}-\frac{1}{2}\cdot e^{-jk\frac{2\pi}{5}}\right]$$

$$=\frac{1}{5}\left[1+j\frac{e^{jk\frac{2\pi}{5}}-e^{-jk\frac{2\pi}{5}}}{j2}\right]=\frac{1}{5}\left[1+j\sin\left({2\pi k}/{5}\right)\right]$$

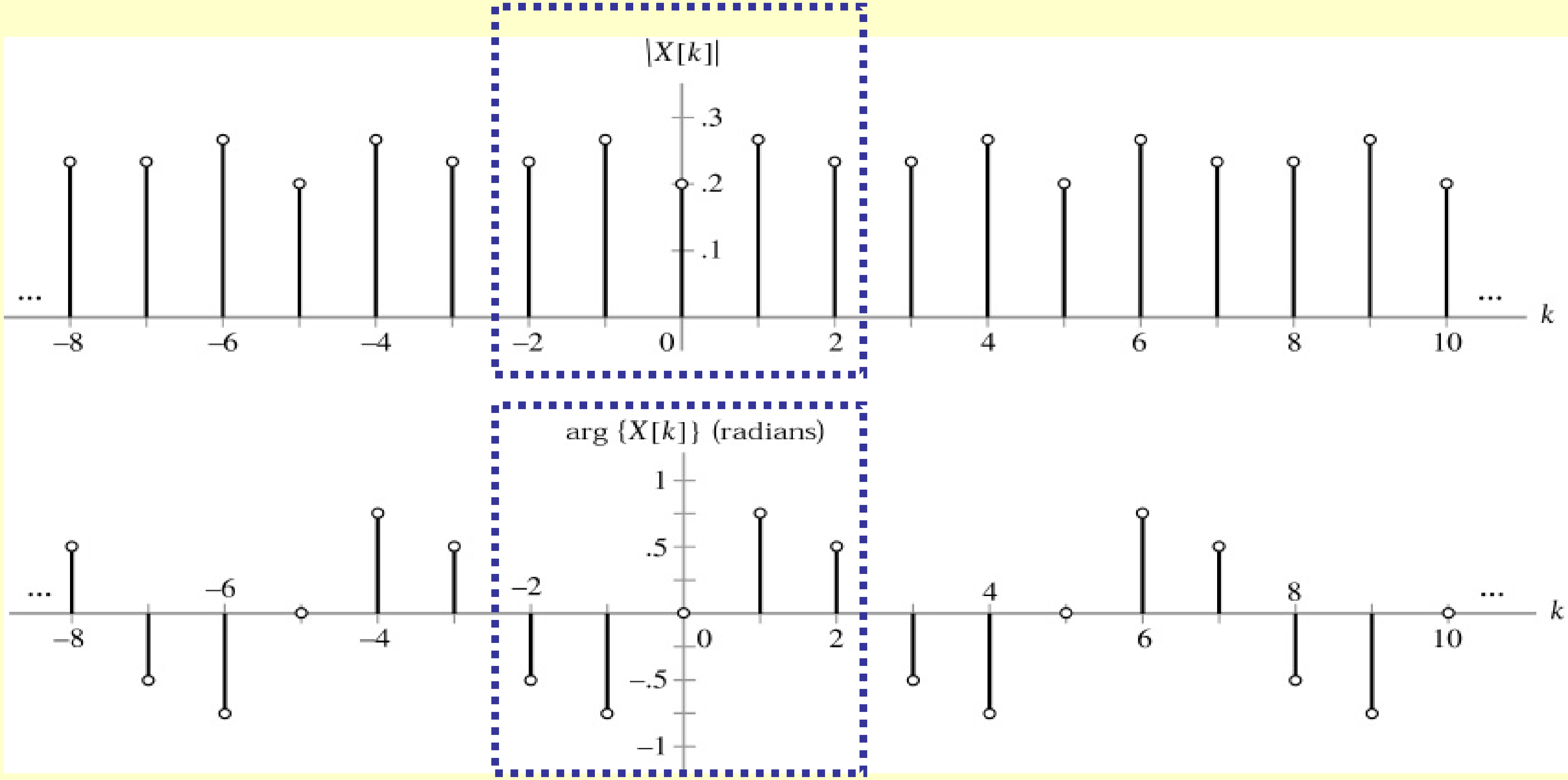
$$\because\ X[k]=\frac{1}{5}\left[1+j\sin\left(2\pi k/5\right)\right]$$

$\therefore$ *Magnitude Spectrum*:

$$\left|X(k)\right|=\frac{1}{5}\left\{\sqrt{1+\sin^2\left(2\pi k/5\right)}\right\}$$

$\therefore$ *Phase Spectrum*:

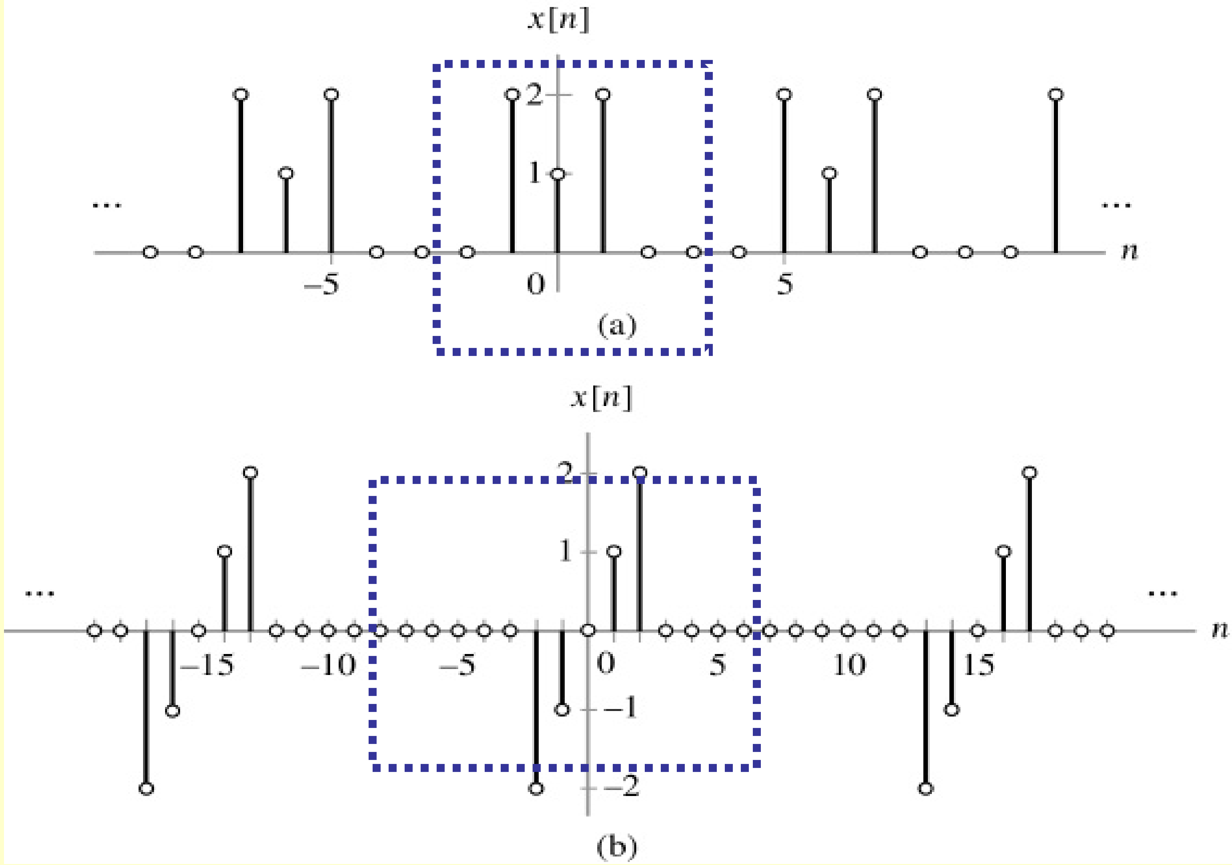
$$\arg\left\{X(k)\right\}=\tan^{-1}\left\{\sin\left(2\pi k/5\right)\right\}$$



Magnitude and phase of the DTFS coefficients for the signal in Fig. 3.5 are periodic with period of N=5.

Problem 3.2: 試求下圖 (a) 和 (b) DTFS 係數 = ?

學生試一試 ？



$N = 6$

$N = 15$

Example 3.3 : 用審視法 (Method of Inspection) 求出 $x[n] = \cos(\pi n/3+\phi)$ 的 DTFS 係數 = ?

$$solution:$$

$$\because \Omega_0 = \frac{2\pi}{N} = \frac{\pi}{3}, \quad \therefore N = 6.$$

$$x[n] = \cos(\pi n/3+\phi) = \frac{1}{2}\left\{ e^{j\left(\frac{\pi n}{3}+\phi\right)} + e^{-j\left(\frac{\pi n}{3}+\phi\right)} \right\}$$

$$= \frac{1}{2} e^{j\phi} e^{j\frac{\pi n}{3}} + \frac{1}{2} e^{-j\phi} e^{-j\frac{\pi n}{3}}$$

$$solution:(cont.)$$

$$x[n]=\frac{1}{2}e^{j\phi}e^{j\frac{\pi n}{3}}+\frac{1}{2}e^{-j\phi}e^{-j\frac{\pi n}{3}}$$

$comparing:....$ N = 6; n = -2 ~ +3

$$x[n]=\sum_{k=-2}^{3}X[k]\,e^{jk\frac{\pi}{3}n}$$

$$=X[-2]e^{-j\frac{2\pi}{3}n}+X[-1]e^{-j\frac{\pi}{3}n}+X[0]$$

$$+X[1]e^{j\frac{\pi}{3}n}+X[2]e^{j\frac{2\pi}{3}n}+X[3]e^{j\frac{3\pi}{3}n}$$

$$\therefore$$

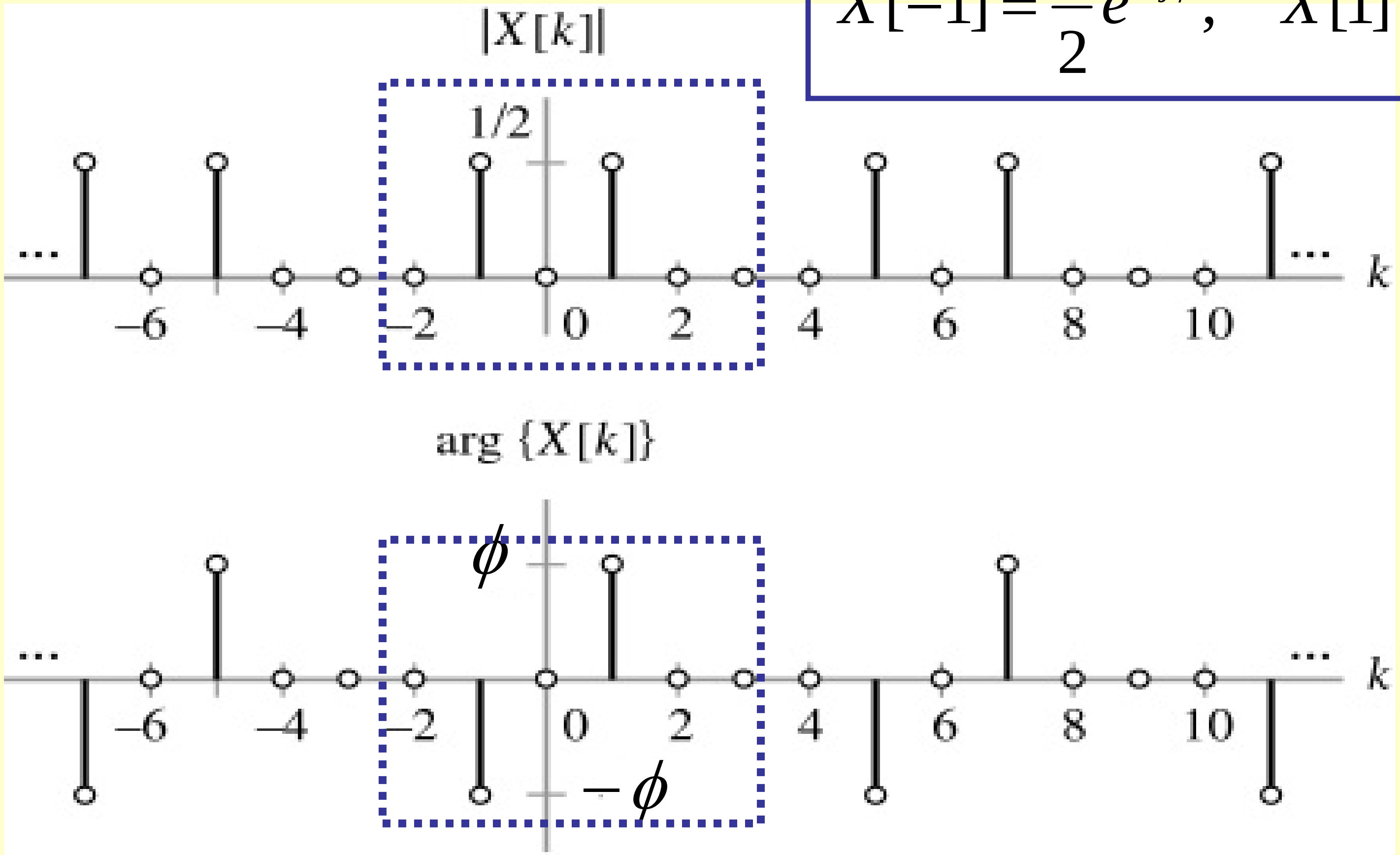
$$X[-2]=X[0]=X[2]=X[3]=0,$$

$$X[-1]=\frac{1}{2}e^{-j\phi},\quad X[1]=\frac{1}{2}e^{j\phi}$$

Magnitude and phase of DTFS coefficients for Example 3.3.

$$X[-2] = X[0] = X[2] = X[3] = 0,$$

$$X[-1] = \frac{1}{2}e^{-j\phi}, \quad X[1] = \frac{1}{2}e^{j\phi}$$



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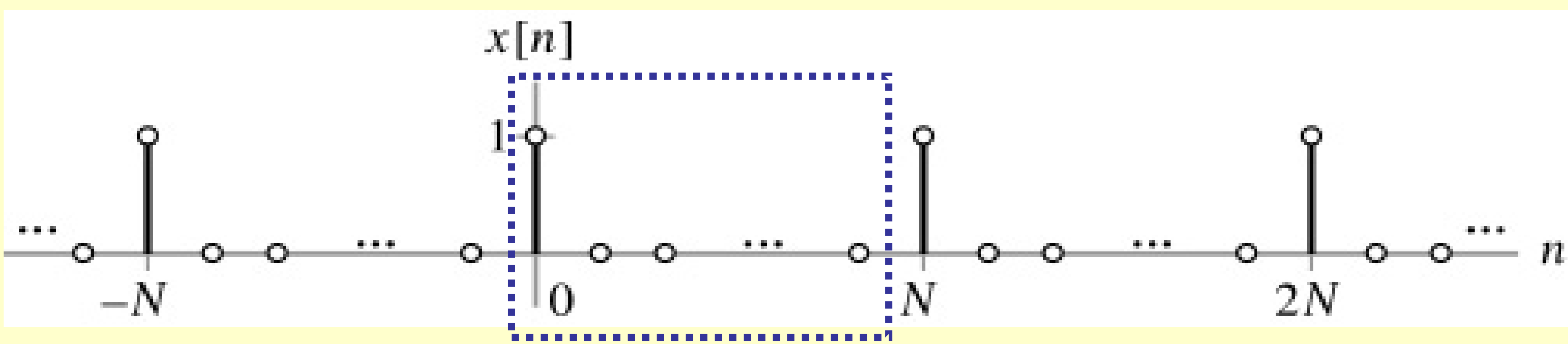
Find the Magnitude and phase of DTFS coefficients for Example 3.4.

離散時間脈衝列

$$x[n]=\sum_{m=-\infty}^{+\infty}\delta\left[n-mN\right]$$



A discrete-time impulse train with period N.

$$\because\ x[n] = \sum_{m=-\infty}^{+\infty} \delta[n - mN]$$

$$\therefore$$

$$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n]\, e^{-jk\frac{2\pi}{N}n}$$

Take one cycle:

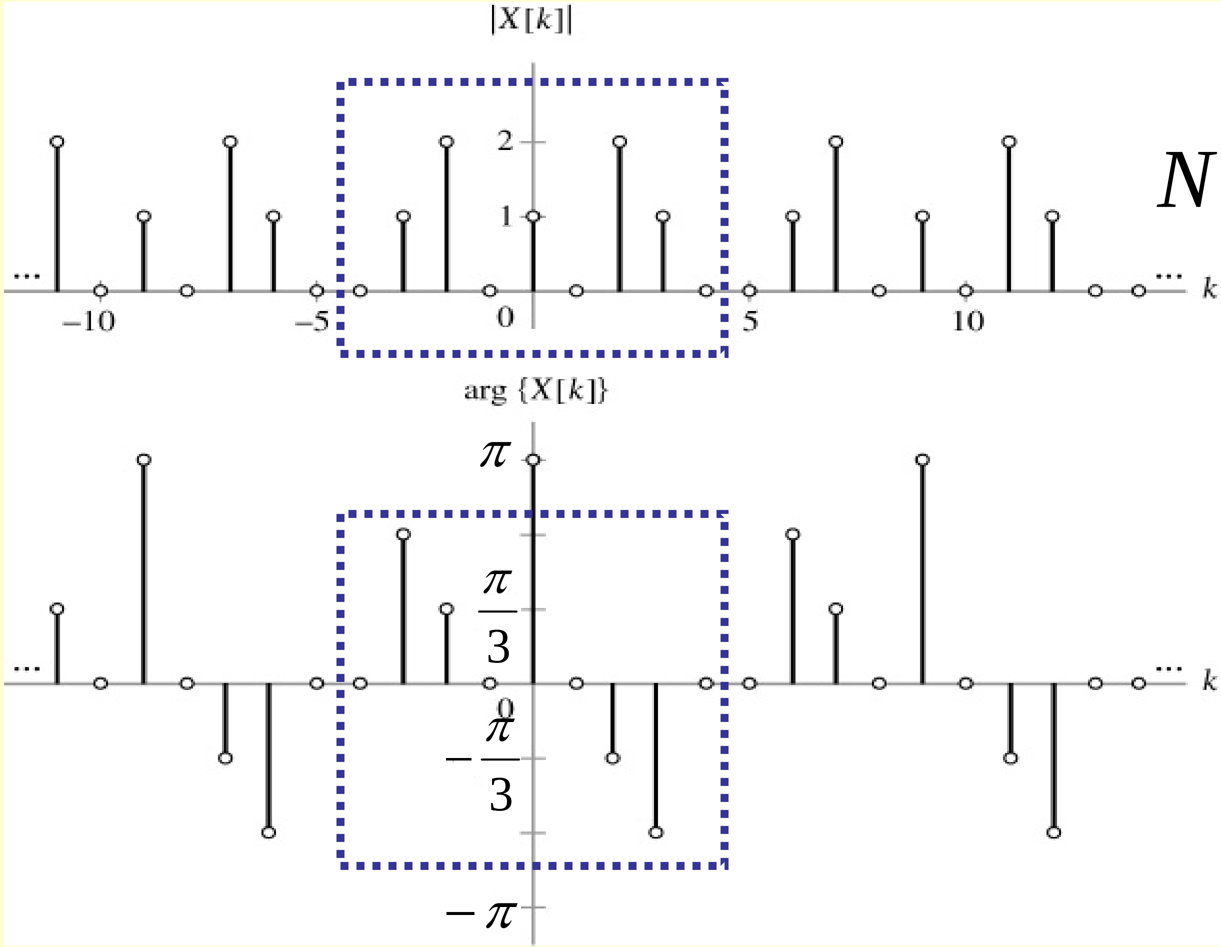
$$= \frac{1}{N} \sum_{n=0}^{N-1} \sum_{m=-\infty}^{+\infty} \delta[n - mN]\, e^{-jkn2\pi/N} = \frac{1}{N} \sum_{n=0}^{N-1} \delta[n]\, e^{-jkn2\pi/N}$$

$$= \frac{1}{N} \left\{ \delta[0] + \delta[1]\, e^{-jk2\pi/N} + \cdots + \delta[N-1]\, e^{-jk2\pi(N-1)/N} \right\}$$

($\delta[1] \to 0$, $\delta[N-1] \to 0$)

$$= \frac{\delta[0]}{N} = \frac{1}{N}$$

EX 3.5: Given the magnitude and phase of DTFS coefficients, please find x[n] = ?



$N=9$

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$$\because N=9, \quad choose\ n=-4\sim+4,$$

$$x[n]=\sum_{n=-4}^{+4} X[k]e^{jk\frac{2\pi}{9}n},$$

代入已知 X[k] 振幅 和相位值

$$=X[-3]e^{-jn\frac{2\pi}{9}3}+X[-2]e^{-jn\frac{2\pi}{9}2}+X[0]+X[2]e^{jn\frac{2\pi}{9}2}+X[3]e^{jn\frac{2\pi}{9}3}$$

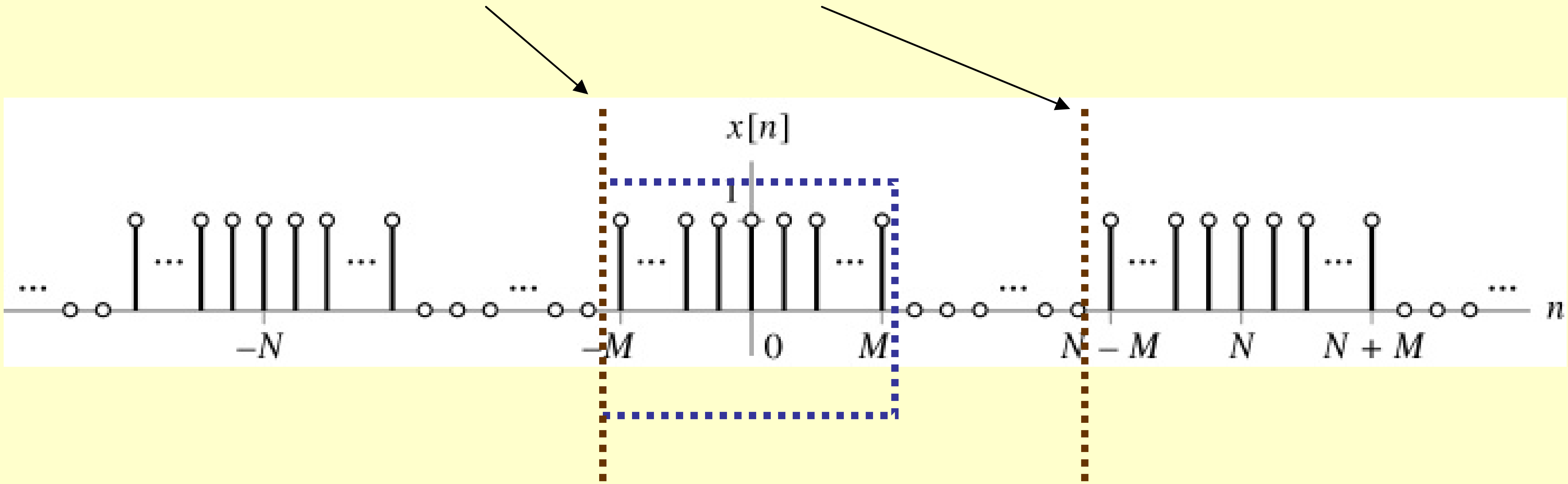
$$=e^{j\frac{2\pi}{3}}e^{-jn\frac{6\pi}{9}}+2e^{j\frac{\pi}{3}}e^{-jn\frac{4\pi}{9}}+e^{-j\pi}+2e^{-j\frac{\pi}{3}}e^{jn\frac{4\pi}{9}}+e^{-j\frac{2\pi}{3}}e^{jn\frac{6\pi}{9}}$$

$$=\left(e^{-j(n\frac{6\pi}{9}-\frac{2\pi}{3})}+e^{j(n\frac{6\pi}{9}-\frac{2\pi}{3})}\right)+(-1)+2\left(e^{-j(n\frac{4\pi}{9}-\frac{\pi}{3})}+e^{j(n\frac{4\pi}{9}-\frac{\pi}{3})}\right)$$

$$=2\cos\left(n\frac{6\pi}{9}-\frac{2\pi}{3}\right)+4\cos\left(n\frac{4\pi}{9}-\frac{\pi}{3}\right)-1$$

EX 3.6: Find the DTFS coefficients of the square wave, or X[k] = ?

$$choose \quad n = -M \quad \sim \quad (N - M - 1)$$



Square wave for Example 3.6.

Applying DTFS Equation (2)

$$choose \quad n = -M \sim N - M - 1$$

$$X[k] = \frac{1}{N} \sum_{n=-M}^{N-M-1} x[n] e^{-jk\frac{2\pi}{N}n} = \frac{1}{N} \sum_{n=-M}^{M} e^{-jk\frac{2\pi}{N}n}$$

$$let \ m = n + M,$$

$$X[k] = \frac{1}{N} \sum_{m=0}^{2M} e^{-jk\frac{2\pi}{N}(m-M)} = \frac{1}{N} e^{jk\frac{2\pi}{N}M} \sum_{m=0}^{2M} e^{-jk\frac{2\pi}{N}m}$$

$$= \frac{1}{N} e^{jk\frac{2\pi}{N}M} \left(1 + e^{-jk\frac{2\pi}{N}} + e^{-jk\frac{4\pi}{N}} + e^{-jk\frac{6\pi}{N}} + \cdots + e^{-jk\frac{4M\pi}{N}} \right)$$

$$if \quad k = 0, \pm N, \pm 2N, \ldots$$

$$X[k] = \frac{1}{N} e^{jk\frac{2\pi}{N}M} \left(1 + e^{-jk\frac{2\pi}{N}} + e^{-jk\frac{4\pi}{N}} + \cdots + e^{-jk\frac{4M\pi}{N}} \right)$$

$$= \frac{1}{N}(1 + 1 + 1 + \cdots + 1)$$

$$= \frac{1}{N}\left(\sum_{m=0}^{2M} 1\right) = \frac{2M+1}{N}$$

$$if \quad k = otherwise,$$

$$X[k] = \frac{1}{N} e^{jk\frac{2\pi}{N}M} \left(1 + e^{-jk\frac{2\pi}{N}} + e^{-jk\frac{4\pi}{N}} + \cdots + e^{-jk\frac{4M\pi}{N}} \right)$$

$$= \frac{1}{N} e^{jk\frac{2\pi}{N}M} \left(\sum_{m=0}^{2M} e^{-jk\frac{2\pi}{N}m} \right)$$

$$= \frac{1}{N} e^{jk\frac{2\pi}{N}M} \left(\frac{1 - e^{-jk\frac{2\pi}{N}(2M+1)}}{1 - e^{-jk\frac{2\pi}{N}}} \right)$$

rewrite :

$$X[k]=\frac{1}{N}e^{jk\frac{2\pi}{N}M}\left(\frac{e^{-jk\frac{\pi(2M+1)}{N}}\left(e^{jk\frac{\pi(2M+1)}{N}}-e^{-jk\frac{\pi(2M+1)}{N}}\right)}{e^{-jk\frac{\pi}{N}}\left(e^{jk\frac{\pi}{N}}-e^{-jk\frac{\pi}{N}}\right)}\right)$$

$$=\frac{1}{N}e^{jk\frac{2\pi}{N}M}\left(\frac{e^{-jk\frac{\pi(2M)}{N}}\left(e^{jk\frac{\pi(2M+1)}{N}}-e^{-jk\frac{\pi(2M+1)}{N}}\right)}{\left(e^{jk\frac{\pi}{N}}-e^{-jk\frac{\pi}{N}}\right)}\right)$$

$$X[k]=\frac{1}{N}\left(\frac{\left(e^{jk\frac{\pi(2M+1)}{N}}-e^{-jk\frac{\pi(2M+1)}{N}}\right)}{\left(e^{jk\frac{\pi}{N}}-e^{-jk\frac{\pi}{N}}\right)}\right)$$

$$=\frac{1}{N}\left(\frac{\left(e^{jk\frac{\pi(2M+1)}{N}}-e^{-jk\frac{\pi(2M+1)}{N}}\right)/\,j2}{\left(e^{jk\frac{\pi}{N}}-e^{-jk\frac{\pi}{N}}\right)/\,j2}\right)=\frac{1}{N}\frac{\sin\left(k\frac{\pi(2M+1)}{N}\right)}{\sin\left(k\frac{\pi}{N}\right)}$$

$\therefore$

$$X[k]=\begin{cases}\dfrac{1}{N}\dfrac{\sin\left(k\dfrac{\pi(2M+1)}{N}\right)}{\sin\left(k\dfrac{\pi}{N}\right)}, & k\neq 0,\quad \pm N,\quad \pm 2N,\ldots\\[2ex] \dfrac{2M+1}{N}, & k=0,\quad \pm N,\quad \pm 2N,\ldots\end{cases}$$

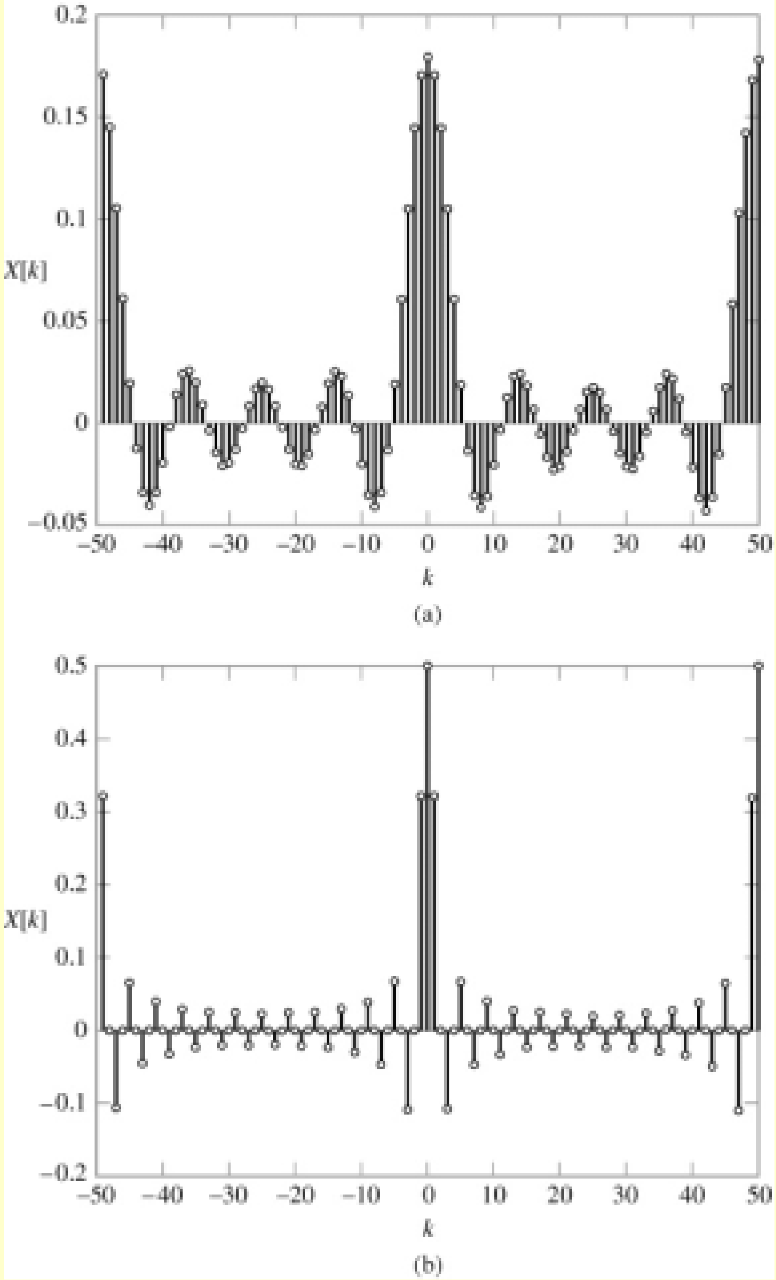
$$Ex:\quad N=50, M=12,\quad \frac{2M+1}{N}=\frac{25}{50}=0.5$$

$$N=50, M=4,\quad \frac{2M+1}{N}=\frac{9}{50}=0.18$$

The DTFS coefficients for the square wave shown in Fig. 3.11, assuming a period $N = 50$:

(a) $M = 4$.

(b) $M = 12$.



** 討論 M 大小不同

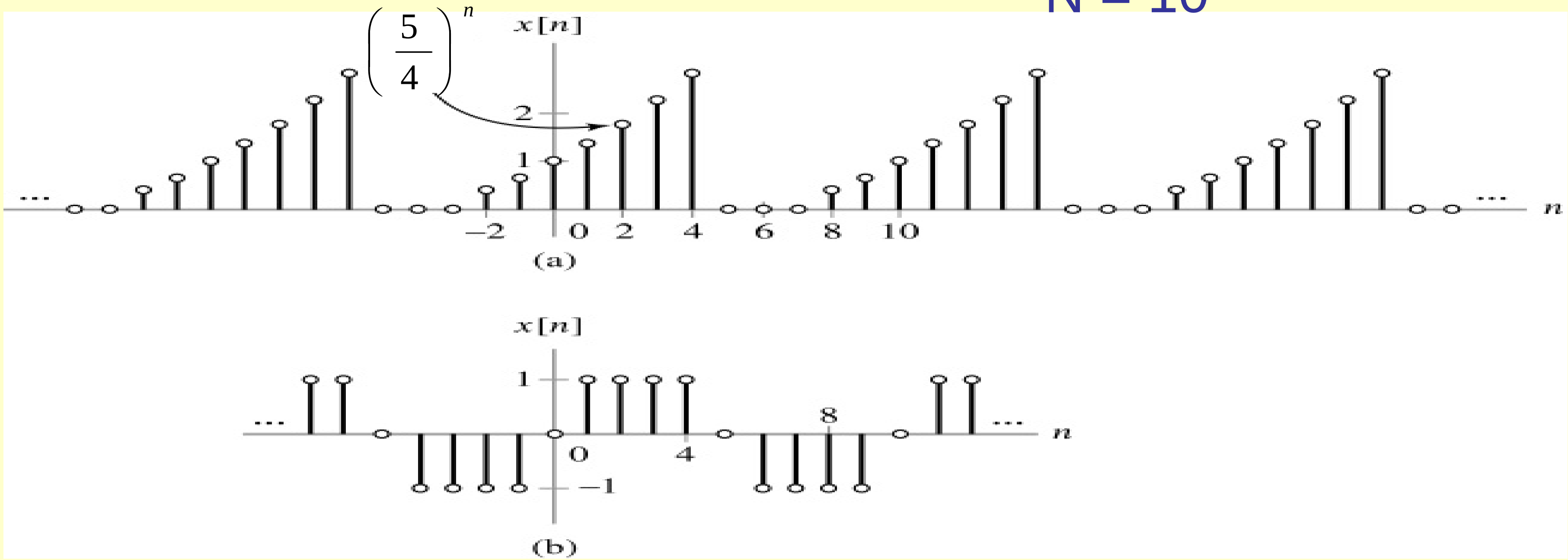
其結果意義為何？

$$Ex: \quad N = 50, M = 12, \quad \frac{2M+1}{N} = \frac{25}{50} = 0.5$$

$$N = 50, M = 4, \quad \frac{2M+1}{N} = \frac{9}{50} = 0.18$$

Problem 3.6: Find the DTFS of the x[n] = ?

N = 10



Signals *x*[*n*] for Problem 3.6.

Example: 3.8 x[n] → X[k] ?

Electrocardiograms for two different heartbeats and the first 60 coefficients of their magnitude spectra.

(a) Normal heartbeat.
(b) Ventricular tachycardia.
(c) Magnitude spectrum for the normal heartbeat.
(d) Magnitude spectrum for ventricular tachycardia.
心室跳動過速
Or 心律不整 (arrhythmia)