Fourier Transform of Signals

訊號的傅立葉轉換

Lecture 3-3

Introduction

- 訊號表示為一組複數弦波的加權疊加
- 把複雜的訊號看成頻率的函數
- 傅立葉表示法
  - 連續時間傅立葉級數 (週期性) : FS
  - 連續時間傅立葉轉換 (非週期性) : FT
  - 離散時間傅立葉級數 (週期性) : DTFS
  - 離散時間傅立葉轉換 (非週期性) : DTFT

Continuous-Time Periodic Signals (CTPS):

The Fourier Series (FS)

基本週期 T， 基本頻率 $\omega_0 = 2\pi/T$ 的週期訊號 x(t) :

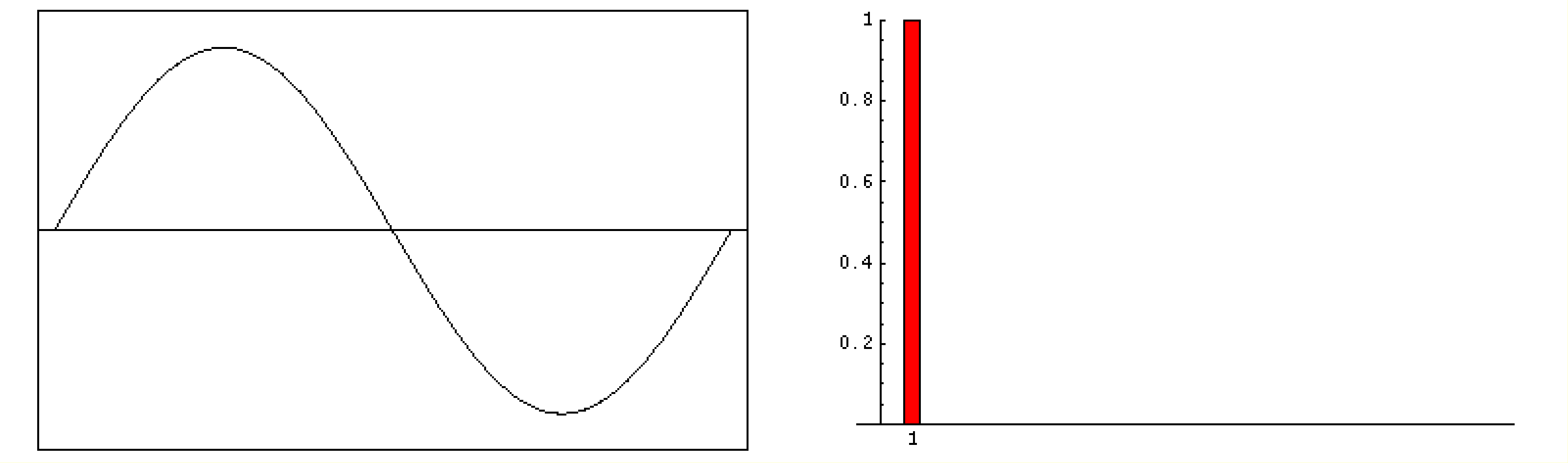
X[k] is discrete spectrum and X[k] is non-periodic.

$$x(t) = \sum_{k=-\infty}^{+\infty} X[k]\, e^{jk\omega_0 t} \qquad (3)$$

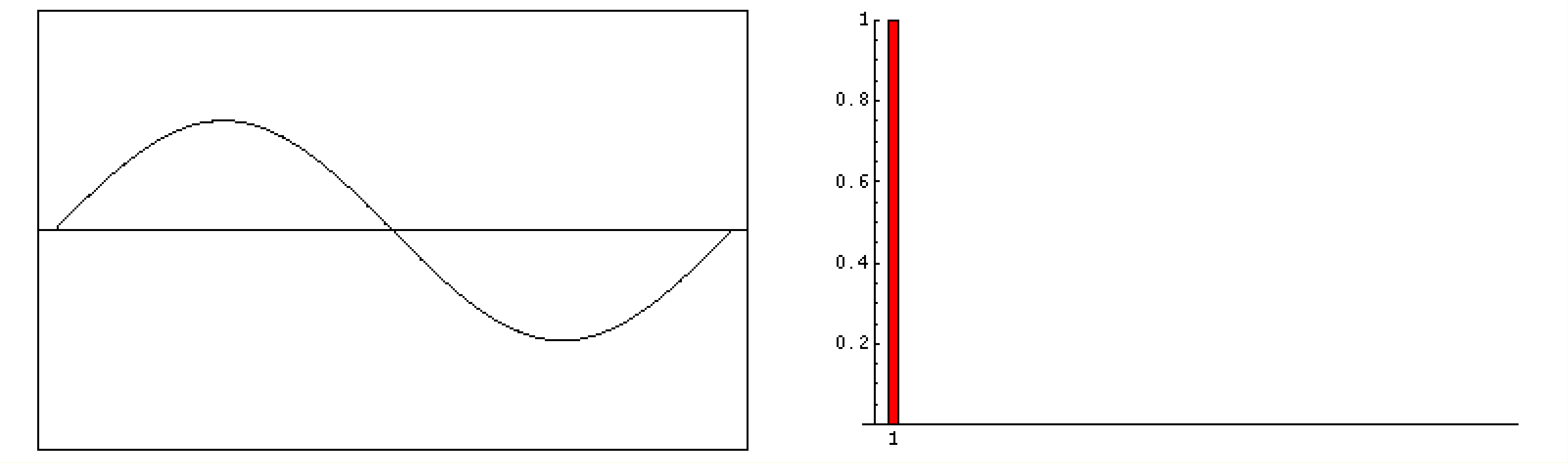
x(t) is continuous signal and x(t) is periodic.

$$X[k] = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt \qquad (4)$$

Square Wave & Spectra

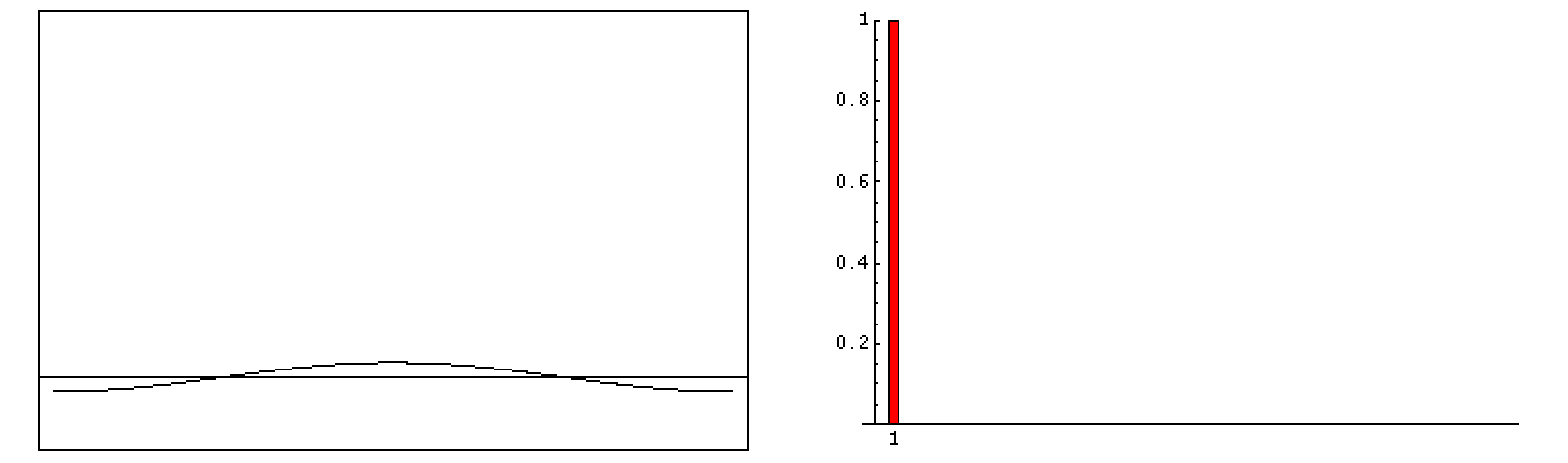


Sawtooth wave & Spectrum



Audio & DSP Lab.

Sinc Wave & Spectrum



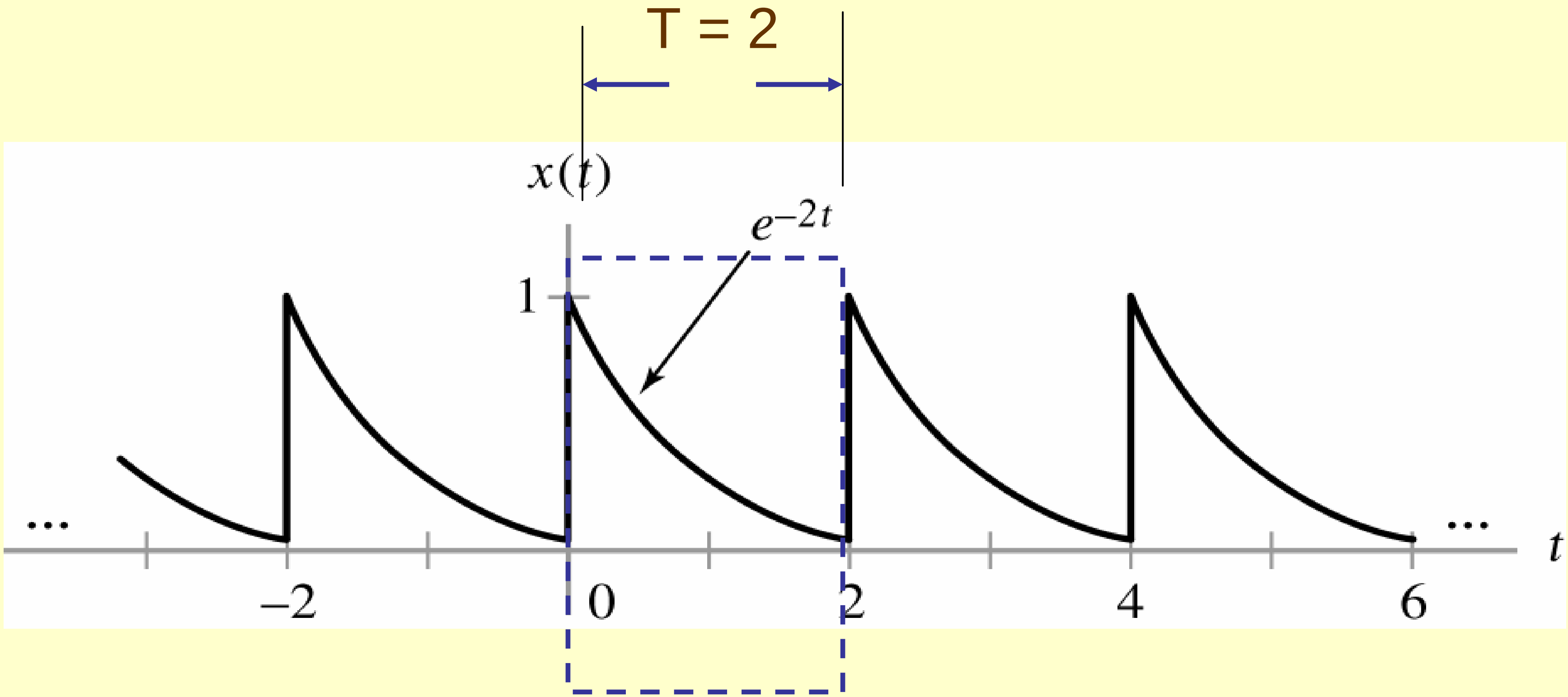
FS – Orthogonal Property

參考 DTFS 範例

請以 正交性特性 推導前頁 FS (4) 公式

Please derive and prove Equation (4)

Example 3.9: Find the FS of the x(t) below:



Time-domain signal for Example 3.9.

Solution for Ex. 3.9

T=2, $\omega_0 = 2\pi/T = 2\pi/2 = \pi$

$$X[k] = \frac{1}{T}\int_0^T e^{-2t}\, e^{-jk\omega_0 t}\, dt = \frac{1}{2}\int_0^2 e^{-2t}\, e^{-jk\pi t}\, dt$$

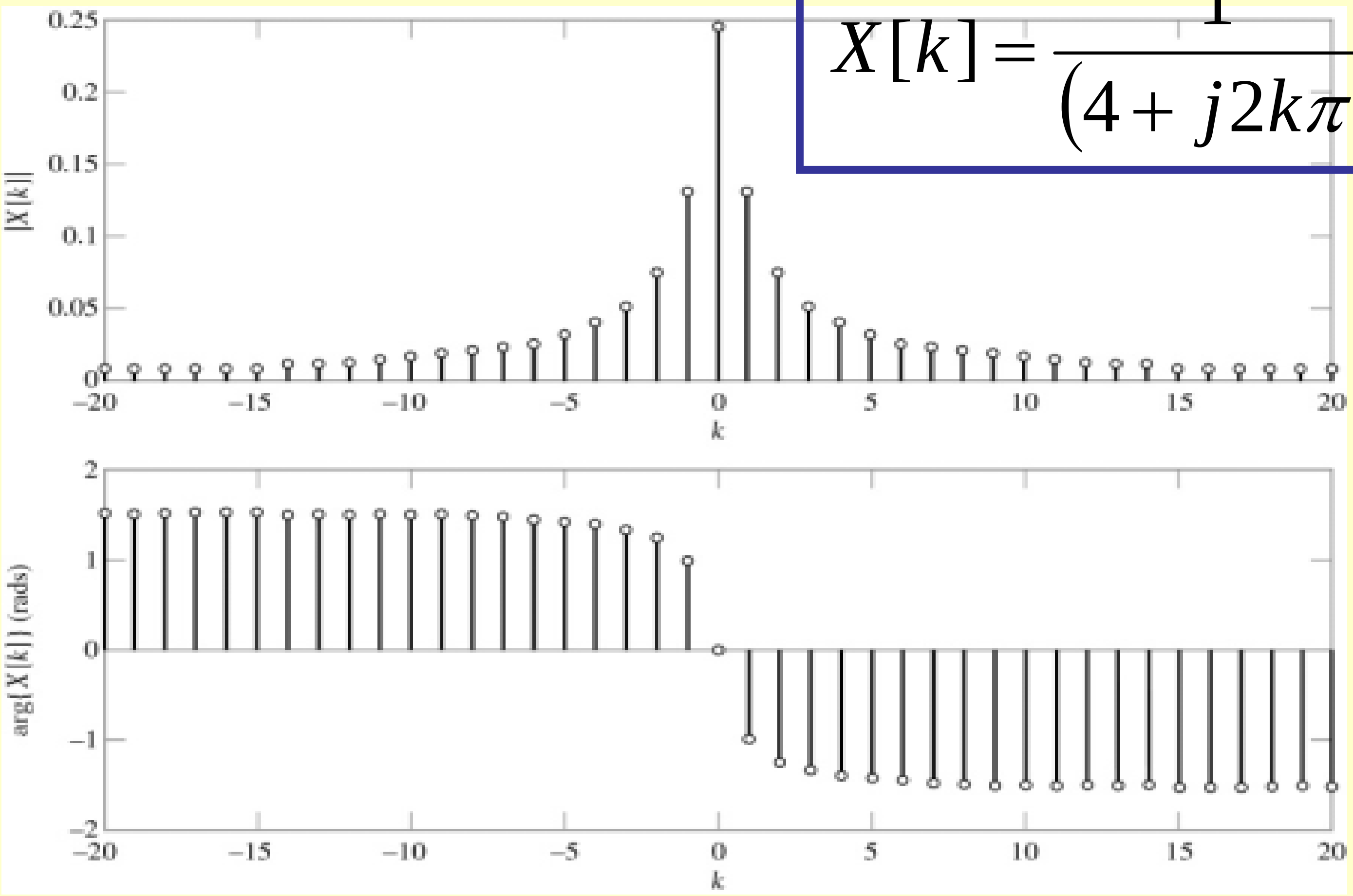
$$= \frac{1}{2}\int_0^2 e^{-(2+jk\pi)t}\, dt = -\frac{1}{2(2+jk\pi)} e^{-(2+jk\pi)t}\Big|_0^2$$

$$= -\frac{1}{(4+j2k\pi)} e^{-(4+j2k\pi)} + \frac{1}{(4+j2k\pi)}$$

$$= \frac{1}{(4+j2k\pi)}\left(1 - e^{-4} e^{-jk2\pi}\right) = \frac{1}{(4+j2k\pi)}\left(1 - e^{-4}\right)$$

Magnitude and Phase Spectra for Ex. 3.9

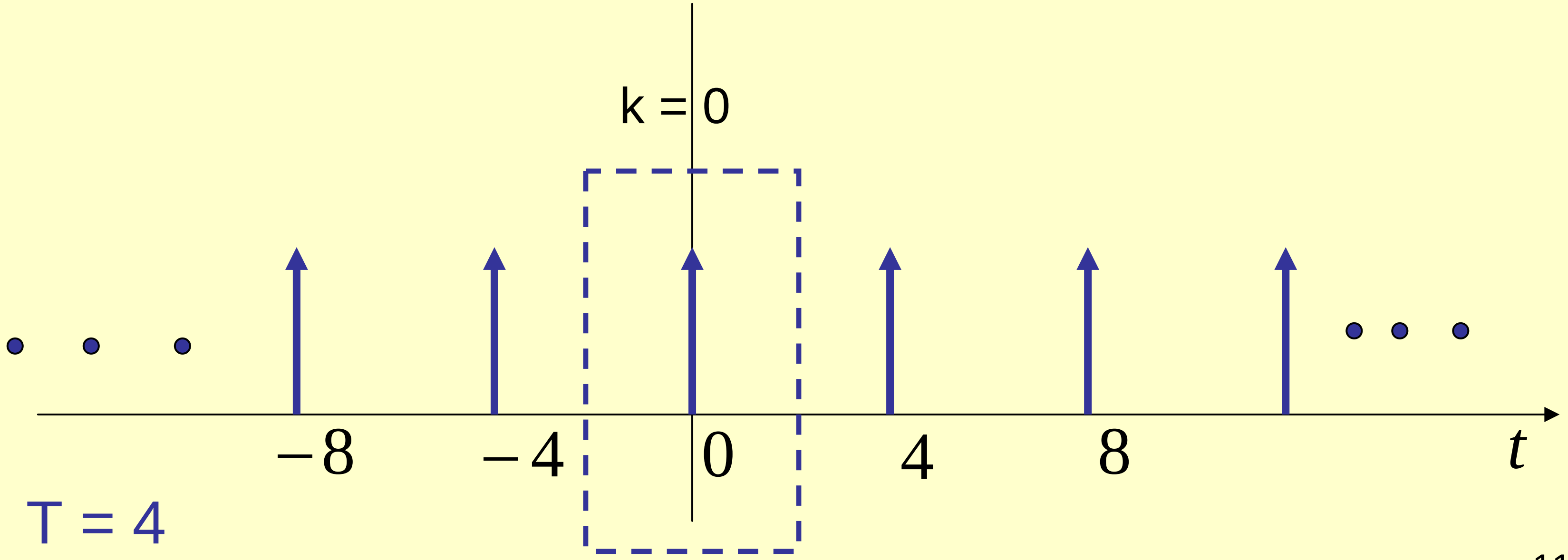
$$X[k]=\frac{1}{(4+j2k\pi)}\left(1-e^{-4}\right)$$



*** 學生試寫出 $|X[k]|$ 和 $\arg\{X[k]\}$

Ex. 3.10 Find the FS for the Impulse Train

$$x(t) = \sum_{k=-\infty}^{+\infty} \delta(t - 4k)$$



T = 4

Solution for Ex. 3.10

T = 4

$$\because\ x(t)=\sum_{k=-\infty}^{+\infty}\delta\left(t-4k\right),$$

$$X[0]=\frac{1}{4}$$

$$\therefore\ X[k]=\frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk\frac{2\pi}{T}t}dt$$

$$=\frac{1}{4}\int_{-2}^{2} \delta(t)\, e^{-jk\frac{\pi}{2}t}dt=\frac{1}{4}$$

X[k] is discrete spectrum and X[k] is non-periodic.

|X[k]| is a constant, and the phase spectrum is zero.

Example 3.11:

Use “the method of inspection” to find the FS of $x(t)=3\cos(\pi t/2 + \pi/4)$. 使用審視法

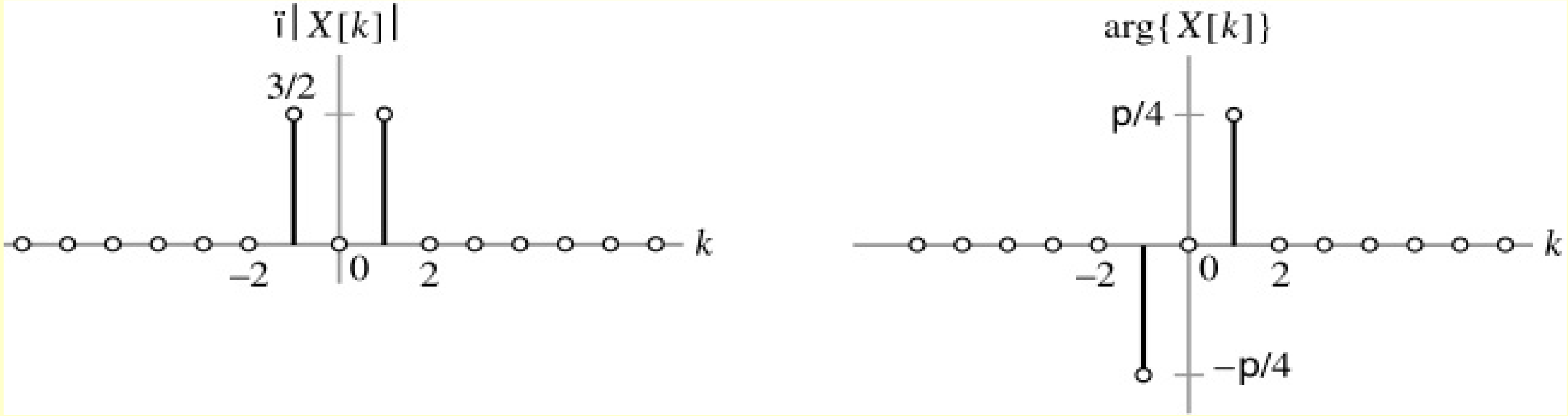
Solution: $\omega_0 = 2\pi/T = \pi/2$, $T = 4$

$$x(t) = 3\cos\left(\frac{\pi t}{2} + \frac{\pi}{4}\right)$$

$$= 3 \cdot \frac{1}{2}\left(e^{j\left(\frac{\pi}{2}t + \frac{\pi}{4}\right)} + e^{-j\left(\frac{\pi}{2}t + \frac{\pi}{4}\right)}\right) = \frac{3}{2}\left(e^{j\frac{\pi}{4}} e^{j\frac{\pi}{2}t} + e^{-j\frac{\pi}{4}} e^{-j\frac{\pi}{2}t}\right)$$

$$= \frac{3}{2} e^{j\frac{\pi}{4}} e^{j\frac{\pi}{2}t} + \frac{3}{2} e^{-j\frac{\pi}{4}} e^{-j\frac{\pi}{2}t}$$

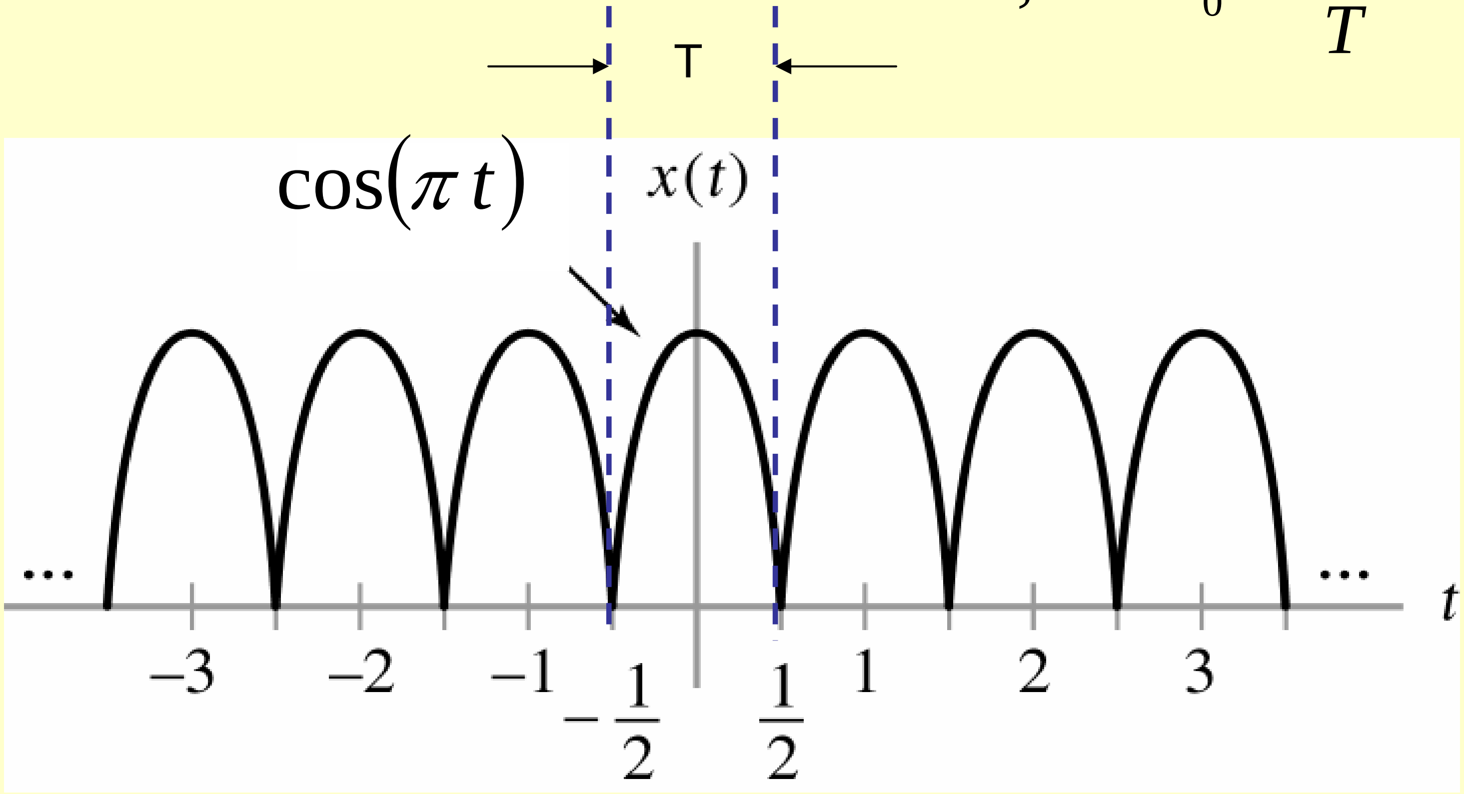
$$X[k]=\begin{cases}\frac{3}{2}e^{j\frac{\pi}{4}}, & k=1\\ \frac{3}{2}e^{-j\frac{\pi}{4}}, & k=-1\\ 0, & otherwise\end{cases}$$



Magnitude and phase spectra for Example 3.11

Problem 3.8: Find the FS of the x(t) below:

$$\because T = 1, \quad \therefore \omega_0 = \frac{2\pi}{T} = 2\pi$$



Full-wave rectified cosine for Problem 3.8

Solution for the P3.8

$$X[k]=\frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk\omega_0 t}dt=\frac{1}{1}\int_{-1/2}^{1/2}\cos(\pi t)\, e^{-jk2\pi t}dt$$

$$=\frac{1}{2}\int_{-1/2}^{1/2}\left(e^{j\pi t}+e^{-j\pi t}\right)e^{-jk2\pi t}dt=\frac{1}{2}\int_{-1/2}^{1/2}\left(e^{j\pi t}e^{-jk2\pi t}+e^{-j\pi t}e^{-jk2\pi t}\right)dt$$

$$=\frac{1}{2}\int_{-1/2}^{1/2}\left(e^{j\pi t}e^{-jk2\pi t}\right)dt+\frac{1}{2}\int_{-1/2}^{1/2}\left(e^{-j\pi t}e^{-jk2\pi t}\right)dt$$

$$=\frac{1}{2}\int_{-1/2}^{1/2}\left(e^{j(1-2k)\pi t}\right)dt+\frac{1}{2}\int_{-1/2}^{1/2}\left(e^{-j(1+2k)\pi t}\right)dt$$

$$=\frac{1}{2j(1-2k)\pi}e^{j(1-2k)\pi t}\Big|_{-\frac{1}{2}}^{+\frac{1}{2}}-\frac{1}{2j(1+2k)\pi}e^{-j(1+2k)\pi t}\Big|_{-\frac{1}{2}}^{+\frac{1}{2}}$$

Solution (cont.)

$$X[k] = \frac{1}{2j(1-2k)\pi} e^{j(1-2k)\pi t}\Big|_{-\frac{1}{2}}^{+\frac{1}{2}} - \frac{1}{2j(1+2k)\pi} e^{-j(1+2k)\pi t}\Big|_{-\frac{1}{2}}^{+\frac{1}{2}}$$

$$= \frac{1}{2j(1-2k)\pi}\left(e^{j(1-2k)\pi/2} - e^{-j(1-2k)\pi/2}\right)$$

$$- \frac{1}{2j(1+2k)\pi}\left(e^{-j(1+2k)\pi/2} - e^{j(1-2k)\pi/2}\right)$$

$$= \frac{1}{(1-2k)\pi} \frac{e^{j(1-2k)\pi/2} - e^{-j(1-2k)\pi/2}}{j2}$$

$$+ \frac{1}{(1+2k)\pi} \frac{e^{j(1+2k)\pi/2} - e^{-j(1+2k)\pi/2}}{j2}$$

$$X[k]=\frac{1}{(1-2k)\pi}\frac{e^{j(1-2k)\pi/2}-e^{-j(1-2k)\pi/2}}{j2}$$

$$+\frac{1}{(1+2k)\pi}\frac{e^{j(1+2k)\pi/2}-e^{-j(1+2k)\pi/2}}{j2}$$

$$=\frac{1}{(1-2k)\pi}\sin\left(\frac{\pi(1-2k)}{2}\right)+\frac{1}{(1+2k)\pi}\sin\left(\frac{\pi(1+2k)}{2}\right)$$

Example 3.12:

[Inverse FS] Find the time-domain x(t) from

Solution: $$X[k] = (1/2)^{|k|} e^{jk\pi/20}, \quad for\ T = 2.$$

$$\omega_0 = 2\pi / T = 2\pi / 2 = \pi$$

$$x(t) = \sum_{k=0}^{+\infty} \left(1/2\right)^{k} e^{jk\pi/20} e^{jk\pi t} + \sum_{k=-1}^{-\infty} \left(1/2\right)^{-k} e^{jk\pi/20} e^{jk\pi t}$$

$$let\ m = -k,$$

$$= \sum_{k=0}^{+\infty} \left(1/2\right)^{k} e^{jk\pi/20} e^{jk\pi t} + \sum_{m=1}^{\infty} \left(1/2\right)^{m} e^{-jm\pi/20} e^{-jm\pi t}$$

$$\sum_{k=0}^{+\infty} (1/2)^k e^{jk\pi/20} e^{jk\pi t}$$

$$= \sum_{k=0}^{+\infty} (1/2)^k e^{jk(\pi/20+\pi t)} = \sum_{k=0}^{+\infty} \left(\frac{1}{2} e^{j\left(\frac{\pi}{20}+\pi t\right)} \right)^k$$

$$= \frac{1}{1 - \frac{1}{2} e^{j\left(\frac{\pi}{20}+\pi t\right)}}$$

$$\sum_{m=1}^{\infty} (1/2)^m e^{-jm\pi/20} e^{-jm\pi t}$$

$$= \sum_{m=0}^{\infty} \left(\frac{1}{2} e^{-j\left(\frac{\pi}{20}+\pi t\right)} \right)^m - 1$$

$$= \frac{1}{1 - \frac{1}{2} e^{-j\left(\frac{\pi}{20}+\pi t\right)}} - 1$$

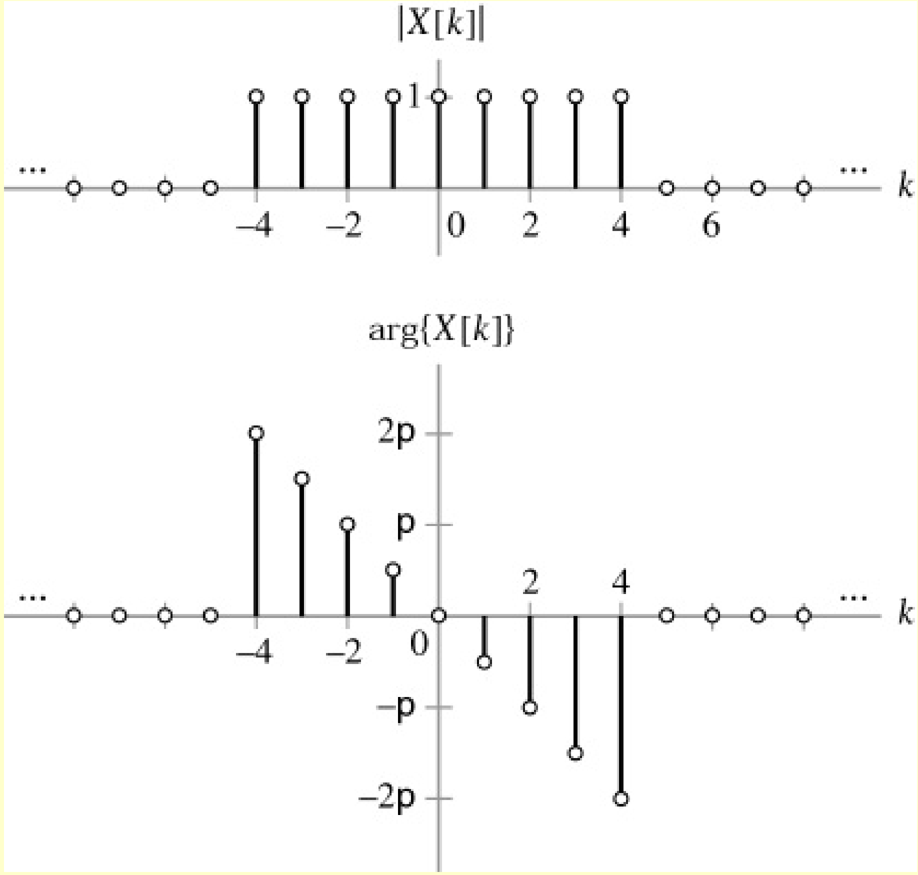
Solution:

$$x(t)=\frac{1}{1-\frac{1}{2}e^{j\left(\frac{\pi}{20}+\pi t\right)}}+\frac{1}{1-\frac{1}{2}e^{-j\left(\frac{\pi}{20}+\pi t\right)}}-1$$

$$=\frac{1-\frac{1}{2}e^{-j\left(\frac{\pi}{20}+\pi t\right)}+1-\frac{1}{2}e^{j\left(\frac{\pi}{20}+\pi t\right)}-\left(1-\frac{1}{2}e^{j\left(\frac{\pi}{20}+\pi t\right)}\right)\left(1-\frac{1}{2}e^{-j\left(\frac{\pi}{20}+\pi t\right)}\right)}{\left(1-\frac{1}{2}e^{j\left(\frac{\pi}{20}+\pi t\right)}\right)\left(1-\frac{1}{2}e^{-j\left(\frac{\pi}{20}+\pi t\right)}\right)}$$

$$=\frac{2-\cos\left(\frac{\pi}{20}+\pi t\right)-\frac{5}{4}+\cos\left(\frac{\pi}{20}+\pi t\right)}{\frac{5}{4}-\cos\left(\frac{\pi}{20}+\pi t\right)}=\frac{\frac{3}{4}}{\frac{5}{4}-\cos\left(\frac{\pi}{20}+\pi t\right)}$$

Problem 3.9: Find the time-domain signal x(t) as the FS coefficients as below.



學生自行推算並驗證課本答案

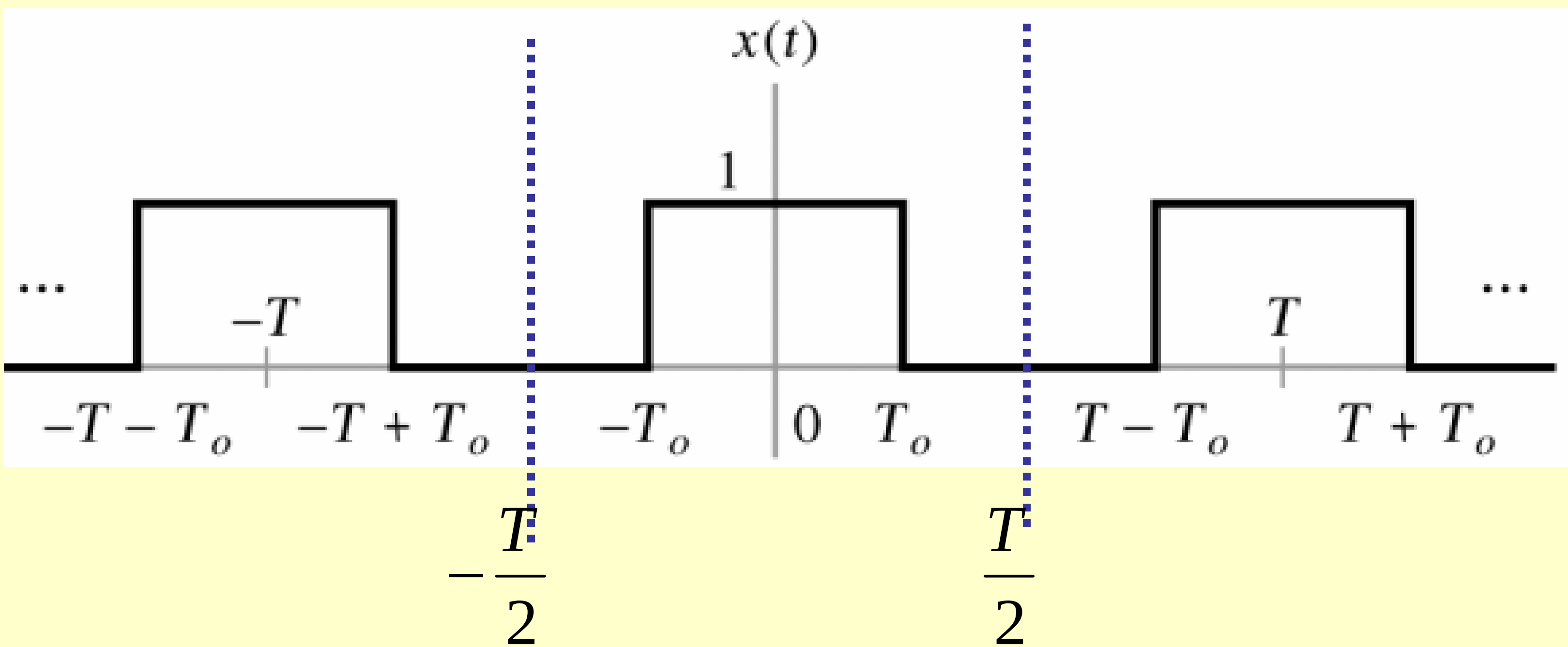
$$x(t) = \sum_{k=-\infty}^{+\infty} X[k]\, e^{jk\omega_0 t}$$

$$X[k] = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

Ex 3.13: Find the FS for the Square wave as below:

$$\omega_0 = 2\pi / T$$

積分區間：-T/2 ~ +T/2



$$for \quad k \neq 0,$$

$$X[k] = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_{-T_0}^{T_0} 1 \cdot e^{-jk\omega_0 t}\, dt = \frac{-1}{Tjk\omega_0} e^{-jk\omega_0 t}\Big|_{-T_0}^{T_0}$$

$$= \frac{-1}{Tjk\omega_0}\left(e^{-jk\omega_0 T_0} - e^{jk\omega_0 T_0}\right) = \frac{1}{Tjk\omega_0}\left(e^{jk\omega_0 T_0} - e^{-jk\omega_0 T_0}\right)$$

$$= \frac{2}{Tk\omega_0}\left(\frac{e^{jk\omega_0 T_0} - e^{-jk\omega_0 T_0}}{j2}\right) = \frac{2}{Tk\omega_0}\sin\left(k\omega_0 T_0\right)$$

$$for \quad k=0,$$

$$X[0]=\frac{1}{T}\int_{-T/2}^{T/2} x(t)\ dt$$

$$=\frac{1}{T}\int_{-T_0}^{T_0} 1\,dt=\frac{1}{T}t\Big|_{-T_0}^{T_0}=\frac{T_0-(-T_0)}{T}=\frac{2T_0}{T}$$

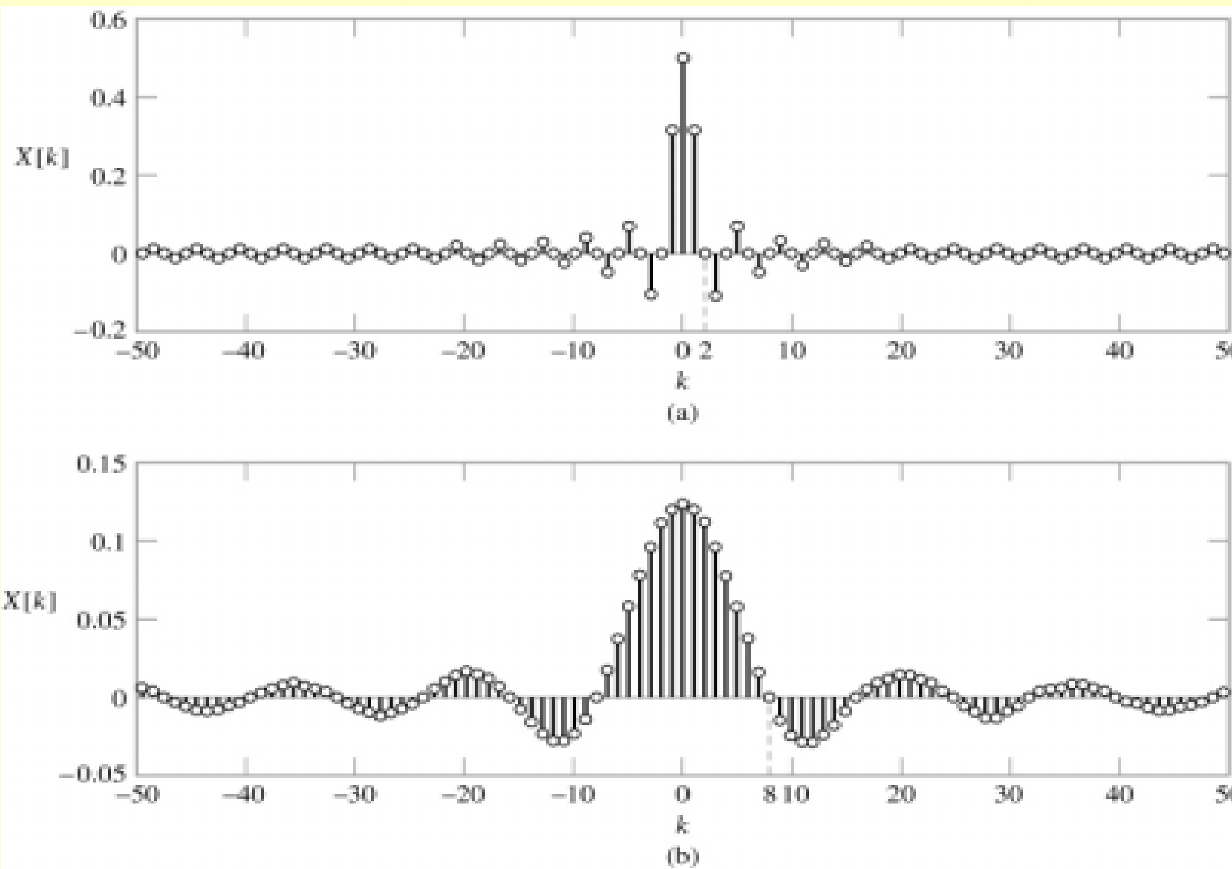
$$for \quad k\neq 0, \quad \omega_0=2\pi/T$$

$$X[k]=\frac{2}{Tk\omega_0}\sin\left(k\omega_0 T_0\right)=\frac{2T}{Tk2\pi}\sin\left(k2\pi T_0/T\right)$$

$$=\frac{1}{k\pi}\sin\left(2\pi k T_0/T\right)$$

The FS coefficients, $X[k]$, $-50 \leq k \leq 50$, for three square waves. (a) $T_0/T = 1/4$. (b) $T_0/T = 1/16$.

$$\frac{1}{k\pi}\sin\left(2\pi k T_0 / T\right)$$



EX : 1st zero crossing at $k = 2$

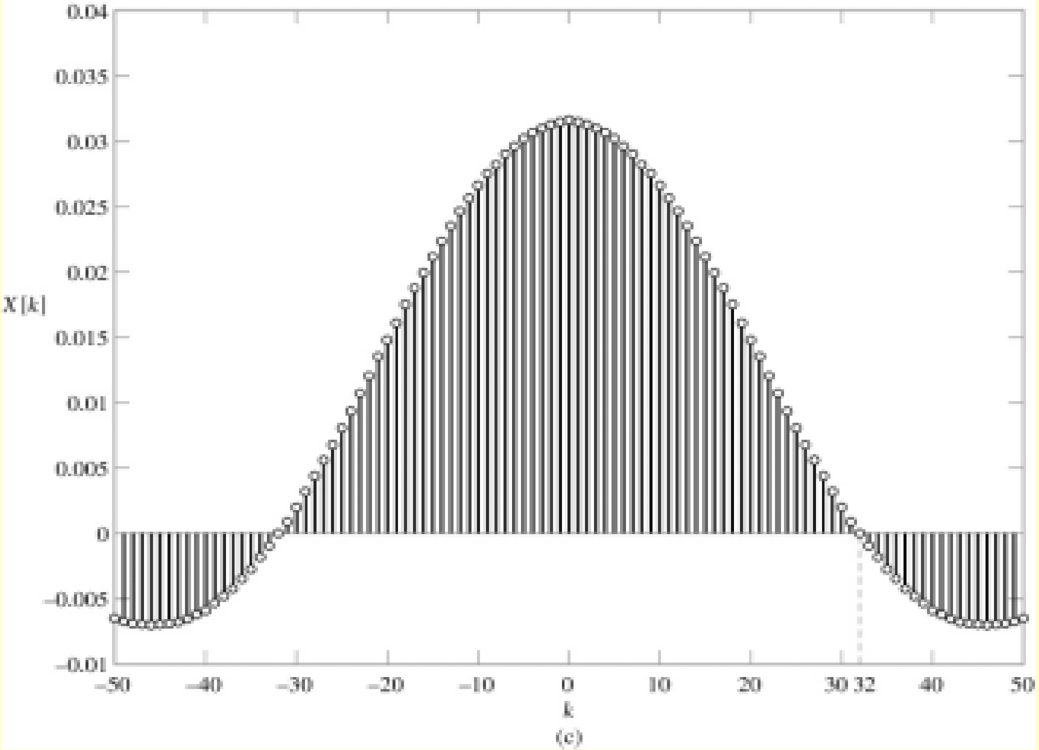
$$\frac{1}{2\pi}\sin\left(2\pi 2\frac{1}{4}\right) = 0$$

EX : 1st zero crossing at $k = 8$

$$\frac{1}{8\pi}\sin\left(2\pi 8\frac{1}{16}\right) = 0$$

(c) $T_0/T = 1/64$.

The first zero-crossing point is at k=32.



$$EX: k = 32$$

$$\frac{1}{8\pi}\sin\left(2\pi 32\frac{1}{64}\right) = 0$$

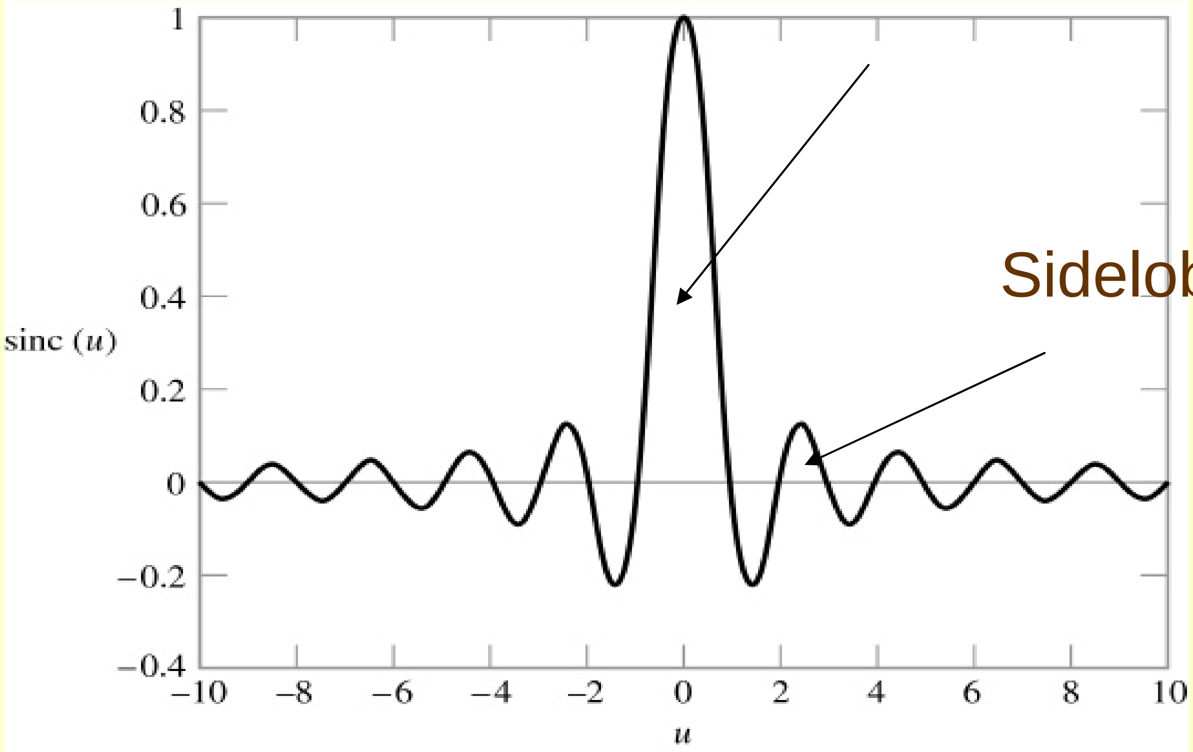
A special function:

Sinc function: $\mathrm{sinc}(u) = \sin(\pi u)/(\pi u)$

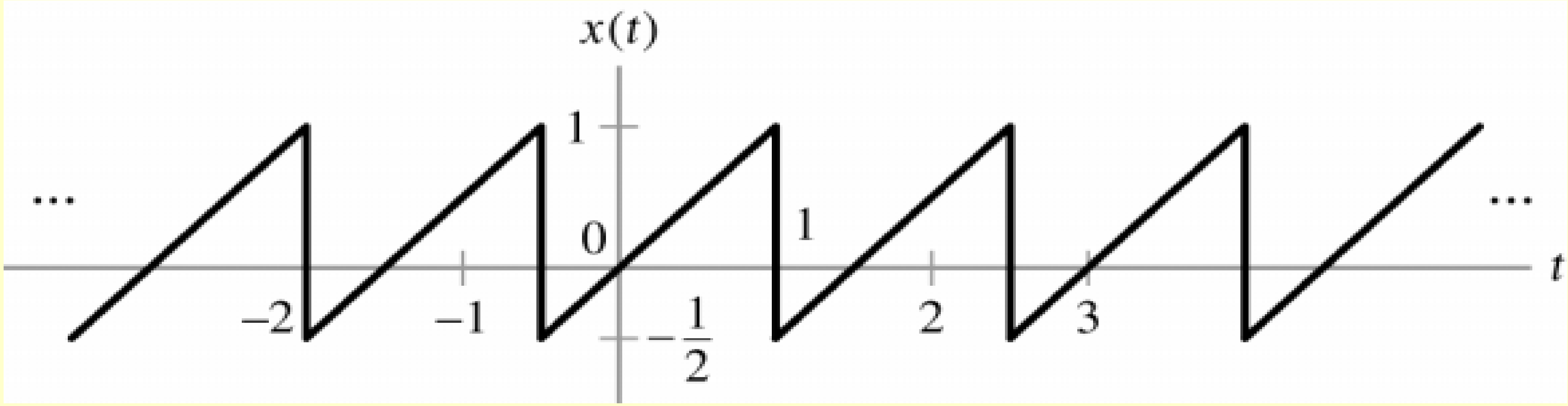
The 1st zero crossing is at u = 1.

Mainlobe (主瓣)

Sidelobe（副瓣）



Problem 3.10: Find the FS for the Sawtooth wave as below:



Periodic signal for Problem 3.10

Solution:

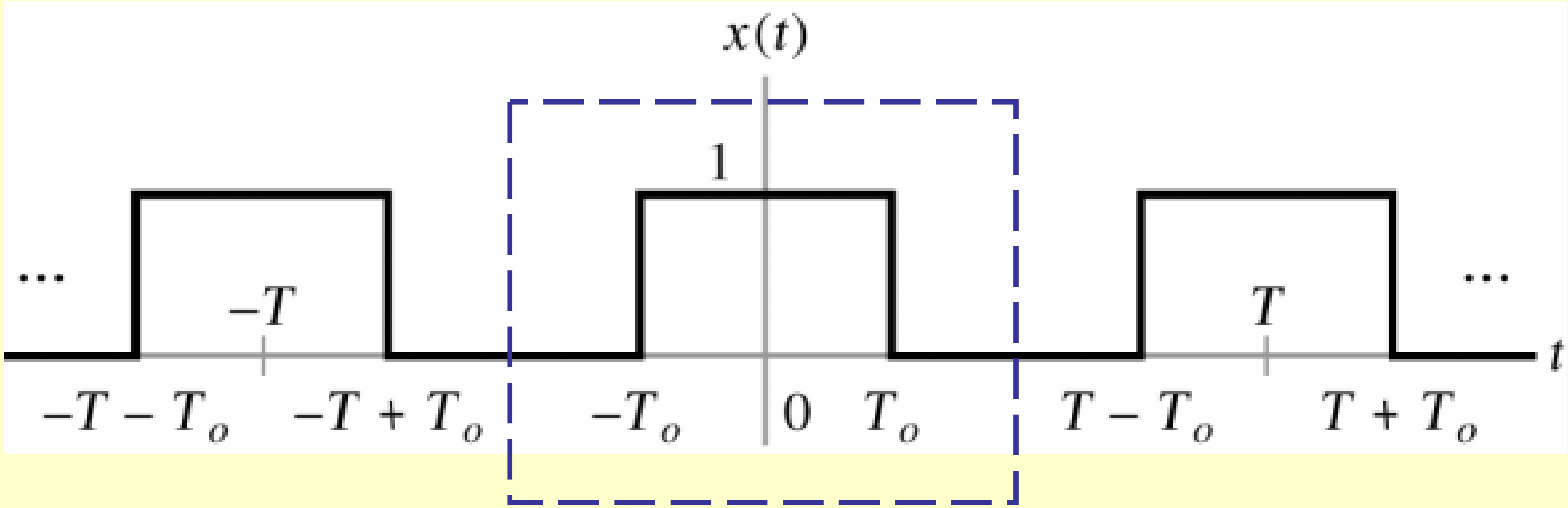
請同學嘗試

Ex 3.15:

Calculating the RC integrator circuit output by means of FS.

參考：

$$X[k] = \frac{1}{k\pi}\sin\left(2\pi k T_0 / T\right)$$



$$impute: \quad x(t) = \sum_{k=-\infty}^{+\infty} X[k]\, e^{jk\omega_0 t},$$

Solution:

$$output: \quad y(t) = \sum_{k=-\infty}^{+\infty} H(jk\omega_0) X[k]\, e^{jk\omega_0 t},$$

$$y(t) \overset{FS}{\leftrightarrow} Y[k] = H(jk\omega_0)\, X[k]$$

$$\because \quad from \quad Example\ 3.1, \quad H(jk\omega_0) = \frac{1/RC}{jk\omega_0 + 1/RC}$$

$$\Rightarrow \quad for \quad RC = 0.1, \quad \omega_0 = 2\pi, \quad H(jk\omega_0) = \frac{10}{j2\pi k + 10}$$

$$from \quad Example\ 3.13, \quad T_0 / T = 1/4, \quad X[k] = \frac{\sin(k\pi/2)}{k\pi}$$

$$\therefore \quad Y[k] = H(jk\omega_0)\, X[k] = \frac{10}{j2\pi k + 10} \frac{\sin(k\pi/2)}{k\pi}$$

y(t)