Fourier Transform of Signals

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Lecture 3-4

Discrete-Time Non-periodic Signals:

The Discrete-Time Fourier Transform (DTFT)

- A discrete-time non-periodic signal is represented by a superposition of complex sinusoids.
- The DTFT involves continuous frequencies on the interval $-\pi < \Omega < \pi$. (頻率連續分佈，週期性頻譜)
- The DTFT pair:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X(e^{j\Omega})\, e^{j\Omega n}\, d\Omega$$

$X(e^{j\Omega})$ is continuous spectrum.

x[n] is discrete signal.

(離散時間、非週期性訊號)

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\Omega n}$$

The Discrete-Time Fourier Transform (DTFT)

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X(e^{j\Omega})\, e^{j\Omega n}\, d\Omega \qquad (5)$$

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\Omega n} \qquad (6)$$

Example 3.17:

Find the DTFT of an exponential signal $x[n] = (\alpha)^n u[n]$.

Solution:

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{+\infty} \alpha^n u[n]\, e^{-j\Omega n} = \sum_{n=0}^{+\infty} \alpha^n\, e^{-j\Omega n}$$

$$= \sum_{n=0}^{+\infty} \left(\alpha\, e^{-j\Omega}\right)^n = \frac{1}{1-\alpha\, e^{-j\Omega}}, \quad |\alpha| < 1$$

If α is real valued, we may expand of the denominator as:

$$\because X(e^{j\Omega}) = \frac{1}{1-\alpha e^{-j\Omega}} = \frac{1}{1-\alpha\cos\Omega + j\alpha\sin\Omega}$$

The magnitude spectrum is denoted by :

$$\therefore \left|X(e^{j\Omega})\right| = \frac{1}{\sqrt{(1-\alpha\cos\Omega)^2 + (\alpha\sin\Omega)^2}}$$

$$= \frac{1}{\sqrt{1-2\alpha\cos\Omega + \alpha^2\cos^2\Omega + \alpha^2\sin^2\Omega}}$$

$$= \frac{1}{\sqrt{1-2\alpha\cos\Omega + \alpha^2}}$$

The phase spectrum is denoted by :

$$\because X(e^{j\Omega}) = \frac{1}{1-\alpha\, e^{-j\Omega}} = \frac{1}{1-\alpha\cos\Omega + j\alpha\sin\Omega}$$

$$\therefore \arg\left\{X(e^{j\Omega})\right\} = -\tan^{-1}\frac{\alpha\sin\Omega}{1-\alpha\cos\Omega}$$

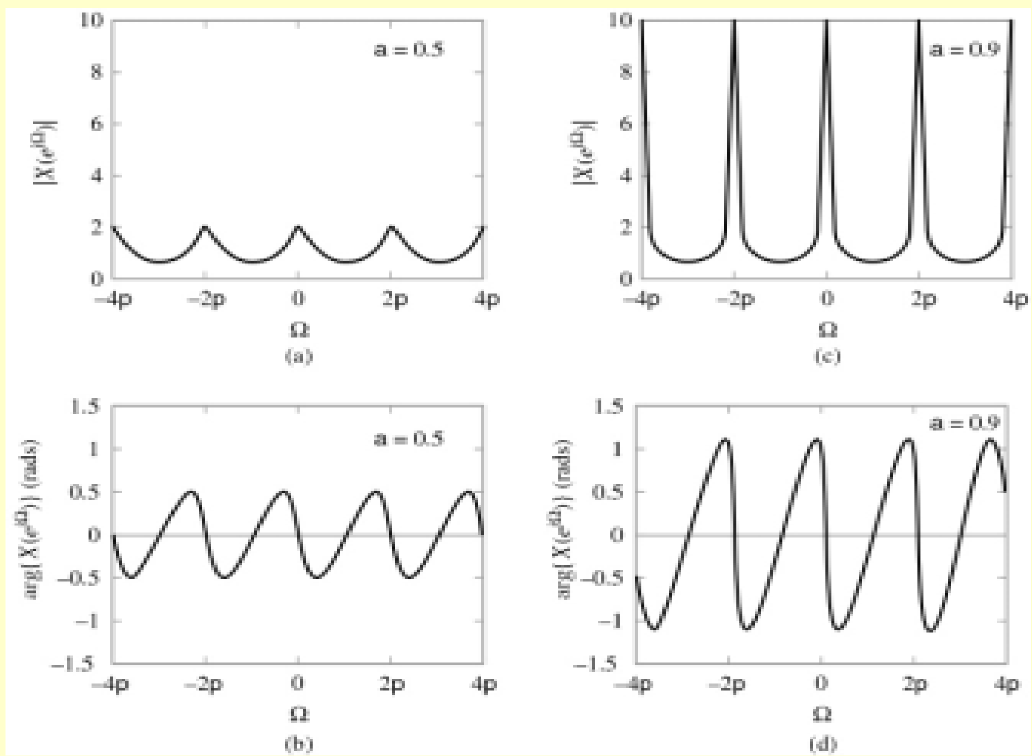
The DTFT of an exponential signal $x[n] = (\alpha)^n u[n]$.

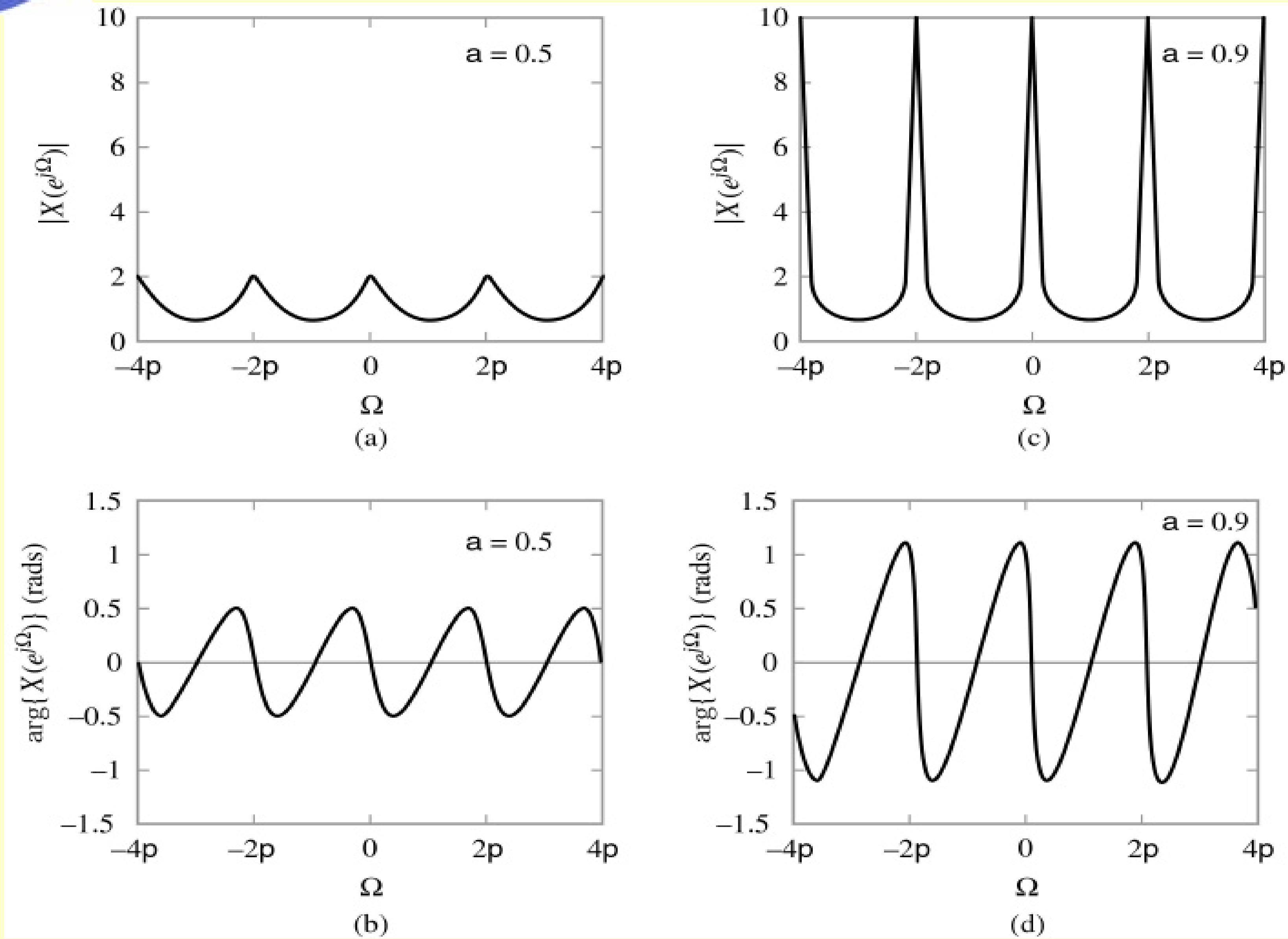
(a) Magnitude spectrum for $\alpha = 0.5$.

(b) Phase spectrum for $\alpha = 0.5$.

(c) Magnitude spectrum for $\alpha = 0.9$.

(d) Phase spectrum for $\alpha = 0.9$.

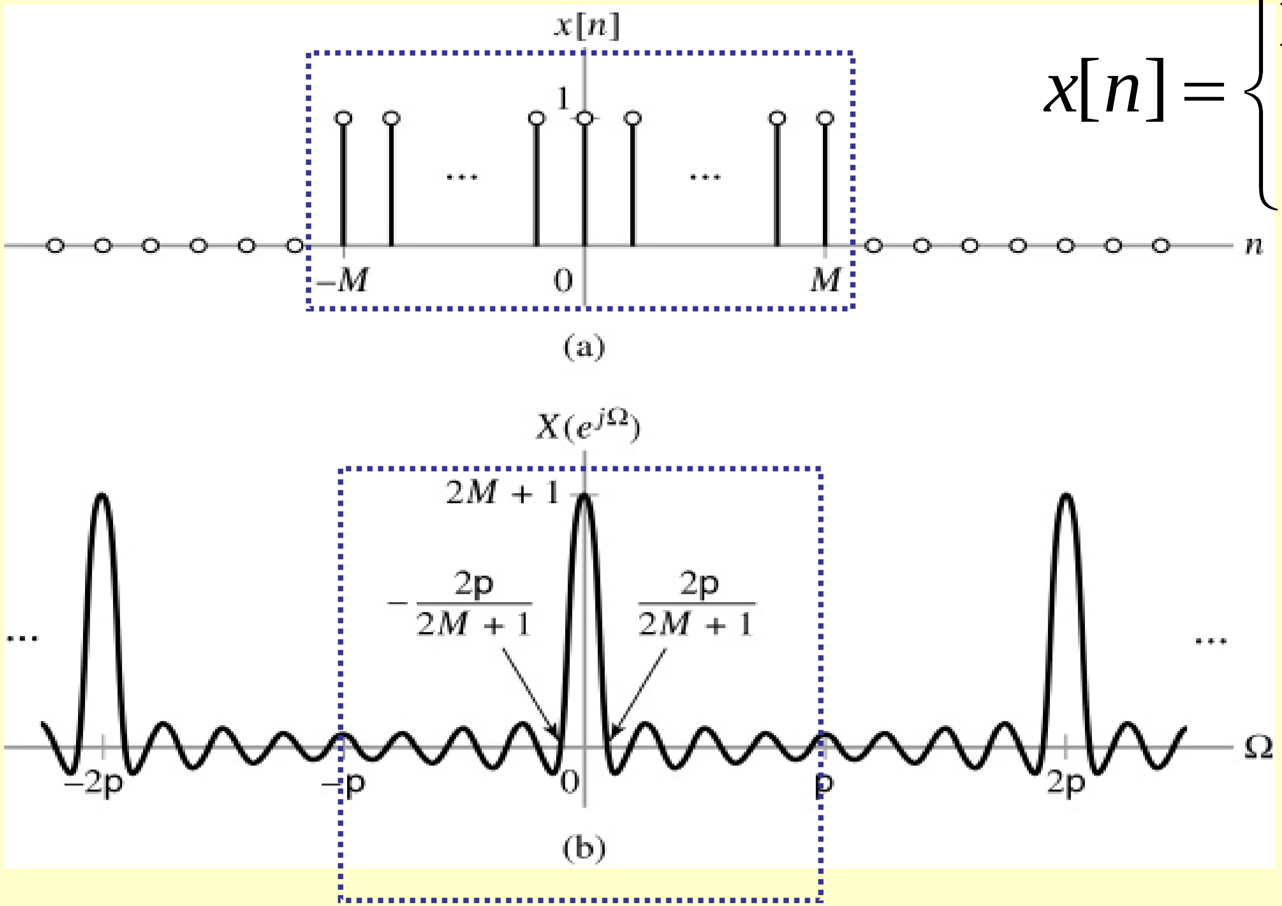




10
8
6
4
2
0
|X(e^jΩ)|
a = 0.5
–4p
–2p
0
2p
4p
Ω
(a)
10
8
6
4
2
0
|X(e^jΩ)|
a = 0.9
–4p
–2p
0
2p
4p
Ω
(c)
1.5
1
0.5
0
–0.5
–1
–1.5
arg{X(e^jΩ)} (rads)
a = 0.5
–4p
–2p
0
2p
4p
Ω
(b)
1.5
1
0.5
0
–0.5
–1
–1.5
arg{X(e^jΩ)} (rads)
a = 0.9
–4p
–2p
0
2p
4p
Ω
(d)

Example 3.18. Find the DTFT of x[n] = ?
(a) Rectangular pulse in the time domain.
(b) DTFT in the frequency domain.

$$x[n] = \begin{cases} 1, & |n| \le M \\ 0, & |n| > M \end{cases}$$



$$p = \pi$$

$$\forall \Omega \neq 0, \quad \pm 2\pi, \quad \pm 4\pi, \cdots$$

$$X(e^{j\Omega}) = \sum_{n=-M}^{+M} x[n]\, e^{-j\Omega n} = \sum_{n=-M}^{+M} 1\, e^{-j\Omega n}$$

$$let\ m = n + M,$$

$$= \sum_{m=0}^{+2M} 1\, e^{-j\Omega(m-M)} = e^{j\Omega M} \sum_{m=0}^{+2M} 1\, e^{-j\Omega m}$$

$$= e^{j\Omega M} \frac{1 - e^{-j\Omega(2M+1)}}{1 - e^{-j\Omega}}$$

$$X(e^{j\Omega}) = e^{j\Omega M}\,\frac{1-e^{-j\Omega(2M+1)}}{1-e^{-j\Omega}}$$

$$= e^{j\Omega M}\,\frac{e^{-j\Omega\frac{2M+1}{2}}\left(e^{j\Omega\frac{2M+1}{2}}-e^{-j\Omega\frac{2M+1}{2}}\right)}{e^{-j\frac{\Omega}{2}}\left(e^{j\frac{\Omega}{2}}-e^{-j\frac{\Omega}{2}}\right)}$$

$$= e^{j\Omega M}\,\frac{e^{-j\Omega\frac{2M+1}{2}}\left(e^{j\Omega\frac{2M+1}{2}}-e^{-j\Omega\frac{2M+1}{2}}\right)}{e^{-j\frac{\Omega}{2}}\left(e^{j\frac{\Omega}{2}}-e^{-j\frac{\Omega}{2}}\right)}$$

$$X(e^{j\Omega}) = e^{j\Omega M} \frac{e^{-j\Omega\frac{2M+1}{2}}\left(e^{j\Omega\frac{2M+1}{2}} - e^{-j\Omega\frac{2M+1}{2}}\right)}{e^{-j\frac{\Omega}{2}}\left(e^{j\frac{\Omega}{2}} - e^{-j\frac{\Omega}{2}}\right)}$$

$$= \frac{\left(e^{j\Omega\frac{2M+1}{2}} - e^{-j\Omega\frac{2M+1}{2}}\right)/\,j2}{\left(e^{j\frac{\Omega}{2}} - e^{-j\frac{\Omega}{2}}\right)/\,j2} = \frac{\sin\left(\frac{(2M+1)\Omega}{2}\right)}{\sin\left(\frac{\Omega}{2}\right)}$$

$$\forall \Omega = 0, \quad \pm 2\pi, \quad \pm 4\pi, \cdots$$

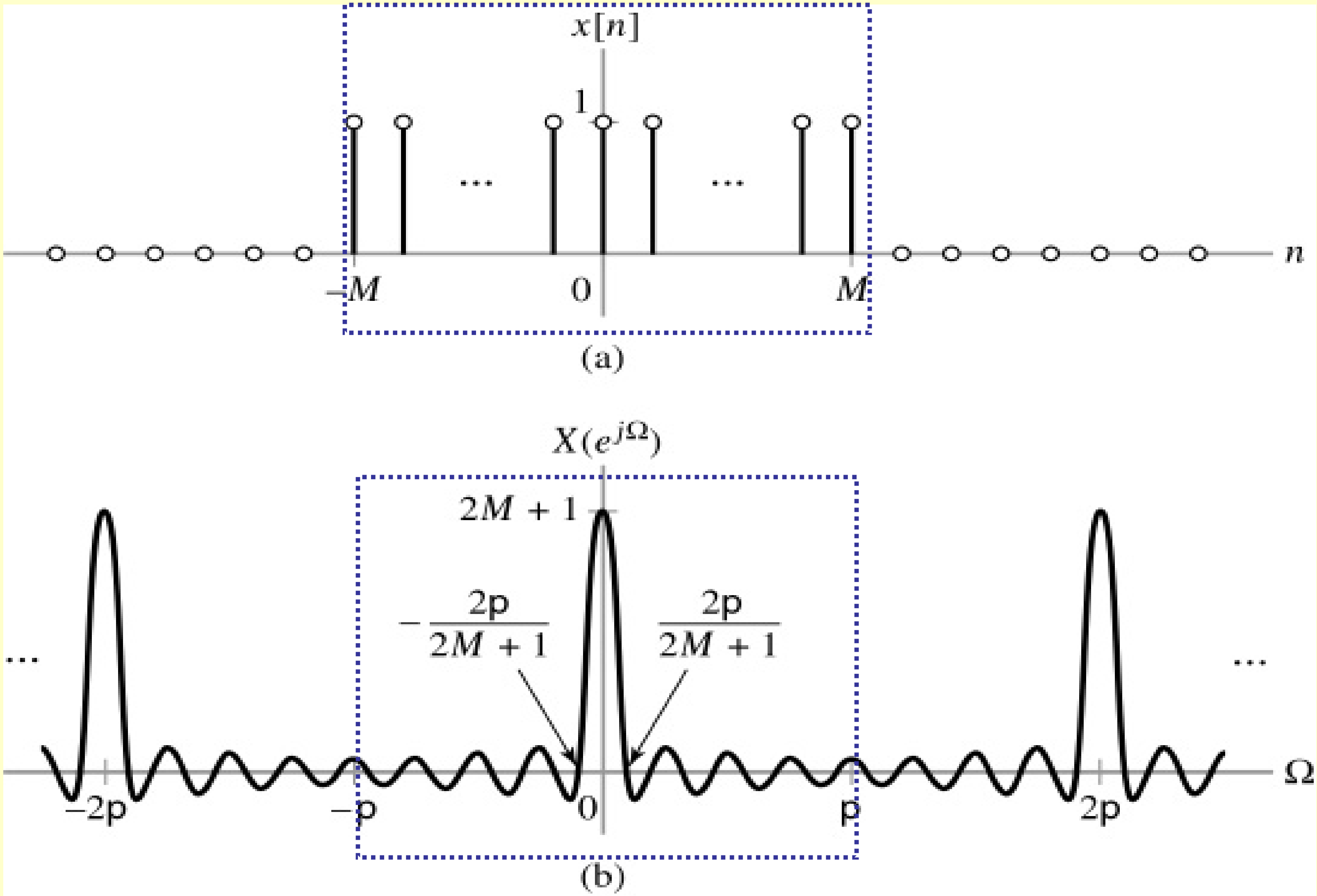
$$X(e^{j\Omega}) = \sum_{n=-M}^{+M} x[n]\, e^{-j\Omega n} = \sum_{n=-M}^{+M} 1 \cdot e^{-j\Omega n}$$

$$let\ m = n + M,$$

$$= \sum_{m=0}^{+2M} 1 \cdot e^{-j\Omega(m-M)} = e^{j\Omega M} \sum_{m=0}^{+2M} 1 \cdot e^{-j\Omega m}$$

$$= e^{j2\pi M} \sum_{m=0}^{+2M} 1 \cdot e^{-j2\pi m} = 1 \cdot \sum_{m=0}^{+2M} 1$$

$$= 2M + 1$$


x[n]
1
…
…
−M
0
M
n
(a)
X(e^{jΩ})
2M + 1
− 2p / (2M + 1)
2p / (2M + 1)
…
…
−2p
−p
0
p
2p
Ω
(b)

The first zero crossing is at ±2π/(2M+1).

$$at \quad \Omega = \frac{2\pi}{2M+1},$$

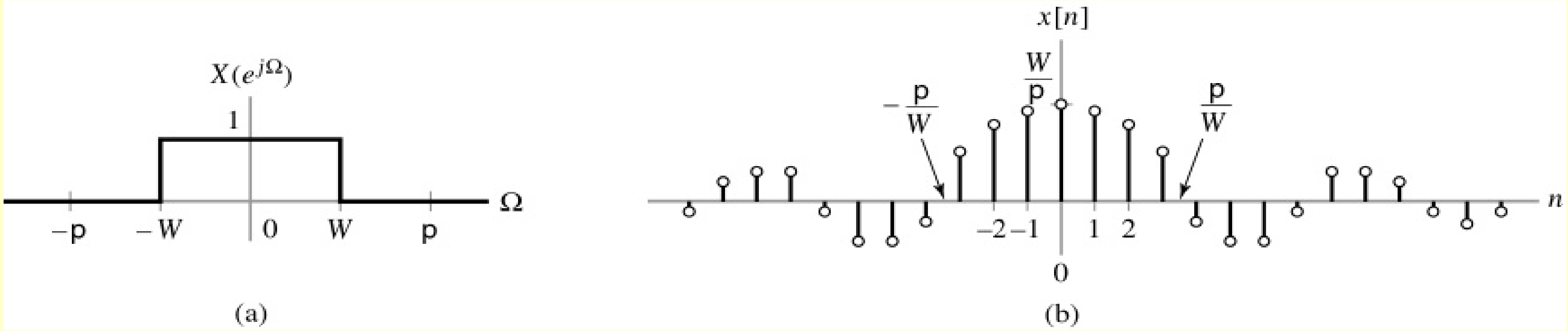
$$X(e^{j\Omega}) = \frac{\sin\left(\frac{(2M+1)\Omega}{2}\right)}{\sin\left(\frac{\Omega}{2}\right)} = \frac{\sin\left(\frac{(2M+1)}{2}\frac{2\pi}{2M+1}\right)}{\sin\left(\frac{1}{2}\frac{2\pi}{2M+1}\right)}$$

$$= \frac{\sin(\pi)}{\sin\left(\frac{\pi}{2M+1}\right)} = 0$$

Example 3.19. Find the x[n] = ?

(a) Rectangular pulse in the frequency domain.
(b) Inverse DTFT in the time domain.

$$X\left(e^{j\Omega}\right)=\begin{cases}1, & |\Omega|<W \\ 0, & W<|\Omega|<\pi\end{cases}$$



Solution: $n \neq 0$ case

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X\left(e^{j\Omega}\right)e^{j\Omega n}d\Omega$$

$$= \frac{1}{2\pi}\int_{-W}^{+W} 1 \cdot e^{j\Omega n}\, d\Omega$$

$$= \frac{1}{2\pi}\left\{\frac{1}{jn}e^{j\Omega n}\Big|_{-W}^{+W}\right\} = \frac{1}{\pi n}\frac{e^{jWn} - e^{-jWn}}{j2}$$

$$= \frac{1}{\pi n}\sin\left(W n\right)$$

Solution: $n = 0$ case

Considering the L'Hospital Rule:

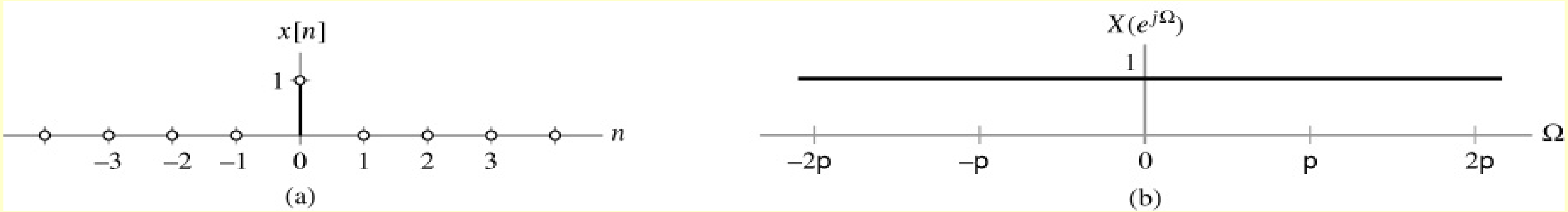
$$x[0] = \lim_{n \to 0} \frac{1}{\pi n} \sin(W n) = \frac{W}{\pi}$$

The first zero crossing is at n = ±π/W.

$$x[\pi / W] = \frac{1}{\pi \frac{\pi}{W}} \sin\left(W \tfrac{\pi}{W}\right) = \frac{W}{\pi^2} \sin(\pi) = 0.$$

Example 3.20. Find DTFT of the $x[n] = \delta[n]$?

(a) Unit impulse in the time domain.
(b) DTFT of unit impulse in the frequency domain.



Solution:

$$\because\ x[n] = \delta[n]$$

$$\therefore\ X\left(e^{j\Omega}\right) = \sum_{n=-\infty}^{+\infty} \delta[n] e^{-j\Omega n} = 1$$

The DTFT vs. Unit Impulse :

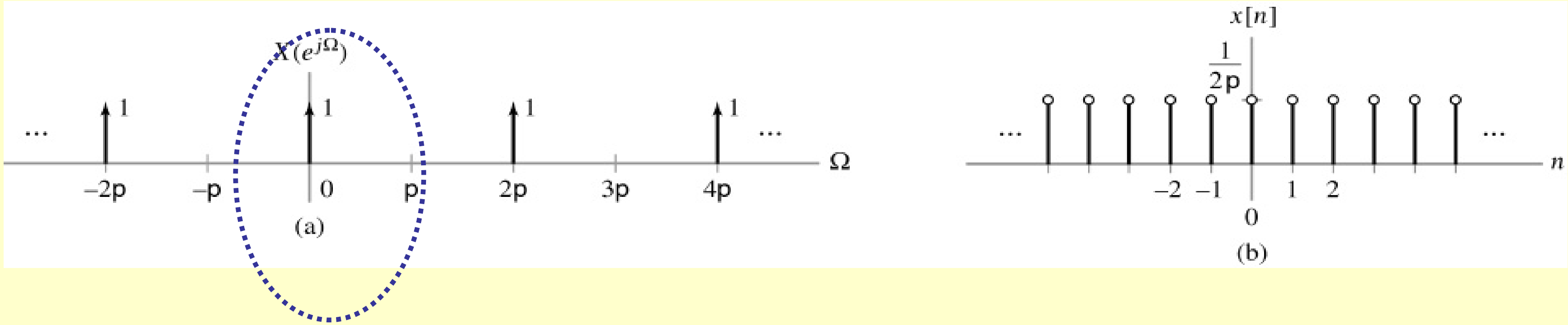
$$\delta[n] \xleftrightarrow{DTFT} 1$$

Example 3.21. Find the x[n] = ?

$$X\left(e^{j\Omega}\right)=\delta(\Omega)$$

(a) Unit impulse in the frequency domain.

(b) Inverse DTFT in the time domain.



頻率連續分佈，週期性頻譜

Solution: defines only one period as,

$$\forall \ X(e^{j\Omega}) = \delta(\Omega), \quad -\pi < \Omega \le \pi$$

$$\therefore x[n] = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \delta(\Omega)\, e^{j\Omega n}\, d\Omega = \frac{1}{2\pi}$$

The DTFT vs. Unit Impulse :

$$\frac{1}{2\pi} \xleftrightarrow{DTFT} \delta(\Omega)$$

Similarly, defines all delta functions as:

$$X\left(e^{j\Omega}\right)=\sum_{k=-\infty}^{\infty}\delta\left(\Omega-2k\pi\right)$$

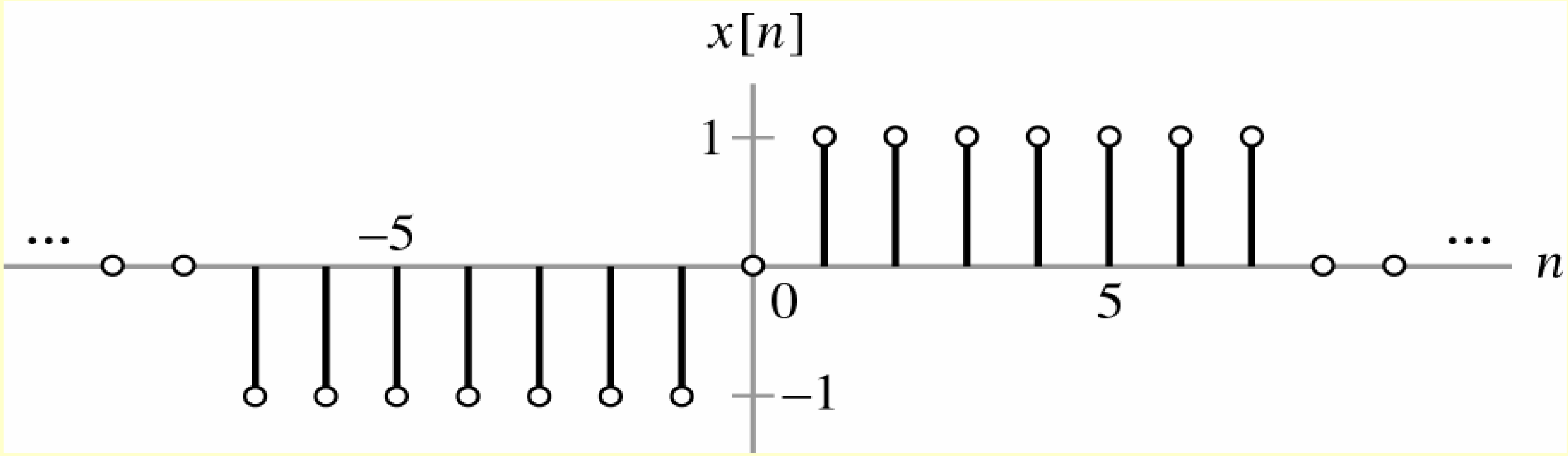
頻率連續分佈，週期性頻譜

The inverse of the DTFT is also impulse train:

$$x[n]=\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\delta[n-k]$$

Problem 3.12.

Find the DTFT of the x[n] = ? $X\left(e^{j\Omega}\right) = ?$



Signal *x*[*n*] for Problem 3.12.

Solution:

$$X\left(e^{j\Omega}\right)=\sum_{n=-\infty}^{\infty} x[n]\,e^{-j\Omega n}=\sum_{n=-7}^{-1}(-1)\,e^{-j\Omega n}+\sum_{m=1}^{7}(1)\,e^{-j\Omega m}$$

$$let\ l=n+7,\quad p=m-1,$$

$$=\sum_{l=0}^{6}(-1)\,e^{-j\Omega(l-7)}+\sum_{p=0}^{6}(1)\,e^{-j\Omega(p+1)}$$

$$=-e^{j7\Omega}\sum_{l=0}^{6}e^{-j\Omega l}+e^{-j\Omega}\sum_{p=0}^{6}e^{-j\Omega p}$$

$$=-e^{j7\Omega}\frac{1-e^{-j\Omega 7}}{1-e^{-j\Omega}}+e^{-j\Omega}\frac{1-e^{-j\Omega 7}}{1-e^{-j\Omega}}$$

Audio & DSP Lab.

Solution:

$$X\left(e^{j\Omega}\right)=\frac{1-e^{-j\Omega 7}}{1-e^{-j\Omega}}\left(e^{-j\Omega}-e^{j7\Omega}\right)$$

$$=\frac{e^{-j\frac{7\Omega}{2}}(e^{j\frac{7\Omega}{2}}-e^{-j\frac{7\Omega}{2}})}{e^{-j\frac{\Omega}{2}}(e^{j\frac{\Omega}{2}}-e^{-j\frac{\Omega}{2}})}\left(e^{-j\Omega}-e^{j7\Omega}\right)$$

$$=\frac{e^{-j3}(e^{j\frac{7\Omega}{2}}-e^{-j\frac{7\Omega}{2}})}{(e^{j\frac{\Omega}{2}}-e^{-j\frac{\Omega}{2}})}\left(e^{-j\Omega}-e^{j7\Omega}\right)$$

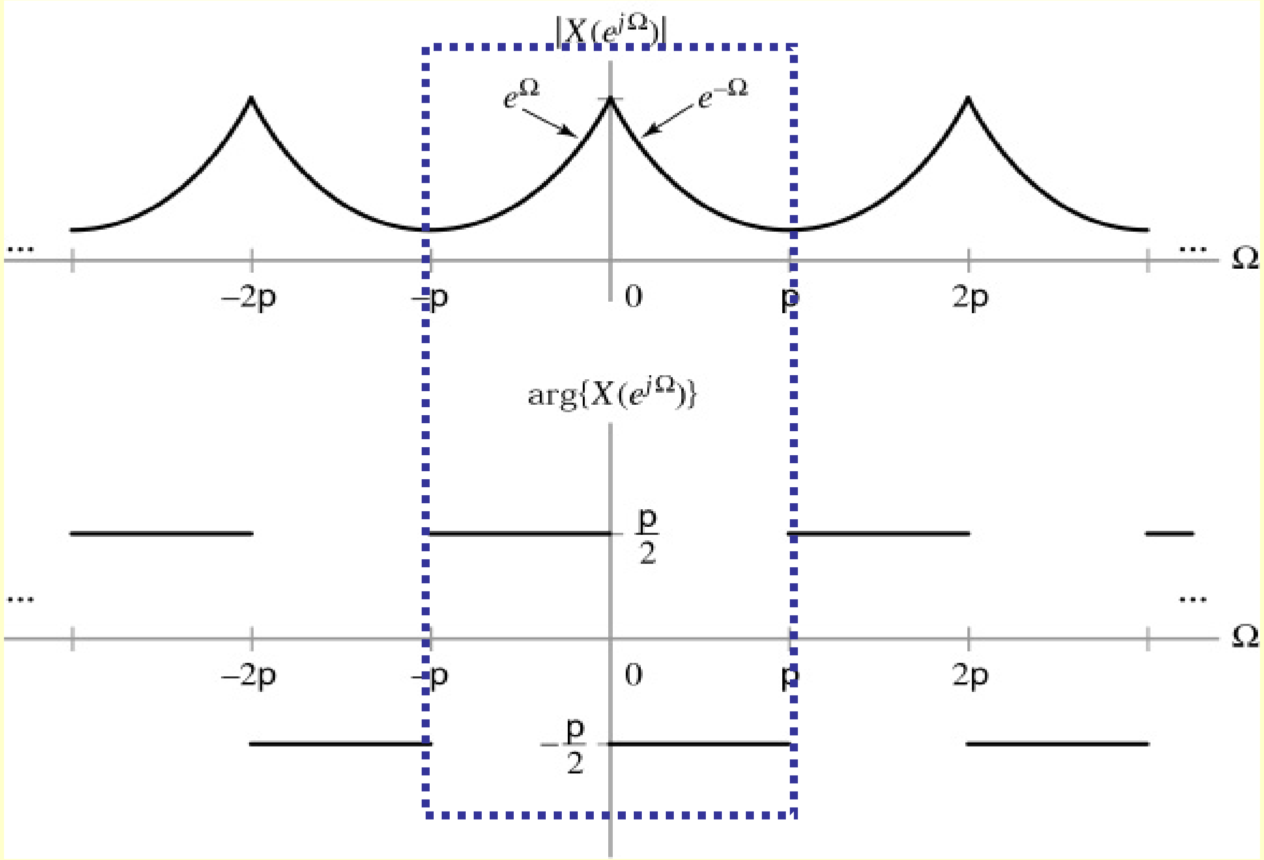
Solution:

$$X\left(e^{j\Omega}\right)=\frac{(e^{j\frac{7\Omega}{2}}-e^{-j\frac{7\Omega}{2}})}{(e^{j\frac{\Omega}{2}}-e^{-j\frac{\Omega}{2}})}\left(e^{-j4\Omega}-e^{j4\Omega}\right)$$

$$=-j2\frac{(e^{j\frac{7\Omega}{2}}-e^{-j\frac{7\Omega}{2}})/\,j2}{(e^{j\frac{\Omega}{2}}-e^{-j\frac{\Omega}{2}})/\,j2}\left(e^{j4\Omega}-e^{-j4\Omega}\right)/\,j2$$

$$=-j2\frac{\sin\left(\frac{7\Omega}{2}\right)\sin(4\Omega)}{\sin\left(\frac{\Omega}{2}\right)}$$

Problem 3.13. Find the inverse DTFT of the following frequency-domain signals: $x[n] = ?$



Frequency-domain signal for Problem 3.13.

Solution:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X(e^{j\Omega})\, e^{j\Omega n}\, d\Omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{0} (e^{\Omega} e^{j\frac{\pi}{2}}) e^{j\Omega n}\, d\Omega + \frac{1}{2\pi}\int_{0}^{+\pi} (e^{-\Omega} e^{-j\frac{\pi}{2}})\, e^{j\Omega n} d\Omega$$

$$= \frac{j}{2\pi}\left\{\int_{-\pi}^{0} \cdot e^{(1+jn)\Omega}\, d\Omega - \int_{0}^{+\pi} \cdot e^{(-1+jn)\Omega}\, d\Omega\right\}$$

$$= \frac{j}{2\pi}\left\{\frac{1}{1+jn} e^{(1+jn)\Omega}\Big|_{-\pi}^{0} + \frac{1}{1-jn} e^{(-1+jn)\Omega}\Big|_{0}^{\pi}\right\}$$

$$x[n]=\frac{j}{2\pi}\left\{\frac{1-e^{-(1+jn)\pi}}{1+jn}+\frac{e^{(-1+jn)\pi}-1}{1-jn}\right\}$$

$$=\frac{j}{2\pi}\left\{\frac{1-e^{-\pi}e^{-jn\pi}}{1+jn}+\frac{e^{-\pi}e^{jn\pi}-1}{1-jn}\right\}=\frac{j}{2\pi}\left\{\frac{1-e^{-\pi}(-1)^n}{1+jn}-\frac{1-e^{-\pi}(-1)^n}{1-jn}\right\}$$

$$=\frac{j}{2\pi}\left\{\frac{[1-e^{-\pi}(-1)^n](1-jn)-[1-e^{-\pi}(-1)^n](1+jn)}{(1+jn)(1-jn)}\right\}$$

$$=\frac{j}{2\pi}\left\{\frac{1-jn-e^{-\pi}(-1)^n+jne^{-\pi}(-1)^n-1+e^{-\pi}(-1)^n-jn+jne^{-\pi}(-1)^n}{(1+n^2)}\right\}$$

$$x[n]=\frac{j}{2\pi}\left\{\frac{-2jn+2jne^{-\pi}(-1)^{n}}{(1+n^{2})}\right\}$$

$$=\frac{1}{2\pi}\left\{\frac{-2j^{2}n+2j^{2}ne^{-\pi}(-1)^{n}}{(1+n^{2})}\right\}$$

$$=\frac{1}{2\pi}\left\{\frac{2n-2ne^{-\pi}(-1)^{n}}{(1+n^{2})}\right\}=\frac{n}{\pi}\left\{\frac{1-e^{-\pi}(-1)^{n}}{(1+n^{2})}\right\}$$

Example 3.22: Consider two moving-average systems described by the equation:

$$y_1[n] = \frac{1}{2}\left(x[n] + x[n-1]\right) \;\; and \;\; y_2[n] = \frac{1}{2}\left(x[n] - x[n-1]\right)$$

It implies that the impulse responses of these systems are:

$$h_1[n] = \frac{1}{2}\left(\delta[n] + \delta[n-1]\right) \;\; and \;\; h_2[n] = \frac{1}{2}\left(\delta[n] - \delta[n-1]\right)$$

Please find the <u>frequency response</u> of each system and plot the <u>magnitude response</u>.

Solution:

$$H_1\left(e^{j\Omega}\right)=\sum_{n=-\infty}^{\infty}h_1[n]\,e^{-j\Omega n}=\sum_{n=0}^{\infty}\left\{\frac{1}{2}\delta[n]+\frac{1}{2}\delta[n-1]\right\}e^{-j\Omega n}$$

$$=\frac{1}{2}+\frac{1}{2}e^{-j\Omega}$$

$$=\frac{1}{2}e^{-j\frac{\Omega}{2}}\left(e^{j\frac{\Omega}{2}}+e^{-j\frac{\Omega}{2}}\right)=e^{-j\frac{\Omega}{2}}\cos\left(\frac{\Omega}{2}\right)$$

$$\left|H_1(e^{j\Omega})\right|=\left|\cos\left(\frac{\Omega}{2}\right)\right|$$

$$\arg\left\{H_1(e^{j\Omega})\right\}=-\frac{\Omega}{2}$$

$$H_2\left(e^{j\Omega}\right) = \sum_{n=-\infty}^{\infty} h_2[n]\, e^{-j\Omega n} = \sum_{n=0}^{\infty} \left\{ \frac{1}{2}\delta[n] - \frac{1}{2}\delta[n-1] \right\} e^{-j\Omega n}$$

$$= \frac{1}{2} - \frac{1}{2} e^{-j\Omega}$$

$$= j \cdot e^{-j\frac{\Omega}{2}} \left(e^{j\frac{\Omega}{2}} - e^{-j\frac{\Omega}{2}} \right) / j2 = j \cdot e^{-j\frac{\Omega}{2}} \sin\left(\frac{\Omega}{2}\right)$$

$$= \begin{cases} j \cdot e^{-j\frac{\Omega}{2}} \sin\left(\frac{\Omega}{2}\right), & \forall \Omega > 0 \\ -j \cdot e^{-j\frac{\Omega}{2}} \sin\left(\frac{\Omega}{2}\right), & \forall \Omega < 0 \end{cases}$$

$$H_2\left(e^{j\Omega}\right) = j \cdot e^{-j\frac{\Omega}{2}} \sin\left(\frac{\Omega}{2}\right), \quad \forall \Omega > 0$$

$$= e^{j\frac{\pi}{2}} \cdot e^{-j\frac{\Omega}{2}} \sin\left(\frac{\Omega}{2}\right) = e^{j(\frac{\pi}{2}-\frac{\Omega}{2})} \cdot \sin\left(\frac{\Omega}{2}\right)$$

$$\left|H_2\left(e^{j\Omega}\right)\right| = \left|\sin\left(\frac{\Omega}{2}\right)\right|$$

$$\arg\left\{H_2\left(e^{j\Omega}\right)\right\} = \frac{\pi}{2} - \frac{\Omega}{2}$$

$$H_2\left(e^{j\Omega}\right) = -j \cdot e^{-j\frac{\Omega}{2}} \sin\left(\frac{\Omega}{2}\right), \quad \forall \Omega < 0$$

$$= e^{-j\frac{\pi}{2}} \cdot e^{-j\frac{\Omega}{2}} \sin\left(\frac{\Omega}{2}\right) = e^{j(-\frac{\pi}{2}-\frac{\Omega}{2})} \cdot \sin\left(\frac{\Omega}{2}\right)$$

$$\left|H_2\left(e^{j\Omega}\right)\right| = \left|\sin\left(\frac{\Omega}{2}\right)\right|$$

$$\arg\left\{H_2\left(e^{j\Omega}\right)\right\} = -\frac{\pi}{2} - \frac{\Omega}{2}$$