Fourier Transform of Signals

Lecture 3-5

Continuous-Time Non-Periodic Signals

The Fourier Transform (FT)

-

- $-\infty \qquad +\infty$

$X(j\omega)$ is continuous spectrum and non-periodic.

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)\, e^{j\omega t}\, d\omega \qquad (7)$$

$x(t)$ is continuous signal and non-periodic.

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \qquad (8)$$

FS (if T $\rightarrow \infty$)

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]\, e^{j\omega_0 k t}$$

$$X[k] = \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, e^{-jk\omega_0 t}\, dt$$

$\rightarrow$ FT   ?

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t}\, d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$
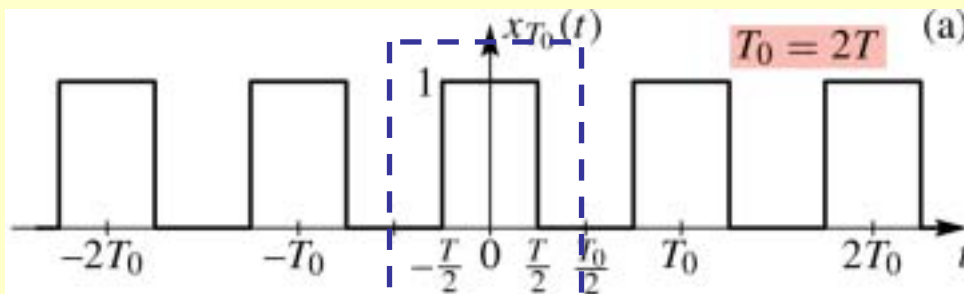
What if Signal *x(t)* is not periodic?

- ## Still Sum of Sinusoids?
  - Non-harmonically related sinusoids
  - Would not be periodic, but would probably be non-zero for all *t.*

- ## Need "Fourier Transform" to Find Spectrum?
  - gives a "sum" (actually an **integral**) that involves **ALL** frequencies
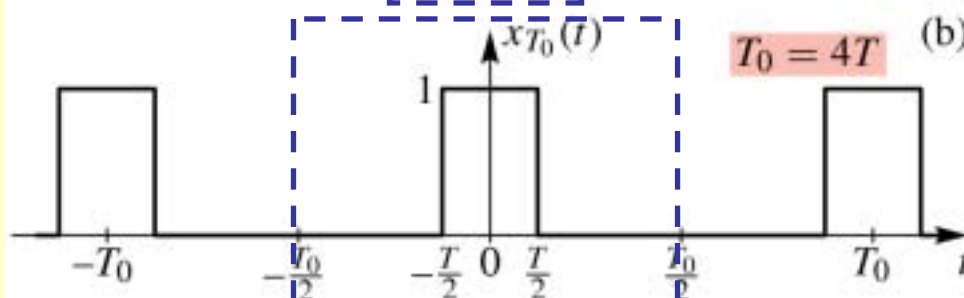 - can represent signals that are identically zero for negative *t.*   !!!!!!!!!
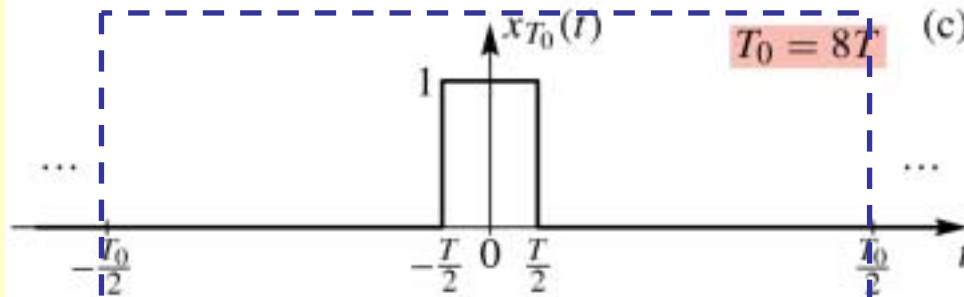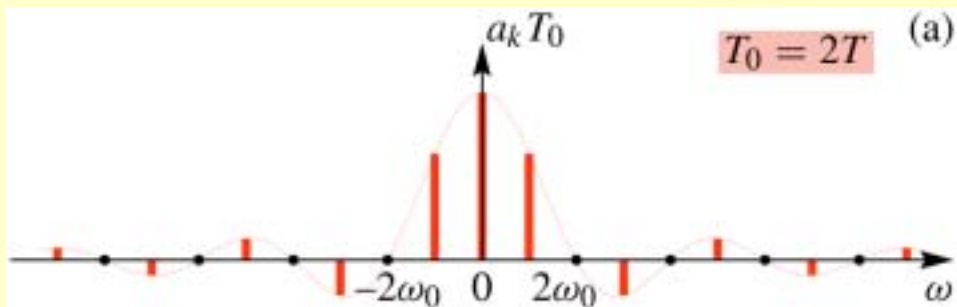
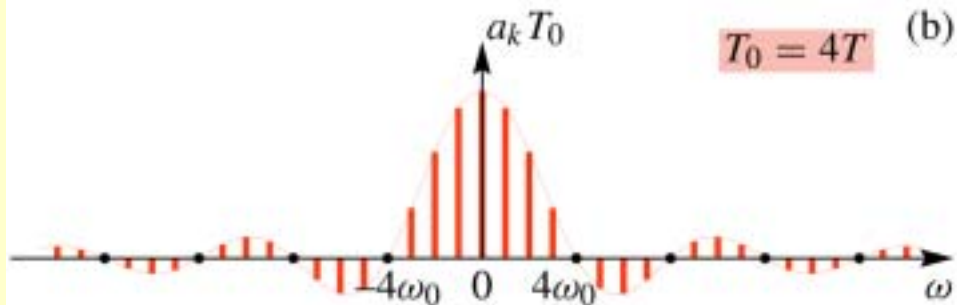# Limiting Behavior of FS

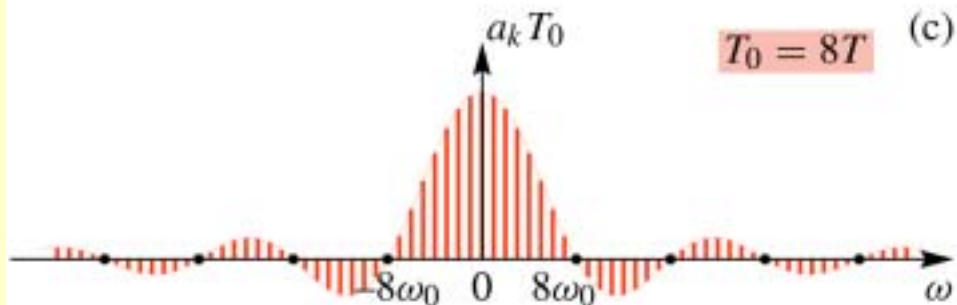$T_0=2T$

$T_0=4T$

$T_0=8T$

Limiting Behavior of Spectrum

$T_0=2T$

$T_0=4T$

$T_0=8T$

FS in the LIMIT (long period)

Fourier Series Formula:

$$x_{T_0}(t) = \sum_{k=-\infty}^{\infty} X[k]e^{j\omega_0 kt}$$

$k\omega_0 = \omega \quad \text{variable}$

$2\pi / T_0 = \omega_0$

$\lim_{T_0 \to \infty} 2\pi / T_0 = d\omega$

$\lim_{T_0 \to \infty} x_{T_0}(t) = x(t) \quad \text{for } -\infty < t < \infty$

$$x_{T_0}(t) = \tfrac{1}{2\pi} \sum_{k=-\infty}^{\infty} \left[T_0 X[k]\right] e^{j(\omega_0 k)t} \left(\tfrac{2\pi}{T_0}\right)$$

$$\Updownarrow$$

$$x(t) = \tfrac{1}{2\pi} \int_{-\infty}^{\infty} \left[X(j\omega)\right] e^{j(\omega)t} d\omega$$

$$x_{T_0}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \left(T_0 X[k]\right) e^{j\omega_0 kt} \left(\frac{2\pi}{T_0}\right) \mapsto x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

$$\lim_{T_0 \to \infty} \frac{2\pi}{T_0} = d\omega$$

$$\lim_{T_0 \to \infty} \frac{2\pi}{T_0} k = \omega$$

$$\lim_{T_0 \to \infty} T_0 \cdot X[k] = X(j\omega)$$

$$X[k] = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x_{T_0}(t) e^{-j\omega_0 kt} dt \mapsto X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

FS

Fourier Analysis
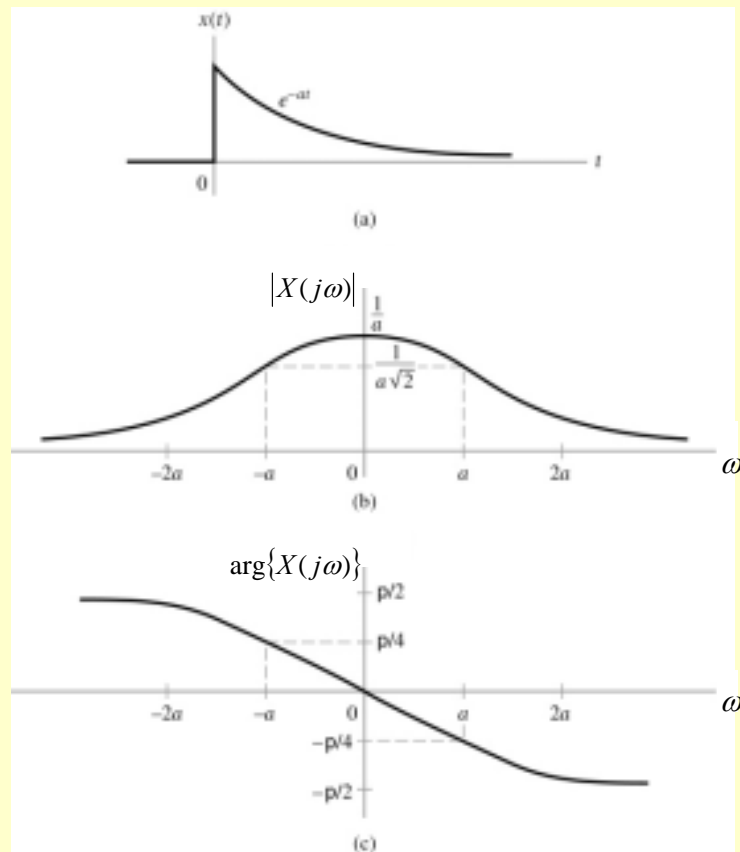
Fourier Transform Defined

- For non-periodic signals

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} d\omega$$
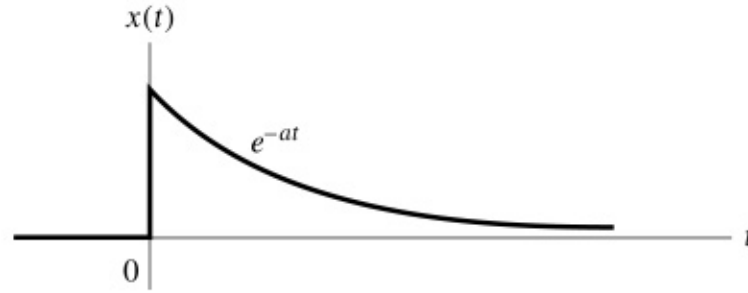
Fourier Synthesis

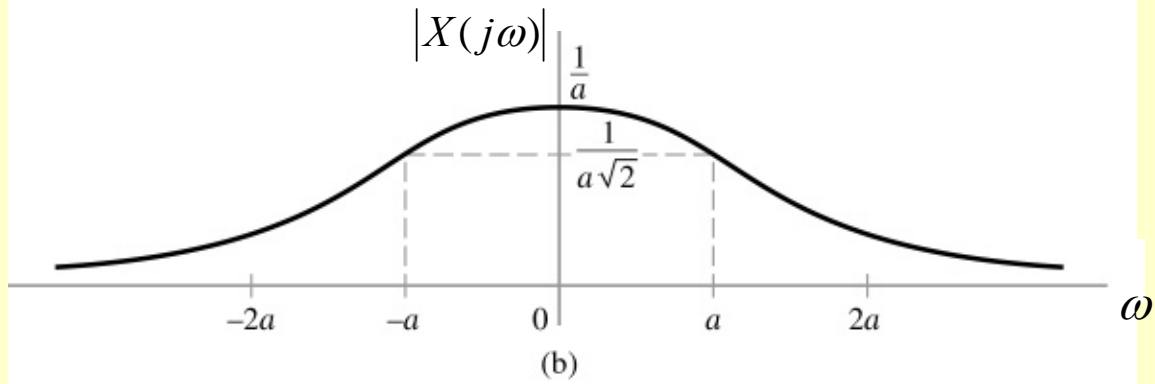$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$$

Fourier Analysis

x(t)

e^{-at}

(a)

$|X(j\omega)|$

$\frac{1}{a}$

$\frac{1}{a\sqrt{2}}$

$-2a$ $-a$ 0 a $2a$ ω

(b)

$\arg\{X(j\omega)\}$

p/2

p/4

$-2a$ $-a$ 0 a $2a$ ω

$-p/4$

$-p/2$

(c)

Ex 3.24 Find the FT of $x(t) = e^{-at}u(t)$:

(a) Real time-domain exponential signal.
(b) Magnitude spectrum.
(c) Phase spectrum.

$x(t)$

e^{-at}

t

0

(a)

$\left|X(j\omega)\right|$

$\dfrac{1}{a}$

$\dfrac{1}{a\sqrt{2}}$

$-2a$ $-a$ 0 a $2a$ ω

(b)

$\arg\{X(j\omega)\}$

p/2

p/4

$-2a$ $-a$ 0 a $2a$ ω

$-$p/4

$-$p/2

(c)

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Solution of Ex 3.24:

$$case\,1: \quad a \leq 0, \qquad \text{(Growing Exponential Signal)}$$

$$\int_0^\infty e^{-at}\,dt = \infty. \qquad not\ absolutely\ summable.$$

$$case\,2: \quad a > 0, \qquad \text{(Decaying Exponential Signal)}$$

$$X(j\omega) = \int_{-\infty}^\infty e^{-at} u(t)\, e^{-j\omega t}\,dt$$

$$= \int_0^\infty e^{-(a+j\omega)t}\,dt = \frac{-1}{a+j\omega} e^{-(a+j\omega)t}\bigg|_0^\infty = \frac{-1}{a+j\omega}(0-1)$$
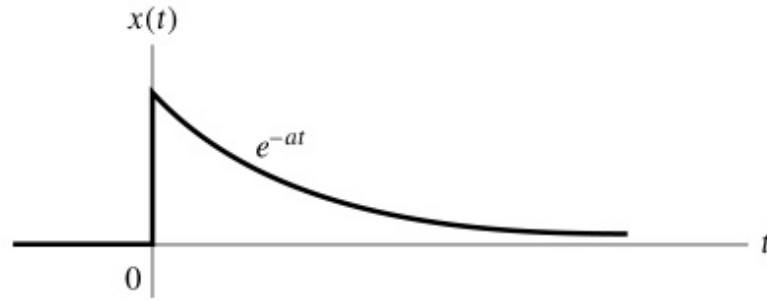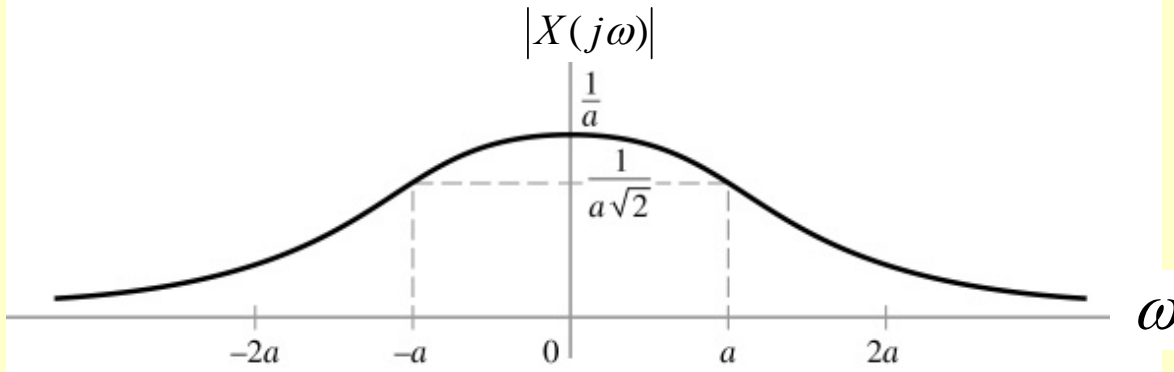
$$= \frac{1}{a+j\omega}$$

Solution: (cont.)

$$X(j\omega) = \frac{1}{a + j\omega}$$

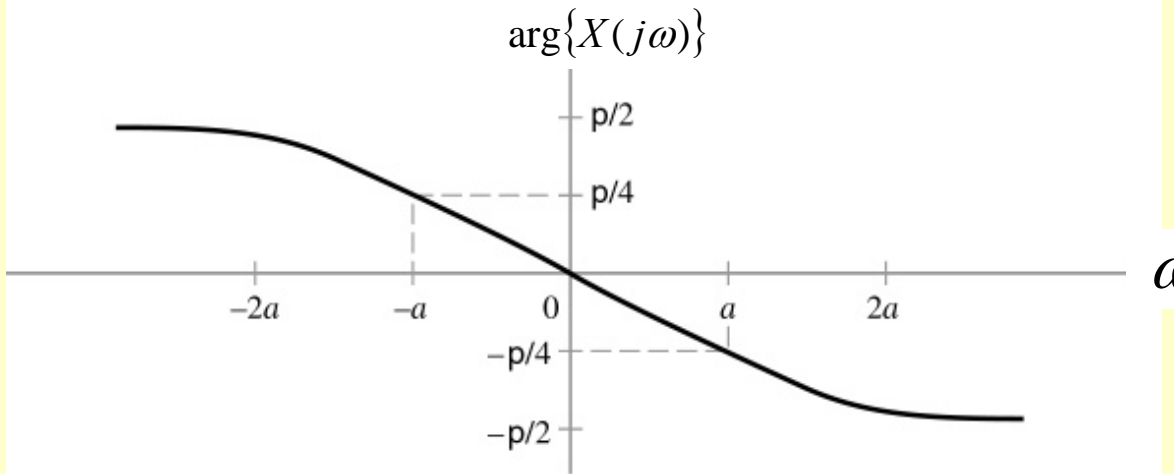$$\left|X(j\omega)\right| = \frac{1}{\sqrt{a^2 + \omega^2}}; \qquad \arg\{X(j\omega)\} = -\tan^{-1}\left(\frac{\omega}{a}\right)$$

(a)

(b)

(c)

Example 3.25:

Consider the rectangular pulse in the figure (a), and find the FT of x(t), $X(j\omega)$ = ?

$$x(t) = \begin{cases} 1, & -T_0 < t < T_0 \\ 0, & |t| > T_0 \end{cases}$$

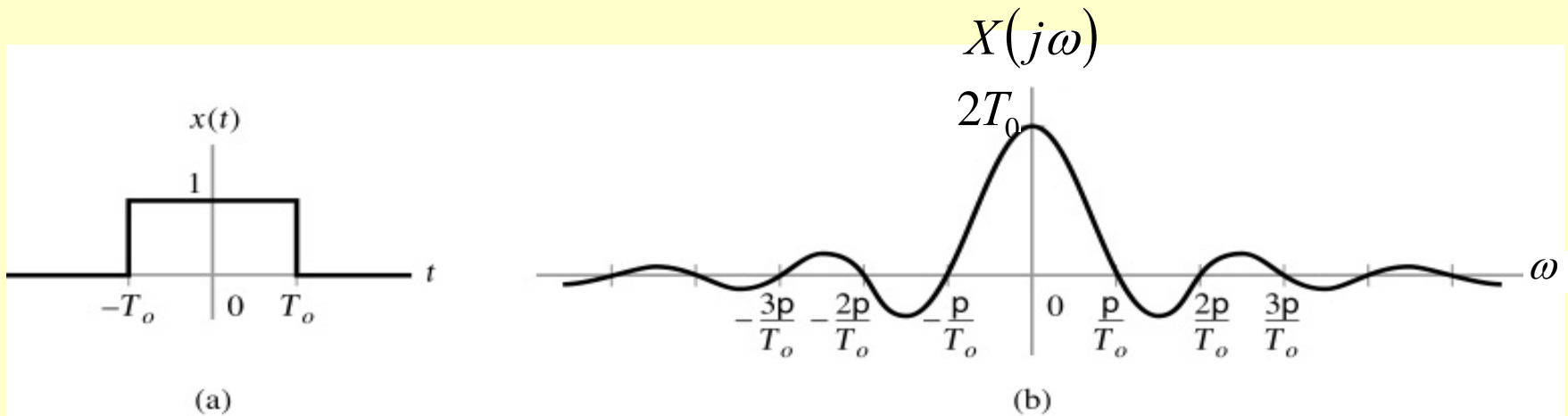(a) Rectangular pulse in the time domain.
(b) FT in the frequency domain.

Audio & DSP Lab.

Solution of Ex 3.25:

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-T_0}^{+T_0} 1 \cdot e^{-j\omega t}\, dt = -\frac{1}{j\omega} e^{-j\omega t} \Big|_{-T_0}^{T_0}$$

$$= -\frac{1}{j\omega}\left(e^{-j\omega T_0} - e^{j\omega T_0}\right) = \frac{1}{j\omega}\left(e^{j\omega T_0} - e^{-j\omega T_0}\right)$$

$$= \frac{2}{\omega}\left(\frac{e^{j\omega T_0} - e^{-j\omega T_0}}{j2}\right) = \frac{2}{\omega}\sin(\omega T_0), \quad \forall\, \omega \neq 0$$
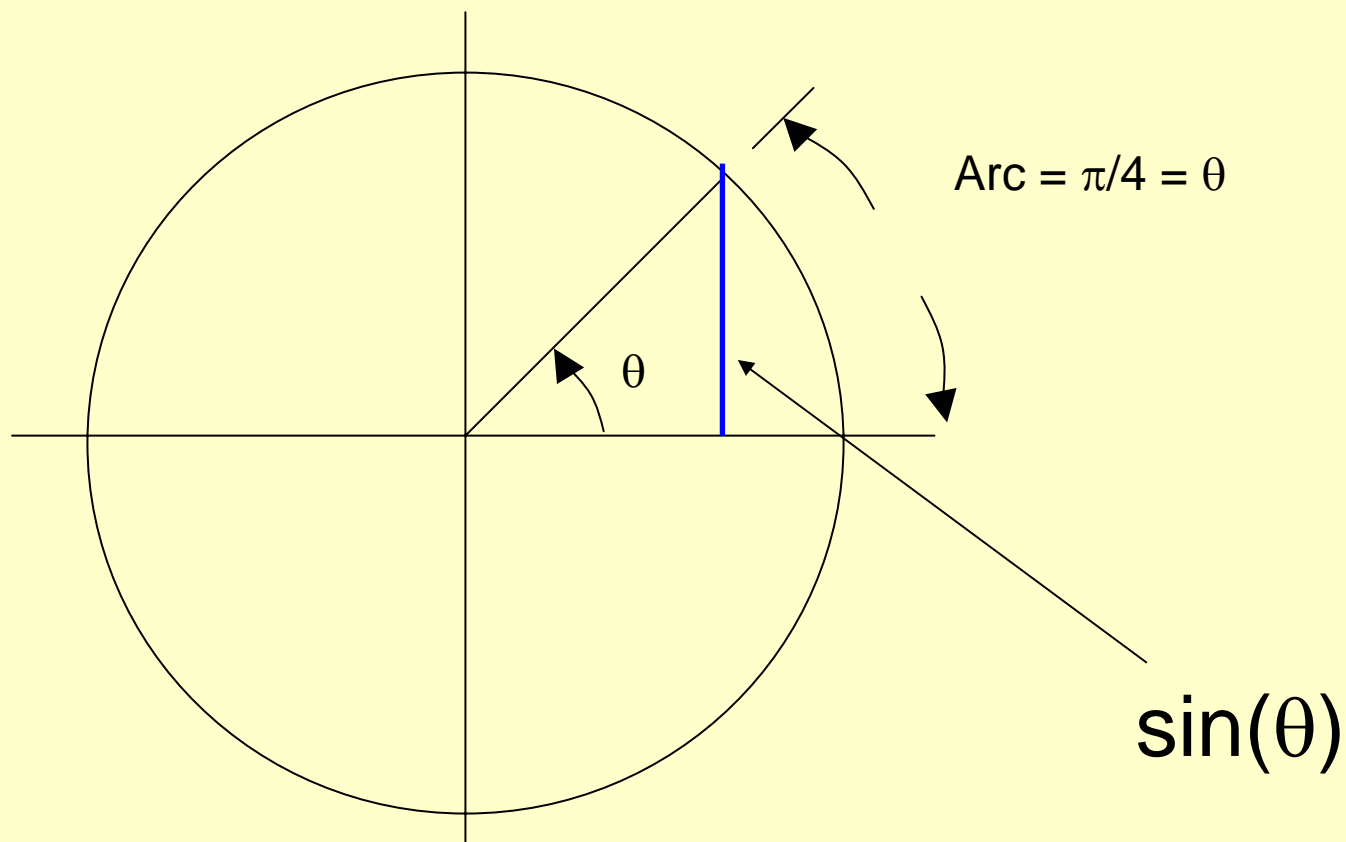
$$\forall \omega = 0,$$

$$\lim_{\omega \to 0} X(j\omega) = \lim_{\omega \to 0} \frac{2}{\omega}\sin(\omega T_0) = \frac{2}{\omega}(\omega T_0) = 2\,T_0$$

(a)

(b)

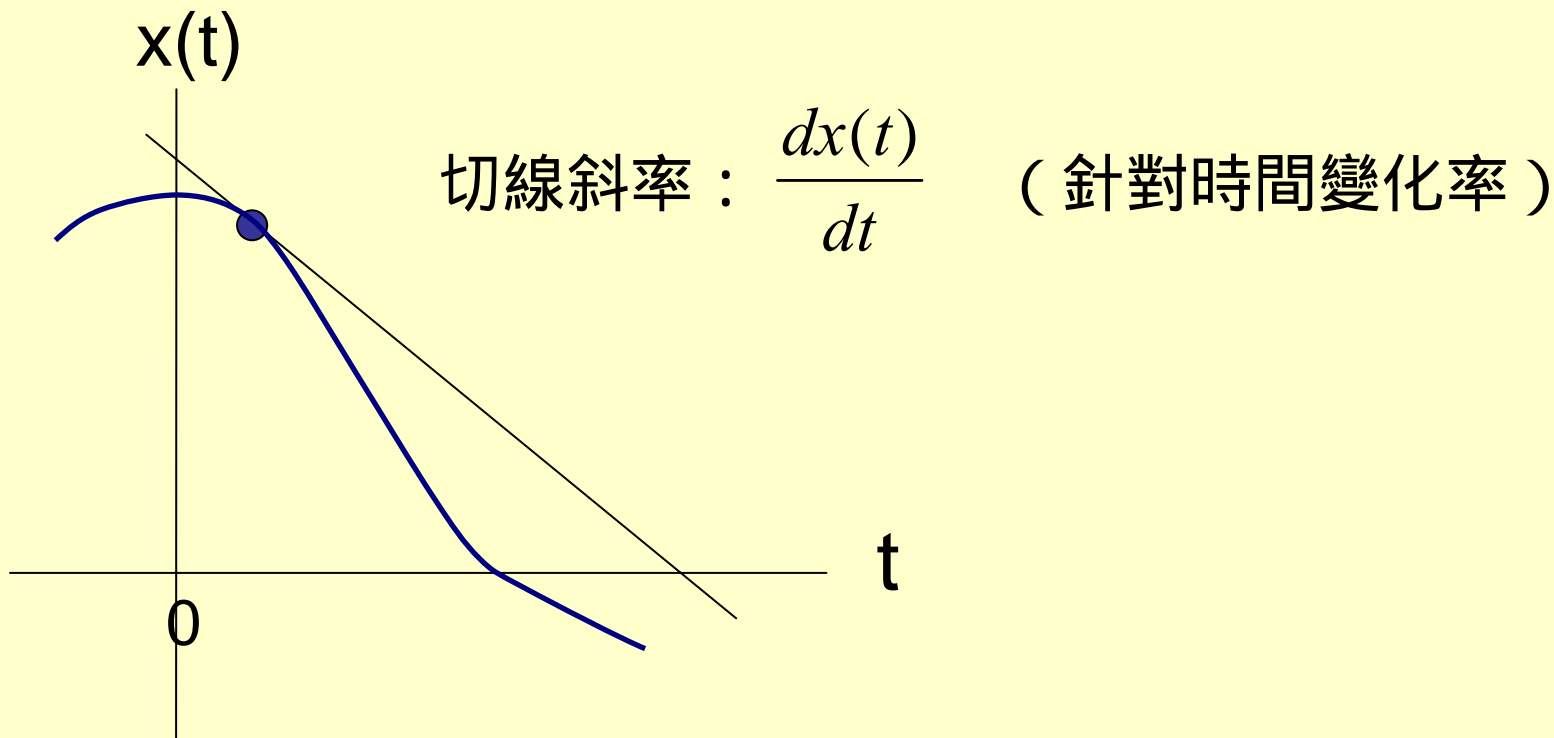- $\sin(\theta) \approx \theta$ if θ is small.

Arc = $\pi/4 = \theta$

θ

$\sin(\theta)$

- Changing Rate
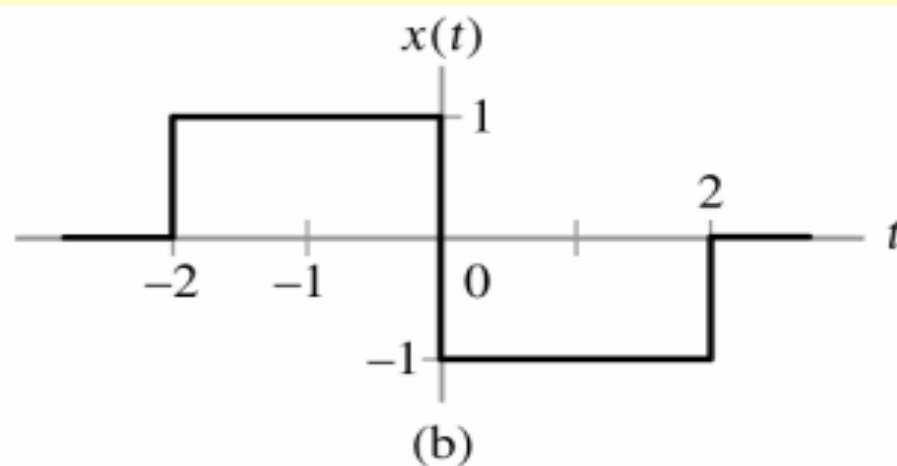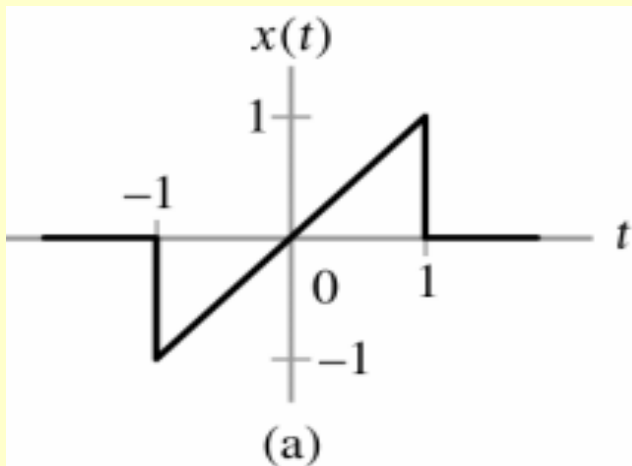
x(t)

$$\frac{dx(t)}{dt}$$

t

0

Time-domain signals for Problem 3.14.

Find the FT of the parts (a) and (b) = ?
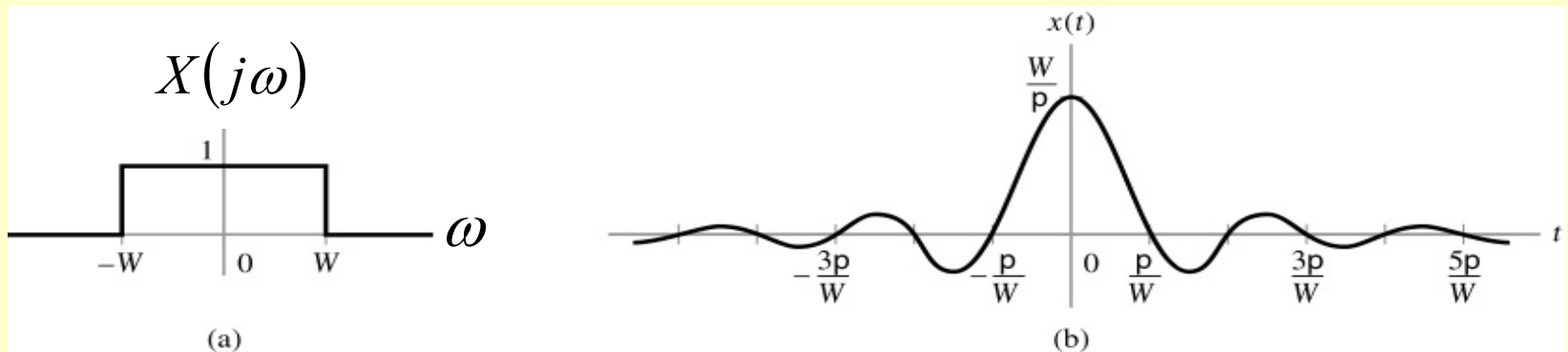

(a)

(b)

Example 3.26:

(a) Rectangular spectrum in the frequency domain.

(b) Inverse FT in the time domain.

$$X(j\omega) = \begin{cases} 1, & -W < \omega < W \\ 0, & |\omega| > W \end{cases}$$

(a)   (b)

Solution of the Ex 3.26:

$$\text{sinc}(W) = \frac{\sin(\pi W)}{\pi W}$$

$$x(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(j\omega) e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi} \int_{-W}^{|W} e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi \, jt} e^{j\omega t} \Big|_{-W}^{W} = \frac{1}{2\pi \, jt} \left(e^{jWt} - e^{-jWt} \right)$$

$$= \frac{1}{\pi t} \frac{e^{jWt} - e^{-jWt}}{j2}$$

$$= \frac{1}{\pi t} \sin(Wt)$$

$$\frac{W}{\pi} \text{sinc}\left(\frac{Wt}{\pi} \right) = \frac{W}{\pi} \frac{\sin(Wt)}{Wt} = \frac{\sin(Wt)}{\pi t}$$

Example 3.27: Find the FT of x(t) = δ(t)

Solution:

$$X(j\omega) = \int_{-\infty}^{+\infty} \delta(t)\, e^{-j\omega t}\, dt = 1$$

$notation:$

$$X(j\omega) = FT\{\delta(t)\} = 1$$

or

$$\delta(t) \quad \xrightarrow{FT} \quad 1$$

Example 3.28: Find the inv FT of $X(j\omega) = 2\pi\delta(\omega)$?

Solution:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} 2\pi\delta(\omega)\, e^{j\omega t}\, d\omega = 1$$

$notation:$ 0

$$x(t) = FT^{-1}\{2\pi\delta(\omega)\} = 1$$

or

$$1 \xleftrightarrow{FT} 2\pi\delta(\omega)$$

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Frequency-domain signals for Problem 3.15.
Find the inverse of FT = ?

$$|X(j\omega)|$$

$$\arg\{X(j\omega)\}$$

$$X(j\omega)$$

(a)

(b)