Fourier Transform of Signals

訊號的傅立葉轉換

Lecture 3-6

Fourier Representations

DTFS: $x[n]$: discrete-time and periodic signal

$X[k]$: discrete and periodic spectrum

FS: $x(t)$: continuous-time and periodic signal

$X[k]$: discrete-time and non-periodic spectrum

DTFT: $x[n]$: discrete-time and non-periodic signal

$X(e^{j\Omega})$: continuous and periodic spectrum

FT: $x(t)$: continuous-time and non-periodic signal

$X(j\omega)$: continuous-time and non-periodic spectrum

Linearity and Symmetry

Linear Properties: (想一想 線性特性的用處？)

$$z(t) = ax(t) + by(t) \xleftrightarrow{FT} Z(j\omega) = aX(j\omega) + bY(j\omega)$$

$$z(t) = ax(t) + by(t) \xleftrightarrow{FS} Z[k] = aX[k] + bY[k]$$

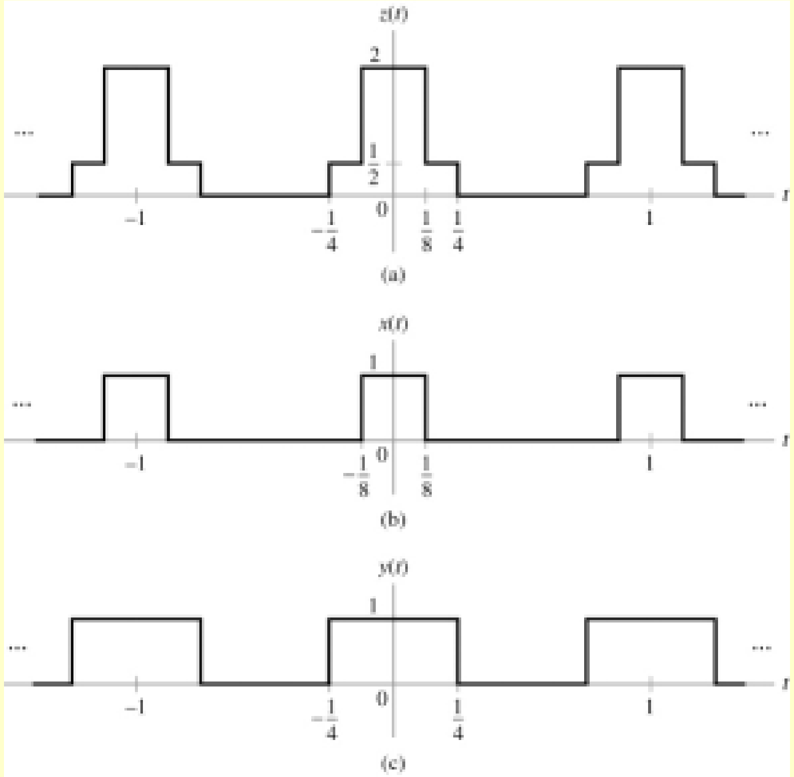
$$z[n] = ax[n] + by[n] \xleftrightarrow{DTFT} Z(e^{j\Omega}) = aX(e^{j\Omega}) + bY(e^{j\Omega})$$

$$z[n] = ax[n] + by[n] \xleftrightarrow{DTFS} Z[k] = aX[k] + bY[k]$$

Representation of the periodic signal $z(t)$ as a weighted sum of periodic square waves:

$z(t) = (3/2)x(t) + (1/2)y(t)$.

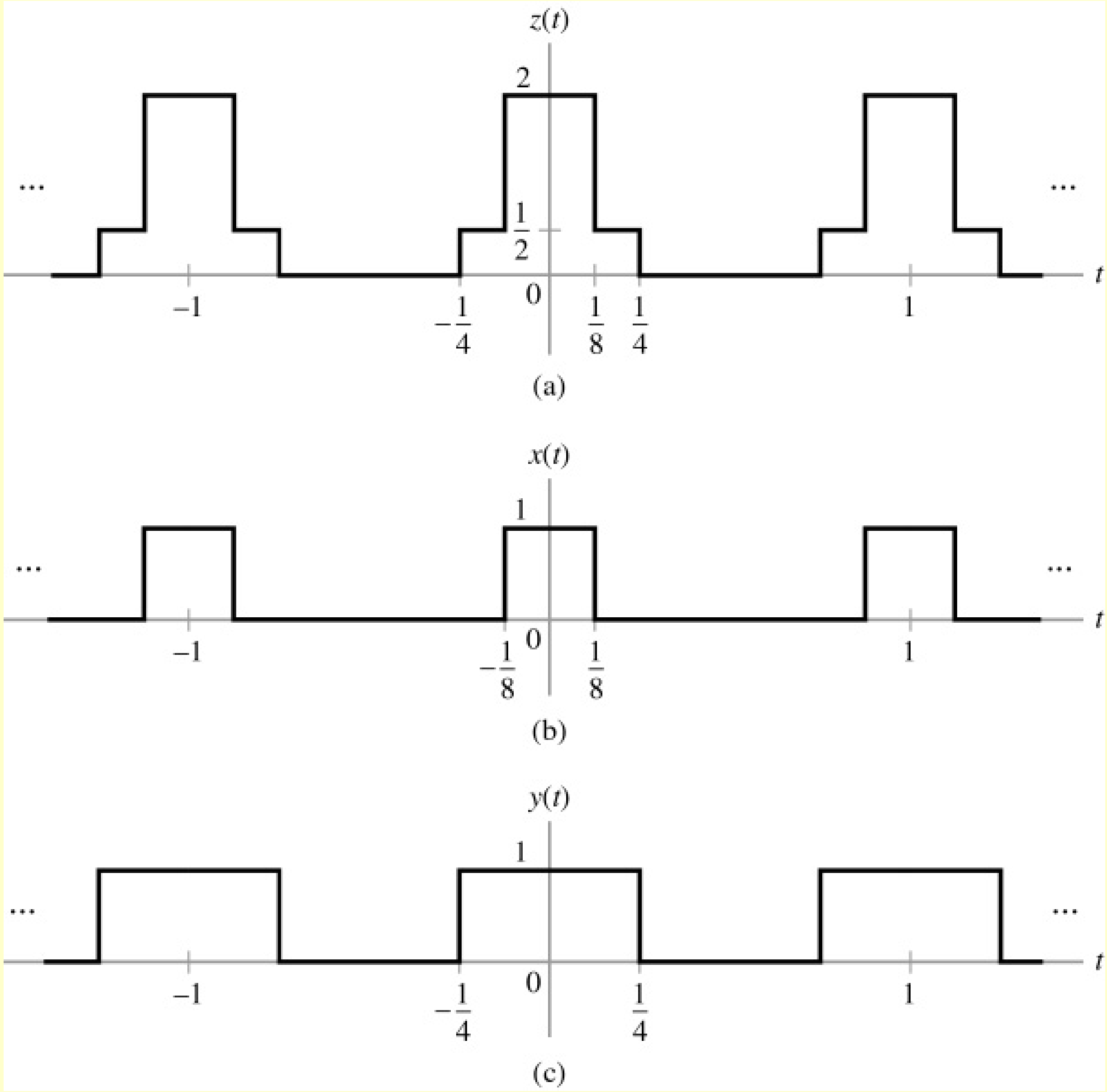
(a) $z(t)$.
(b) $x(t)$.
(c) $y(t)$.



Z[k] = (3/2)X[k] +(1/2)Y[k]

X[k] = ?

Y[k] = ?



$z(t) = (3/2)x(t) + (1/2)y(t).$

Problem 3.16:

利用線性性質求取下述訊號傅立葉表示法

$$x(t) = 2e^{-t}u(t) - 3e^{-2t}u(t)$$

Solution:

$$X(j\omega) = 2 \cdot FT\left\{e^{-t}u(t)\right\} - 3 \cdot FT\left\{e^{-2t}u(t)\right\}$$

$$= 2\left(\frac{1}{1+j\omega}\right) - 3\left(\frac{1}{2+j\omega}\right)$$

$$= \frac{2}{1+j\omega} - \frac{3}{2+j\omega} = ?$$

請繼續完成 ...

Symmetry Properties: Real Signals

$$\because x(t) = x^*(t)$$ x(t) 是實數

$$X^*(j\omega) = \left[\int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t} dt\right]^* = \int_{-\infty}^{+\infty} x^*(t)\, e^{j\omega t} dt$$

$$= \int_{-\infty}^{+\infty} x(t)\, e^{j\omega t} dt = \int_{-\infty}^{+\infty} x(t)\, e^{-j(-\omega)t} dt$$

$$= X(-j\omega)$$

$\because x(t) = x^*(t)$ x(t) 是實數

$$X^*(j\omega) = X(-j\omega)$$

X (jω) 為 複數共軛對稱 (complex-conjugate symmetric)

$\because$ X (jω) = Re{X(jω)} + j Im{X(jω)},

X*(jω) = Re{X(jω)} - j Im{X(jω)},

X (-jω) = Re{X(-jω)} + j Im{X(-jω)},

→ Re {X(jω)} = Re {X(-jω)}

→ Im {X(jω)} = -Im {X(-jω)}

x(t) FT後 實數部份是頻率偶函數(Even function)

x(t) FT後 虛數部份是頻率奇函數(Odd function)

振幅頻譜是*偶對稱* 相位頻譜是*奇對稱*

Symmetry Properties: Imaginary Signals

$$\because -x(t) = x^*(t)$$

x(t) 是虛數

$$X^*(j\omega) = \left[\int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t} dt\right]^* = \int_{-\infty}^{+\infty} x^*(t)\, e^{j\omega t} dt$$

$$= -\int_{-\infty}^{+\infty} x(t)\, e^{j\omega t} dt = -\int_{-\infty}^{+\infty} x(t)\, e^{j\omega t} dt$$

$$= -\int_{-\infty}^{+\infty} x(t)\, e^{-j(-\omega)t} dt = -X(-j\omega)$$

$\because -x(t) = x^*(t)$ x(t) 是虛數 $X^*(j\omega) = -X(-j\omega)$

複數共軛對稱 (complex-conjugate symmetric)

$\because$ X(jω) = Re{X(jω)} + j Im{X(jω)},

X*(jω) = Re{X(jω)} - j Im{X(jω)},

-X(-jω) = -Re{X(-jω)} - j Im{X(-jω)},

→Re{X(jω)} = -Re{X(-jω)}

→Im{X(jω)} = Im{X(-jω)}

x(t) FT後實數部份是頻率奇函數(Odd function)

x(t) FT後虛數部份是頻率偶函數(Even function)

振幅頻譜是*偶對稱* 相位頻譜是*奇對稱*

x(t) 是實數

FT後 實數部份是<u>頻率偶函數</u>(Even function)

FT後 虛數部份是<u>頻率奇函數</u>(Odd function)

振幅頻譜是<u>*偶對稱*</u>　　　　相位頻譜是<u>*奇對稱*</u>

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x(t) 是虛數

FT後 實數部份是<u>頻率奇函數</u>(Odd function)

FT後 虛數部份是<u>頻率偶函數</u>(Even function)

振幅頻譜是<u>*偶對稱*</u>　　　　相位頻譜是<u>*奇對稱*</u>

Symmetry Property 應用範例

若實數輸入訊號：$x(t) = A\cos(\omega_0 t - \phi) = \dfrac{Ae^{-j\phi}e^{j\omega_0 t} + Ae^{j\phi}e^{-j\omega_0 t}}{2}$

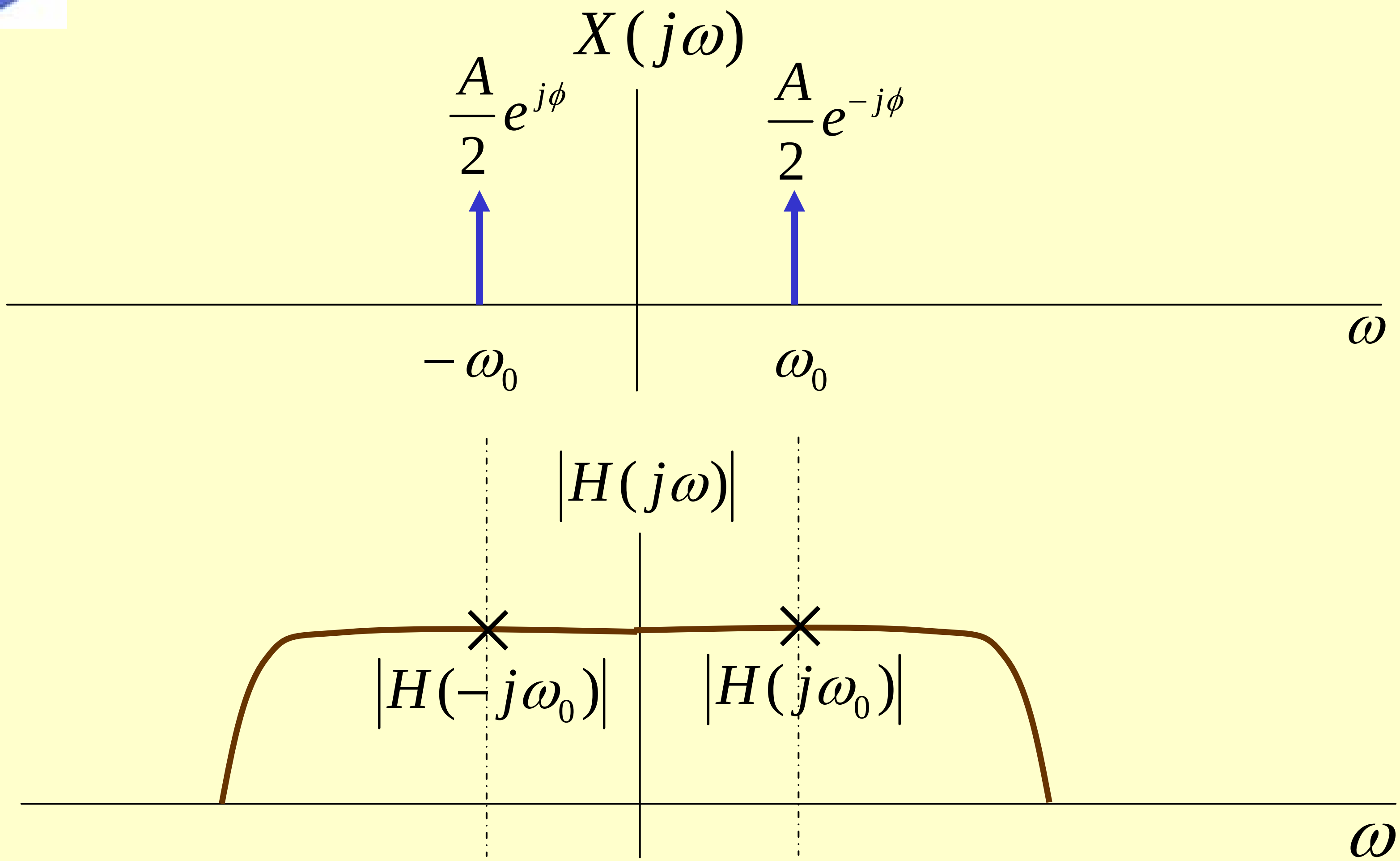
系統振幅響應：$|H(j\omega)|$

系統相位響應： $\arg[H(j\omega)]$

系統輸出： $y(t) = \dfrac{A}{2}e^{-j\phi}e^{j\omega_0 t} \cdot H(j\omega_0) + \dfrac{A}{2}e^{j\phi}e^{-j\omega_0 t} \cdot H(-j\omega_0)$

重新整理系統輸出：

$$y(t) = \frac{A}{2}e^{-j\phi}e^{j\omega_0 t} \cdot \left\{|H(j\omega_0)|e^{j\arg[H(j\omega_0)]}\right\} + \frac{A}{2}e^{j\phi}e^{-j\omega_0 t} \cdot \left\{|H(-j\omega_0)|e^{j\arg[H(-j\omega_0)]}\right\}$$



請補上“ 相位響應 ” 後，繪出“ 輸出頻譜 ” $Y(j\omega)$

系統振幅響應是***偶對稱***：$\left|H(j\omega)\right| = \left|H(-j\omega)\right|$

系統相位響應是***奇對稱***：$\arg\left[H(j\omega)\right] = -\arg\left[H(-j\omega)\right]$

整理系統輸出：

$$y(t) = \frac{A}{2}e^{-j\phi}e^{j\omega_0 t}\left\{\left|H(j\omega_0)\right|e^{j\arg[H(j\omega_0)]}\right\} + \frac{A}{2}e^{j\phi}e^{-j\omega_0 t}\left\{\left|H(-j\omega_0)\right|e^{j\arg[H(-j\omega_0)]}\right\}$$

$$= \frac{A}{2}e^{-j\phi}e^{j\omega_0 t}\left\{\left|H(j\omega_0)\right|e^{j\arg[H(j\omega_0)]}\right\} + \frac{A}{2}e^{j\phi}e^{-j\omega_0 t}\left\{\left|H(-j\omega_0)\right|e^{-j\arg[H(j\omega_0)]}\right\}$$

$$= \frac{A}{2}\left|H(j\omega_0)\right|\left\{e^{-j\phi}e^{j\omega_0 t}e^{j\arg[H(j\omega_0)]} + e^{j\phi}e^{-j\omega_0 t}e^{-j\arg[H(j\omega_0)]}\right\}$$

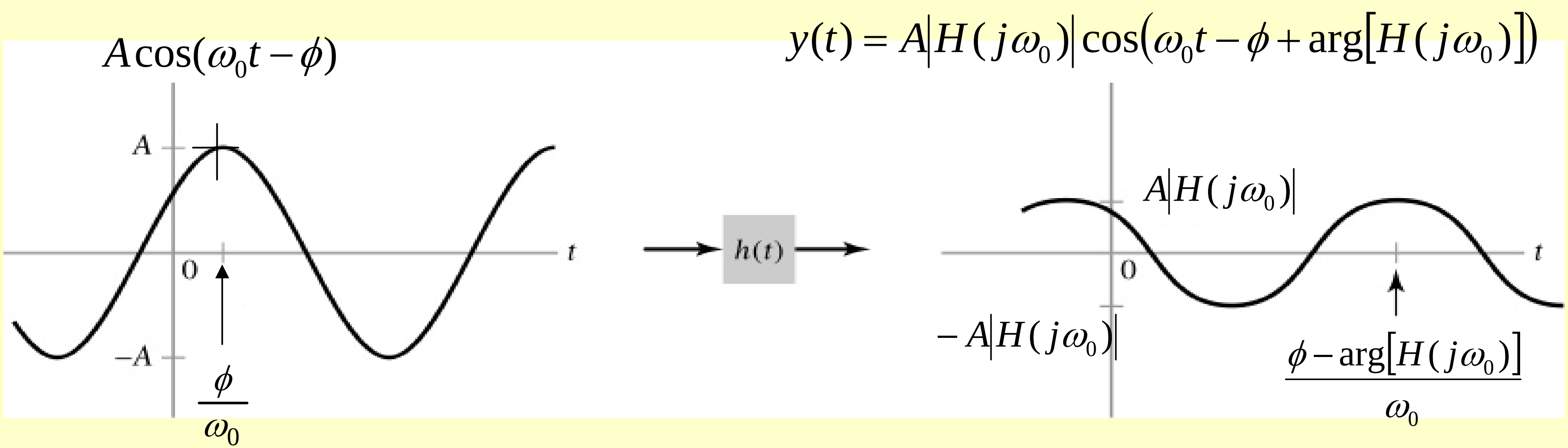
$$= A\left|H(j\omega_0)\right|\left\{\left(e^{j(\omega_0 t-\phi+\arg[H(j\omega_0)])} + e^{-j(\omega_0 t-\phi+\arg[H(j\omega_0)])}\right)/2\right\}$$

$$= A\left|H(j\omega_0)\right|\cos\left(\omega_0 t - \phi + \arg\left[H(j\omega_0)\right]\right)$$

A sinusoidal input to an LTI system:

It results in a sinusoidal output of the same frequency.

The amplitude and phase are modified by the system's frequency response.



Symmetry Properties: Even Signals

Even function : $x(-t) = x(t)$

Real Function: $x^*(t) = x(t)$

Even and Real Function: $x(t)^* = x(-t)$

$$X^*(j\omega) = \left[\int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t} dt\right]^* = \int_{-\infty}^{+\infty} x^*(t)\, e^{j\omega t} dt$$

$$= \int_{-\infty}^{+\infty} x(-t)\, e^{j\omega t} dt = \int_{-\infty}^{+\infty} x(-t)\, e^{-j\omega(-t)} dt, \quad let \quad \tau = -t$$

$$= \int_{-\infty}^{+\infty} x(\tau)\, e^{-j\omega\tau} d\tau$$

$$= X(j\omega)$$

$X(j\omega)$ *is real.*

Symmetry Properties: Odd Signals

Odd function : $x(-t) = -x(t)$

Real Function: $x^*(t) = x(t)$

Odd and Real Function: $x(t)^* = -x(-t)$

$$X^*(j\omega) = \left[\int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t} dt\right]^* = \int_{-\infty}^{+\infty} x^*(t)\, e^{j\omega t} dt$$

$$= -\int_{-\infty}^{+\infty} x(-t)\, e^{j\omega t} dt = -\int_{-\infty}^{+\infty} x(-t)\, e^{-j\omega(-t)}\, dt, \quad let \quad \tau = -t$$

$$= -\int_{-\infty}^{+\infty} x(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$= -X(j\omega)$$

$X(j\omega)$ *is imaginary.*

補充資料: Odd Signals

Odd function : x(-t) = -x(t)

Imaginary Function: $x^*(t) = -x(t)$

Odd and Imaginary Function: $x(t)^* = x(-t)$

$$X^*(j\omega) = \left[\int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t} dt\right]^* = \int_{-\infty}^{+\infty} x^*(t)\, e^{j\omega t} dt$$

$$= \int_{-\infty}^{+\infty} x(-t)\, e^{j\omega t} dt = \int_{-\infty}^{+\infty} x(-t)\, e^{-j\omega(-t)} dt, \quad let \quad \tau = -t$$

$$= \int_{-\infty}^{+\infty} x(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$= X(j\omega)$$

$X(j\omega)$ *is real.*

補充資料: Even Signals

Even function : $x(-t) = x(t)$

Imaginary Function: $x^*(t) = -x(t)$

Even and Imaginary Function: $x(t)^* = -x(-t)$

$$X^*(j\omega) = \left[\int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t} dt\right]^* = \int_{-\infty}^{+\infty} x^*(t)\, e^{j\omega t} dt$$

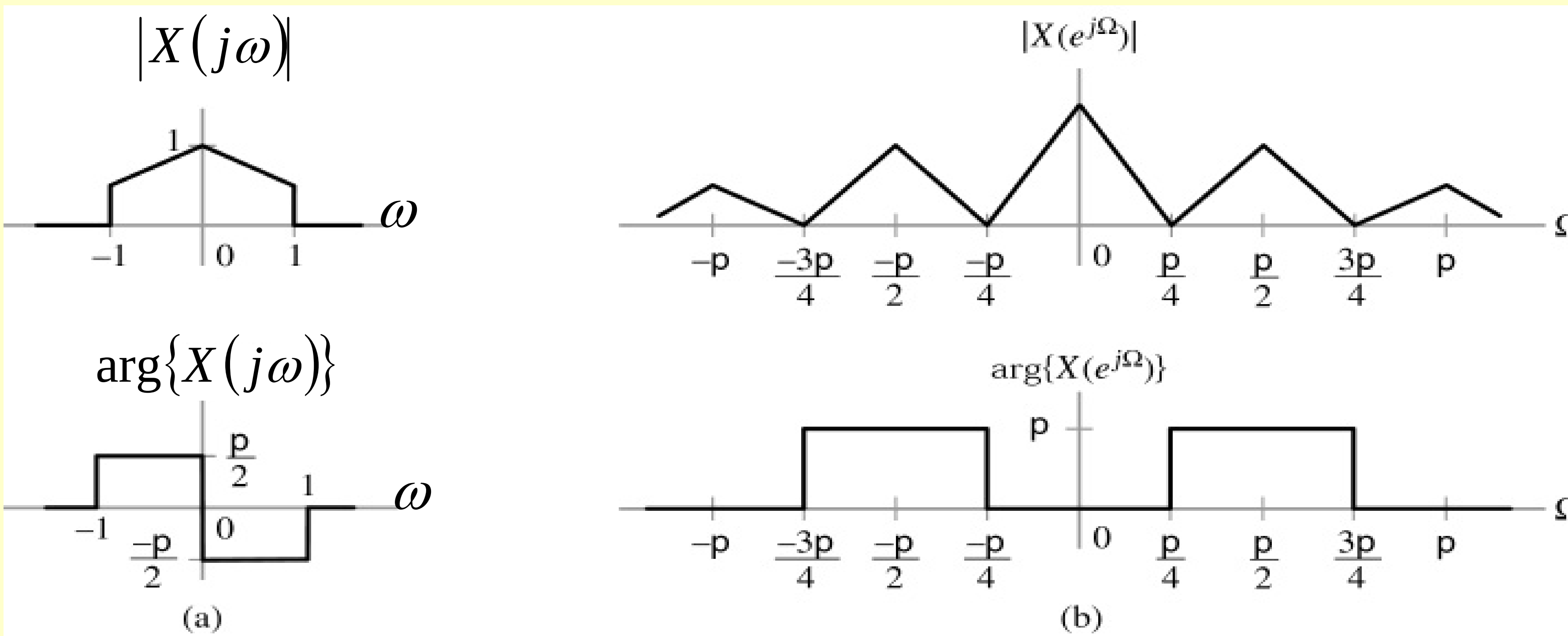
$$= -\int_{-\infty}^{+\infty} x(-t)\, e^{j\omega t} dt = -\int_{-\infty}^{+\infty} x(-t)\, e^{-j\omega(-t)} dt, \quad let \quad \tau = -t$$

$$= -\int_{-\infty}^{+\infty} x(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$= -X(j\omega)$$

$X(j\omega)$ *is imaginary.*

Problem 3.17. 判定(a)及(b) 所對應時域訊號是實、複數以及是偶、奇函數？



(a) (b)

(a) The Magnitude Response is even, and the Phase Response is odd functions.

→ The x(t) is a real and odd function.

時域中(實/奇)訊號 → 偶振幅奇相位/頻域虛數→ /相位 $\pm\pi/2$ 變化

(b) The Magnitude Response is even, and the Phase Response is even functions.

→ The x[n] is a real and even function.

時域中(實/偶)訊號 →偶振幅偶相位/頻域實數 →相位 0 和 π 變化

Convolution Property 褶積性質

$$y(t) = h(t) * x(t) \xleftrightarrow{FT} Y(j\omega) = H(j\omega) \cdot X(j\omega)$$

$$y[n] = h[n] * x[n] \xleftrightarrow{DTFT} Y(e^{j\Omega}) = H(e^{j\Omega}) \cdot X(e^{j\Omega})$$

Convolution of Non-periodic Signal
非週期訊號的摺積

$$y(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$\Updownarrow FT$$

$$Y(j\omega) = H(j\omega) \cdot X(j\omega)$$

where

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[Y(j\omega)\right] e^{j\omega t} d\omega$$

Audio & DSP Lab.

摺積 vs. 乘積

$$y(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$\because\ x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{j\omega t} d\omega$$

$$\therefore\ x(t-\tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{j\omega(t-\tau)} d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega) e^{-j\omega\tau}\ e^{j\omega t} d\omega$$

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)\, [\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{-j\omega\tau} e^{j\omega t} d\omega] d\tau$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} h(\tau)\, e^{-j\omega\tau} d\tau\right] X(j\omega) e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left[H(j\omega)\right] X(j\omega) e^{j\omega t} d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left[H(j\omega)\cdot X(j\omega)\right] e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left[Y(j\omega)\right] e^{j\omega t} d\omega$$

Example 3.31:

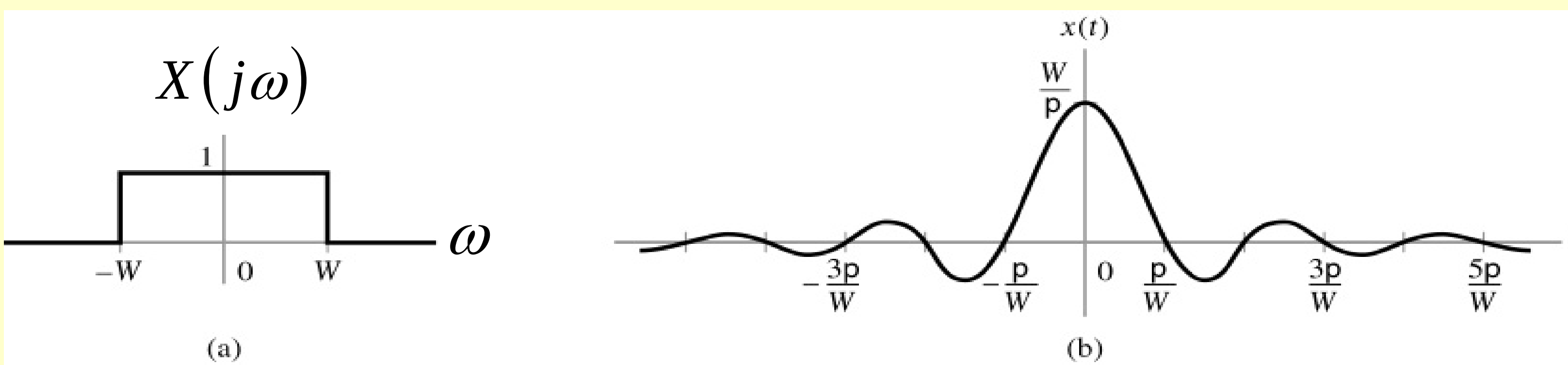
$x(t)$ = (1/πt) sin(πt), h(t) = (1/πt) sin(2πt),

$$y(t) = x(t) * h(t) = ?$$

Solution: $y(t) = (1/\pi t)\sin(\pi t)$

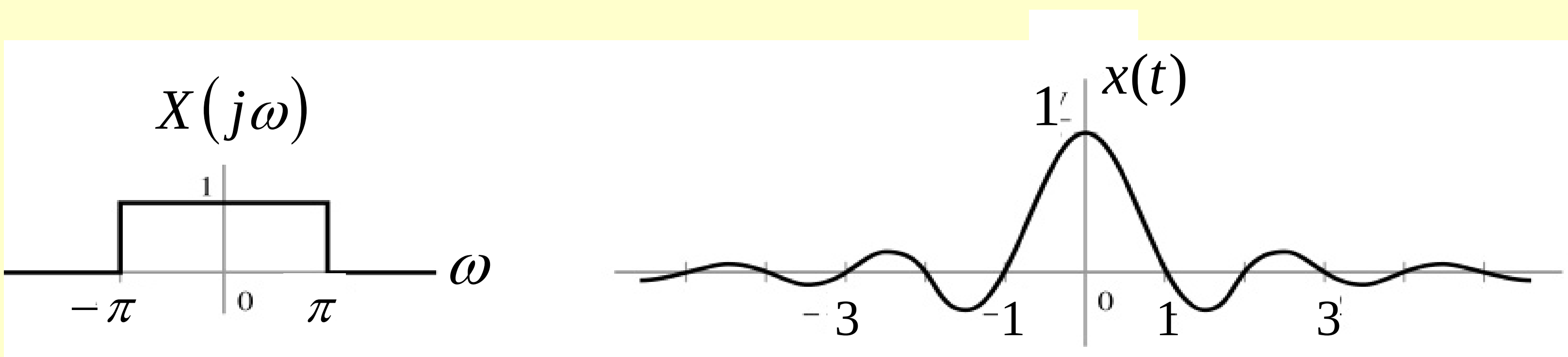
回想

$$x(t) = \frac{1}{\pi t}\sin(Wt) \overset{FT}{\longleftrightarrow} X(j\omega) = \begin{cases} 1, & -W < \omega < W \\ 0, & |\omega| > W \end{cases}$$



$x(t) = (1/\pi t)\sin(\pi t)$,

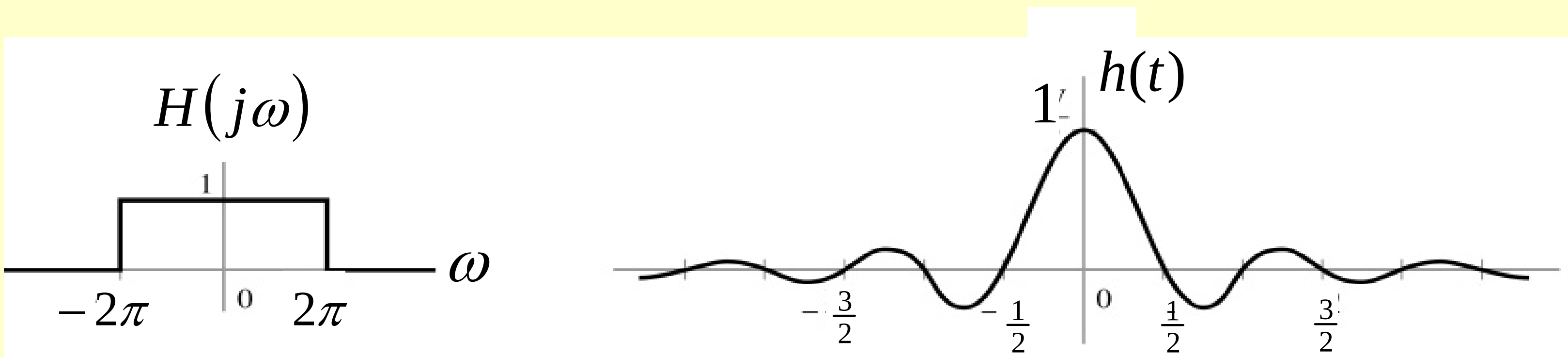
$$x(t) = \frac{1}{\pi t}\sin\left(\pi t\right) \leftrightarrow X(j\omega) = \begin{cases} 1, & -\pi < \omega < \pi \\ 0, & \left|\omega\right| > \pi \end{cases}$$



(a) (b)

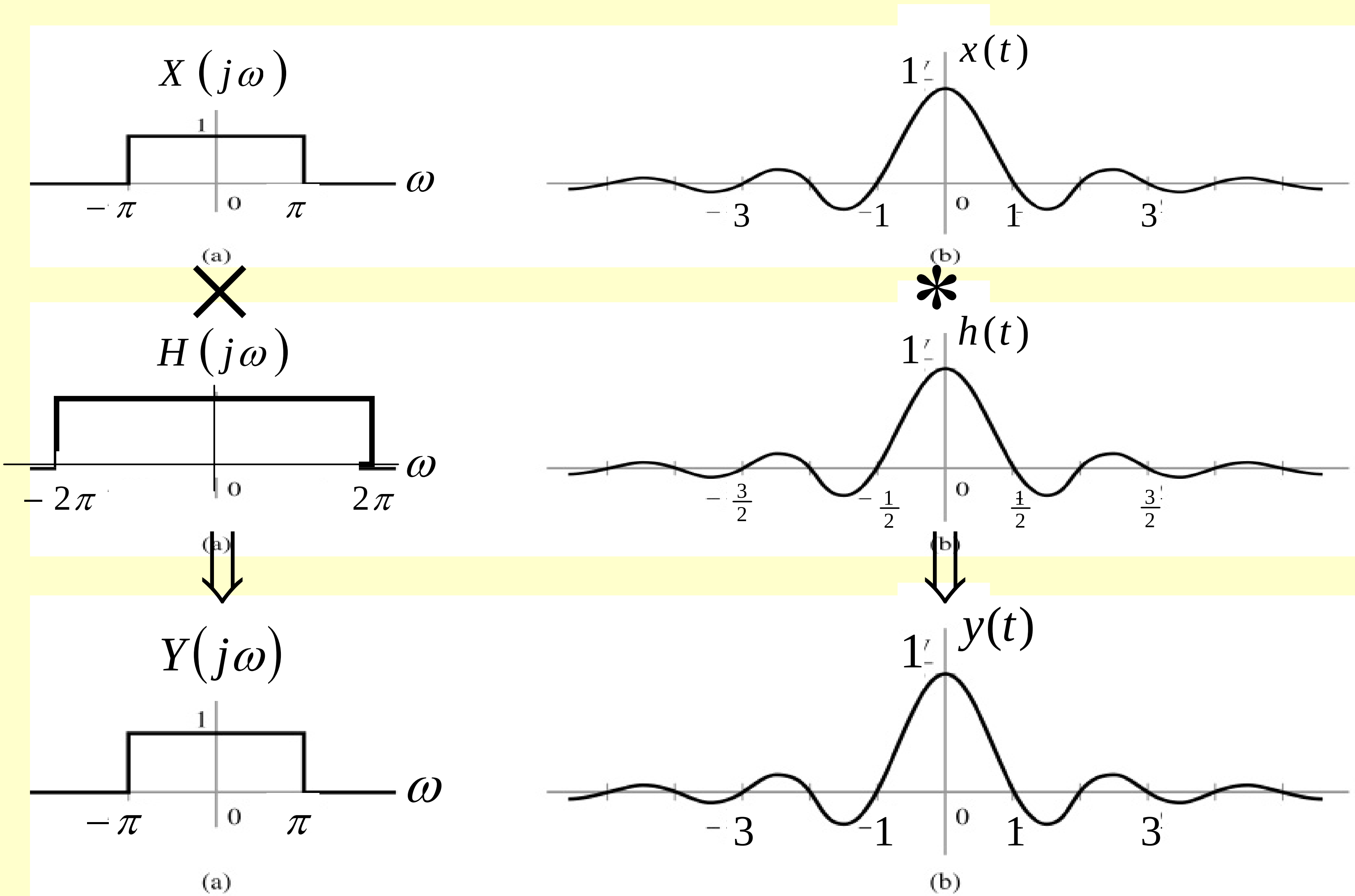
$h(t) = (1/\pi t)\sin(2\pi t)$,

$$h(t)=\frac{1}{\pi t}\sin\left(2\pi t\right)\leftrightarrow H(j\omega)=\begin{cases}1, & -2\pi<\omega<2\pi\\ 0, & |\omega|>2\pi\end{cases}$$



(a) (b)

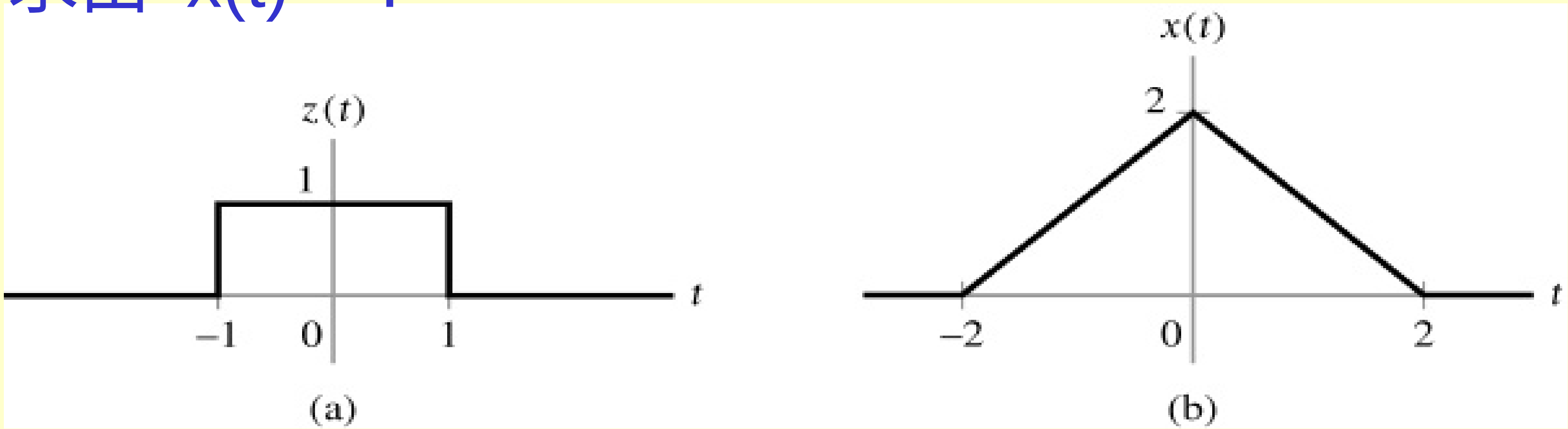
$$y(t)=x(t)*h(t) \quad \leftrightarrow \quad X(j\omega)\cdot H(j\omega)=Y(j\omega)$$



Example 3.32: $X(j\omega) = (4/\omega^2)\sin^2(\omega)$; $x(t) = ?$

Hint: $X(j\omega) = (2/\omega)\sin(\omega) \bullet (2/\omega)\sin(\omega)$

求出 $x(t) = ?$



Example 3.32.
(a) Rectangular pulse $z(t)$.
(b) Convolution of $z(t)$ with itself gives $x(t)$.