# Fourier Transform of Signals

## 訊號的傅立葉轉換

## Lecture 3-7

# Fourier Representation

DTFS: x[n]: discrete-time and periodic signal

X[k]: discrete and periodic spectrum

FS: x(t): continuous-time and periodic signal

X[k]: discrete-time and non-periodic spectrum

DTFT: x[n]: discrete-time and non-periodic signal

$X(e^{j\Omega})$: continuous and periodic spectrum

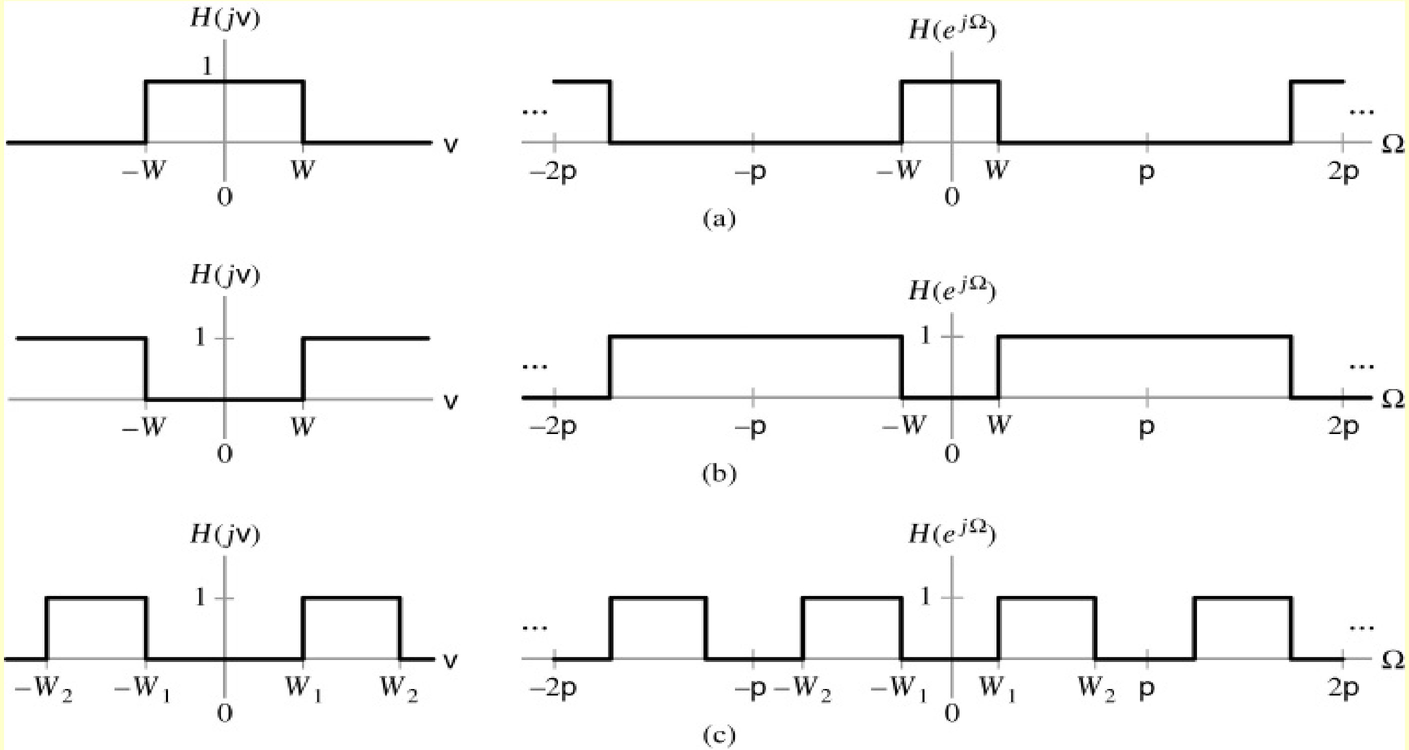
FT: x(t): continuous-time and non-periodic signal

$X(j\omega)$: continuous-time and non-periodic spectrum

# Filtering 濾波

- 任一輸入訊號中含有不同頻率的分量
- 系統透過對輸入訊號中不同頻率的分量做出不同的響應
- 濾波將訊號中部份頻率分量消除，讓其他部份不受影響
    - Low Pass Filter
    - Band Pass Filter
    - High Pass Filter
- 濾波器頻帶
    - 通頻帶 (Pass band)
    - 阻頻帶 (Stop Band)
    - 過渡頻帶 (Transition Band)

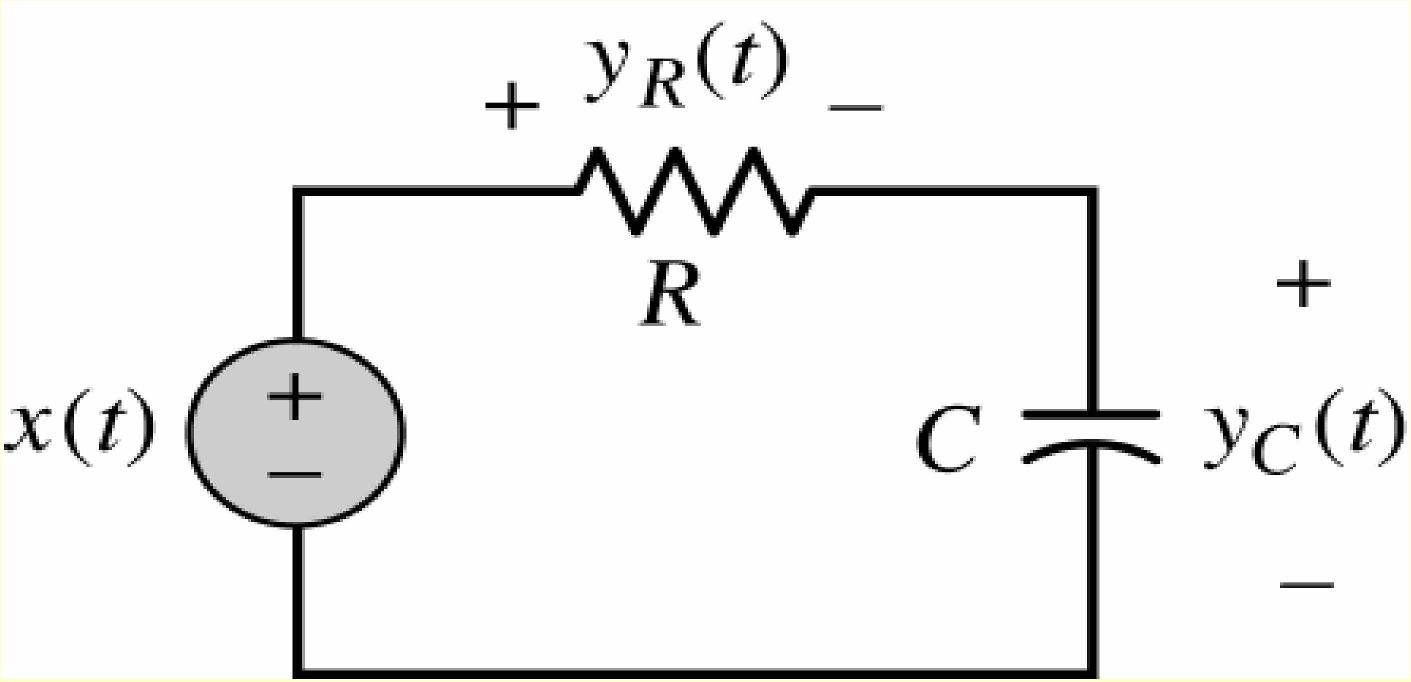
Frequency response of ideal continuous- (left panel) and discrete-time (right panel) filters:
(a) Low-pass characteristic,
(b) High-pass characteristic,
(c) Band-pass characteristic.



Ex 3.33

*RC* circuit with input $x(t)$ and outputs $y_{c(t)}$ and $y_R(t)$.

請分述兩種系統頻率響應與濾波性質



$$\because h_C(t) = \frac{1}{RC} e^{-t/(RC)} u(t)$$

$$\therefore H_C(j\omega) = \frac{1}{1 + j\omega RC}$$

$$\left| H_C(j\omega) \right| = \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}}, \quad \arg\{H_C(j\omega)\} = -\tan^{-1}(\omega RC)$$

---

$$\because h_R(t) = \delta(t) - \frac{1}{RC} e^{-t/(RC)} u(t)$$

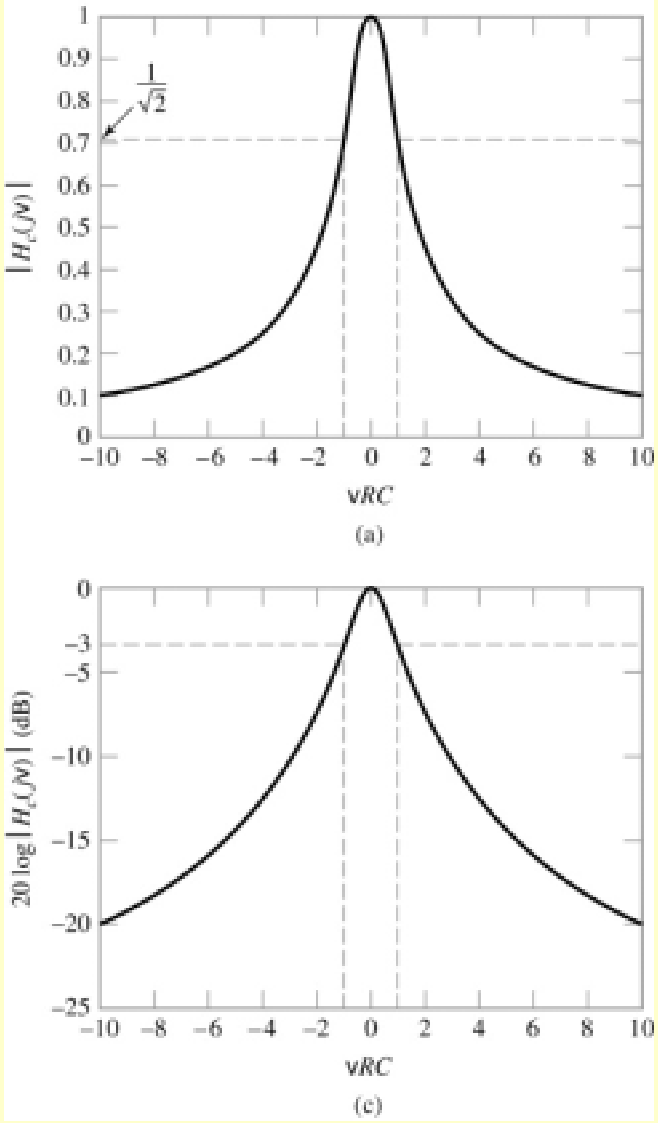
$$\therefore H_R(j\omega) = 1 - \frac{1}{1 + j\omega RC} = \frac{j\omega RC}{1 + j\omega RC}$$

$$\left| H_R(j\omega) \right| = \frac{\omega RC}{\sqrt{1 + \omega^2 R^2 C^2}}, \quad \arg\{H_R(j\omega)\} = \frac{\pi}{2} - \tan^{-1}(\omega RC)$$

*RC* circuit magnitude responses as a function of normalized frequency $\omega RC$.

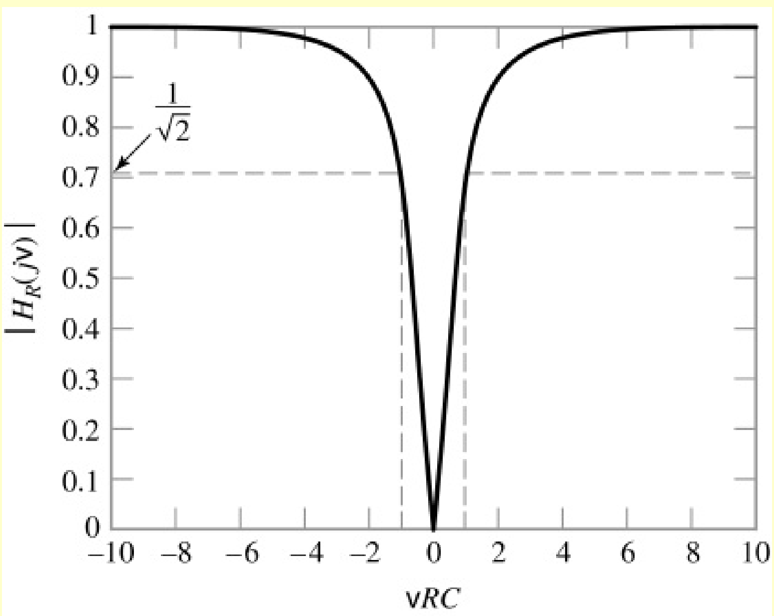
(a) Frequency response of the system corresponding to $y_C(t)$, linear scale.

(c) Frequency response of the system corresponding to $y_C(t)$, dB scale.

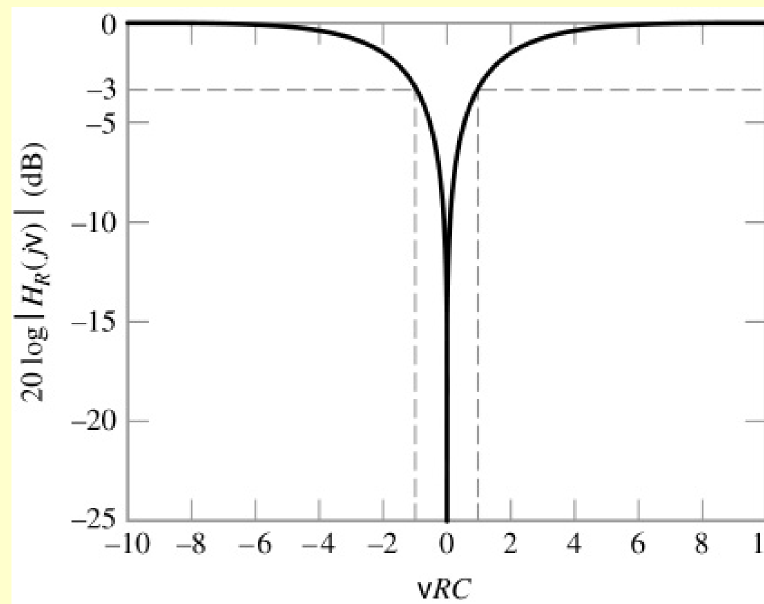


*RC* circuit magnitude responses as a function of normalized frequency ω*RC*.

(b) Frequency response of the system corresponding to *y*R(*t*), linear scale.
(d) Frequency response of the system corresponding to *y*R(*t*), dB scale, shown on the range from 0 dB to –25 dB.



(b)



(d)

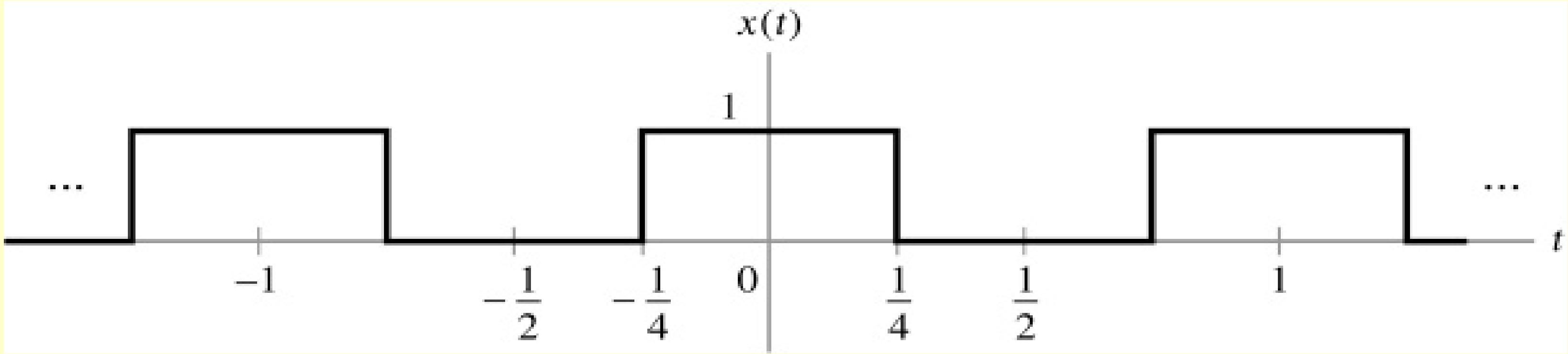
# Convolution of Periodic Signal

## 週期訊號的摺積

- 週期性訊號摺積不會出現在使用計算系統輸出上，因為系統脈衝響應若為週期性函數，系統即為不穩定系統。
- 週期性訊號摺積可能會出現在訊號分析與操作上。
- 週期性訊號摺積定義：

$$y(t) = x(t) \circledast z(t) = \int_0^T x(\tau)\, z(t-\tau)\, d\tau$$

Example 3.36 計算 $z(t) = 2\cos(2\pi t)+\sin(4\pi t)$ 和方波 $x(t)$ 作 週期褶積。



$$z(t) = 2\cos(2\pi t) + \sin(4\pi t)$$

Solution :

$$= 2\left(\frac{e^{j(2\pi)t} + e^{-j(2\pi)t}}{2}\right) + \left(\frac{e^{2(2\pi)t} - e^{-2(2\pi)t}}{j2}\right)$$

$$= e^{j(2\pi)t} + e^{-j(2\pi)t} + \frac{1}{j2}e^{2(2\pi)t} - \frac{1}{j2}e^{-2(2\pi)t}$$

$$\therefore Z[k] = \begin{cases} 1, & k = \pm 1 \\ \frac{1}{j2}, & k = +2 \\ -\frac{1}{j2}, & k = -2 \\ 0, & k = others \end{cases}$$

(FS Spectrum)

$$Given\ \ from\ example\ 3.13$$

$$X[k]=\frac{2}{2\pi k}\sin\left(\frac{k\pi}{2}\right),$$

$$\therefore Y[k]=Z[k]\cdot X[k]$$

$$=\begin{cases}\frac{2}{2\pi k}\sin\left(\frac{k\pi}{2}\right)=\frac{1}{\pi}, & k=\pm 1\\ \frac{1}{j2}\frac{2}{2\pi k}\sin\left(\frac{k\pi}{2}\right)=0, & k=+2\\ -\frac{1}{j2}\frac{2}{2\pi k}\sin\left(\frac{k\pi}{2}\right)=0, & k=-2\\ 0, & others\end{cases}$$

$$Y[k]=\begin{cases}\dfrac{2}{2\pi k}\sin\left(\dfrac{k\pi}{2}\right)=\dfrac{1}{\pi}, & k=\pm 1\\ 0, & others\end{cases}$$

$\therefore$ [ The fundamental frequency: $2\pi$ ]

$$y(t)=Y[1]e^{j(2\pi)t}+Y[-1]e^{-j(2\pi)t}$$

$$=\frac{1}{\pi}\left(e^{j(2\pi)t}+e^{-j(2\pi)t}\right)$$

$$=\frac{2}{\pi}\left(\frac{e^{j(2\pi)t}+e^{-j(2\pi)t}}{2}\right)=\frac{2}{\pi}\cos(2\pi t)$$

# Differentiation & Integration Properties

# 微分與積分性質

## Differentiation in Time:

$$\because \quad x(t) \xleftrightarrow{FT} X(j\omega)$$

$$\frac{d}{dt}x(t) \xleftrightarrow{FT} j\omega \cdot X(j\omega)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

$\therefore$

$$\frac{d}{dt}x(t) = \frac{d}{dt}\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{j\omega t}\, d\omega\right]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\left(\frac{d}{dt}e^{j\omega t}\right) d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\left(j\omega\cdot e^{j\omega t}\right) d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left(j\omega\cdot X(j\omega)\right) e^{j\omega t}\, d\omega$$

$\therefore$

$$\frac{d}{dt}x(t) \xleftrightarrow{FT} j\omega\cdot X(j\omega)$$

# Differentiation in Time (cont.)

∵

$$\frac{d}{dt}x(t) \xleftrightarrow{FT} j\omega \cdot X(j\omega)$$

物理意義：

微分運算是計算變化量

時域上微分會移除直流(DC)分量

# Ex. 3.37

## 驗證

$$\frac{d}{dt}\left(e^{-at}u(t)\right) \xleftrightarrow{FT} \frac{j\omega}{a+j\omega}$$

Solution:

$$\frac{d}{dt}\left(e^{-at}u(t)\right)=\left(\frac{d}{dt}e^{-at}\right)u(t)+e^{-at}\left(\frac{d}{dt}u(t)\right)$$

$$=-a\cdot e^{-at}\cdot u(t)+e^{-at}\cdot\delta(t)=-a\cdot e^{-at}u(t)+\delta(t)$$

$$\therefore$$

$$FT\left\{\frac{d}{dt}\left(e^{-at}u(t)\right)\right\}=FT\left\{-a\cdot e^{-at}u(t)+\delta(t)\right\}$$

$$=FT\left\{-a\cdot e^{-at}u(t)\right\}+FT\left\{\delta(t)\right\}$$

$$=\frac{-a}{a+j\omega}+1=\frac{-a+a+j\omega}{a+j\omega}=\frac{j\omega}{a+j\omega}$$

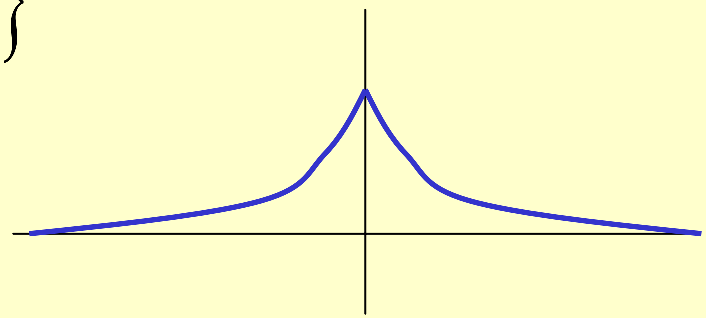
驗證無誤

# Problem 3.22: 利用微分性質求取下列訊號的 FT

$$(a)\quad x(t)=\frac{d}{dt}e^{-2|t|};\quad (b)\quad x(t)=\frac{d}{dt}\left(2te^{-2t}u(t)\right)$$

Solution:

$$(a)\quad FT\left\{\frac{d}{dt}e^{-2|t|}\right\}=j\omega\cdot FT\left\{e^{-2|t|}\right\}$$



$$FT\left\{e^{-2|t|}\right\}=\int_{-\infty}^{+\infty}e^{-2|t|}\,e^{-j\omega t}dt$$

$$=\int_{-\infty}^{0}e^{2t}\,e^{-j\omega t}dt+\int_{0}^{+\infty}e^{-2t}\,e^{-j\omega t}dt$$

Solution: (cont.)

$$= \int_{-\infty}^{0} e^{2t}\, e^{-j\omega t} dt + \int_{0}^{+\infty} e^{-2t}\, e^{-j\omega t} dt$$

$$= \int_{-\infty}^{0} e^{(2-j\omega)t}\, dt + \int_{0}^{+\infty} e^{-(2+j\omega)t}\, dt$$

$$= \frac{1}{2-j\omega} e^{(2-j\omega)t} \Big|_{-\infty}^{0} + \frac{-1}{2+j\omega} e^{-(2+j\omega)t} \Big|_{0}^{+\infty}$$

$$= \frac{1}{2-j\omega}(1-0) - \frac{1}{2+j\omega}(0-1)$$

$$= \frac{1}{2-j\omega} + \frac{1}{2+j\omega} = \frac{2+j\omega+2-j\omega}{(2+j\omega)(2-j\omega)} = \frac{4}{4+\omega^2}$$

## Solution: (cont.)

$$FT\left\{e^{-2|t|}\right\} = \frac{4}{4+\omega^2},$$

$$\therefore$$

$$FT\left\{\frac{d}{dt}e^{-2|t|}\right\} = j\omega \cdot \frac{4}{4+\omega^2} = \frac{4j\omega}{4+\omega^2}$$

Solution:

$$(b)\quad FT\left\{\frac{d}{dt}2te^{-2t}u(t)\right\} = j\omega \cdot FT\left\{2te^{-2t}u(t)\right\}$$

$$FT\left\{2te^{-2t}u(t)\right\} = \int_0^{+\infty} 2te^{-2t}\, e^{-j\omega t} dt$$

$$= \int_{-\infty}^{0} 2t\, e^{-(2+j\omega)t}\, dt\ ; \quad let\ u = 2t,\ dv = e^{-(2+j\omega)t} dt$$

$$\therefore v = \frac{-1}{2+j\omega} e^{-(2+j\omega)t},\ du = 2dt$$

$$= uv - \int vdu = 2t\frac{-1}{2+j\omega}e^{-(2+j\omega)t} - \int_0^{+\infty} \frac{-1}{2+j\omega}e^{-(2+j\omega)t} 2dt$$

Solution: (cont.)

$$FT\left\{2te^{-2t}u(t)\right\}$$

$$= 2t\frac{-1}{2+j\omega}e^{-(2+j\omega)t} - 2\int_0^{+\infty}\frac{-1}{2+j\omega}e^{-(2+j\omega)t}dt$$

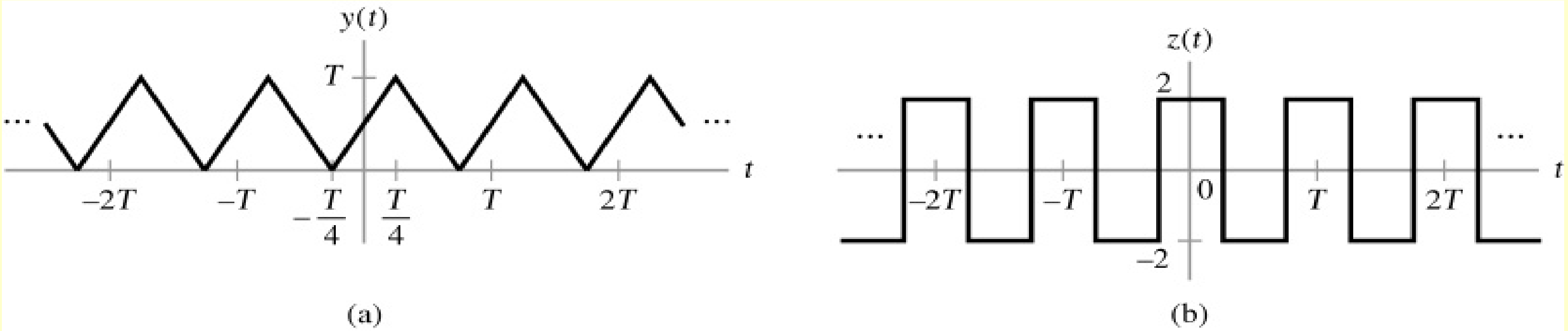
$$= \frac{-2t}{2+j\omega}e^{-(2+j\omega)t}\Big|_0^{+\infty} - \frac{2}{(2+j\omega)^2}e^{-(2+j\omega)t}\Big|_0^{+\infty}$$

$$= (0-0) - \frac{2}{(2+j\omega)^2}(0-1) = \frac{2}{(2+j\omega)^2}$$

$$\therefore FT\left\{\frac{d}{dt}2te^{-2t}u(t)\right\} = \frac{2j\omega}{(2+j\omega)^2}$$

# Ex. 3.39 利用微分性質求取 y(t) 訊號的 FT

想一想 怎麼做 ？ 直接求 FT of y(t) 是否比較難 ？




(a) Triangular wave *y*(*t*).

(b) The derivative of *y*(*t*) is the square wave *z*(*t*).

# Solution:

學生試一試

# Differentiation in Frequency

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

$$\therefore$$

$$\frac{d}{d\omega} X(j\omega) = \frac{d}{d\omega}\left[\int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt\right]$$

$$= \int_{-\infty}^{+\infty} x(t)\left(\frac{d}{d\omega} e^{-j\omega t}\right) dt = \int_{-\infty}^{+\infty} x(t)\left(-jt \cdot e^{-j\omega t}\right) dt$$

$$= \int_{-\infty}^{+\infty} \left(-jt \cdot x(t)\right) e^{-j\omega t}\, dt$$

$$\therefore$$

$$\frac{d}{d\omega} X(j\omega) \xleftrightarrow{FT} -jt \cdot x(t)$$

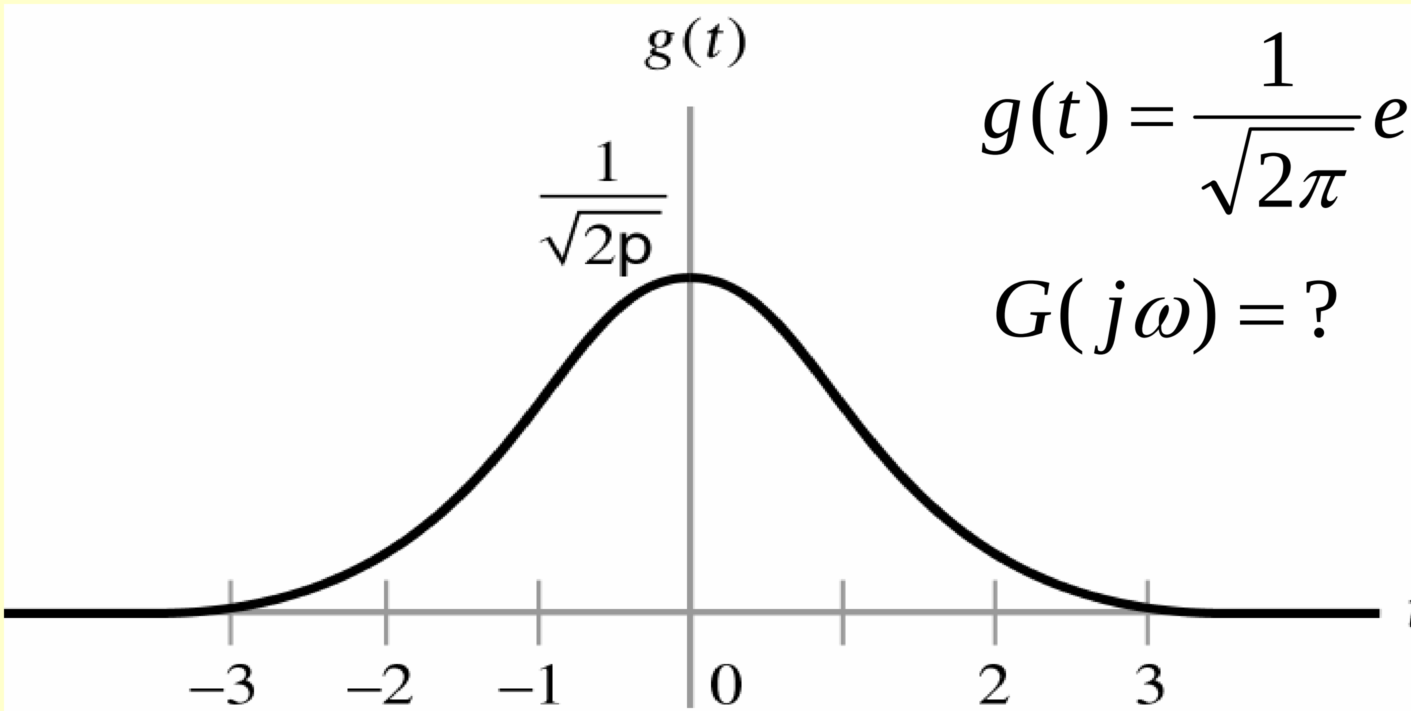
# Differentiation in Frequency:

$\because$

$$\boxed{\frac{d}{d\omega}X(j\omega) \xleftrightarrow{FT} -jt\cdot x(t)}$$

- 頻域上微分 對應在時域上把訊號乘上 $-jt$

# Ex. 3.40: Find the FT of the Gaussian pulse *g*(*t*)?




利用對時間與對頻率微分性質求取 高斯脈波 (Gaussian pulse), g(t) 訊號的 FT.

Solution:

$$given \quad g(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2},$$

$$\therefore \frac{d}{dt} g(t) = (-t)\frac{1}{\sqrt{2\pi}} e^{-t^2/2} = -t \cdot g(t)$$

應用對時間微分性質： $\frac{d}{dt} g(t) \overset{FT}{\leftrightarrow} j\omega \cdot G(j\omega)$

代入上面推導式：

$$\therefore \boxed{-t \cdot g(t) \overset{FT}{\leftrightarrow} j\omega \cdot G(j\omega)}$$

應用對頻率微分性質：

$$-jt \cdot g(t) \overset{FT}{\leftrightarrow} \frac{d}{d\omega} G(j\omega) \Rightarrow -t \cdot g(t) \overset{FT}{\leftrightarrow} \frac{1}{j}\frac{d}{d\omega} G(j\omega)$$

## Solution: (cont.)

$\therefore$

$$-t \cdot g(t) \overset{FT}{\leftrightarrow} j\omega \cdot G(j\omega)\,; \qquad -t \cdot g(t) \overset{FT}{\leftrightarrow} \frac{1}{j}\frac{d}{d\omega}G(j\omega)$$

$$\therefore \quad j\omega \cdot G(j\omega) = \frac{1}{j}\frac{d}{d\omega}G(j\omega)$$

$$\Rightarrow \quad (j)\,j\omega \cdot G(j\omega) = (j)\frac{1}{j}\frac{d}{d\omega}G(j\omega)$$

$$\Rightarrow \boxed{-\omega \cdot G(j\omega) = \frac{d}{d\omega}G(j\omega)}$$

Solution: (cont.)

$$\boxed{\frac{d}{dt}g(t) = -t \cdot g(t)} \qquad \boxed{-\omega \cdot G(j\omega) = \frac{d}{d\omega}G(j\omega)}$$

對照上述關係式，g(t) 和 G(jω) 應該有相同 函數式：

$$\therefore \quad G(j\omega) = c\, e^{-\omega^2/2}$$

當 ω = 0 時，G(j0) = c 同時又可得：

$$G(j0) = \int_{-\infty}^{+\infty} g(t)\,dt = \frac{1}{\sqrt{2\pi}} \underline{\int_{-\infty}^{+\infty} e^{-t^2/2}\,dt} = 1 = c$$

因此 G(jω)：$G(j\omega) = e^{-\omega^2/2}$

$$\boxed{\because \int_{-\infty}^{+\infty} e^{-t^2/2}\,dt = \sqrt{2\pi}}$$

如何 ＝1 ?試推導 ！

## 範例

$$\int_{-\infty}^{+\infty} e^{-x^2/2}\, dx = \sqrt{2\pi} \quad ???$$

$$\int_{-\infty}^{+\infty} e^{-x^2/2}\, dx = \sqrt{\int_{-\infty}^{+\infty} e^{-x^2/2}\, dx \int_{-\infty}^{+\infty} e^{-x^2/2}\, dx}, \quad let\ y = x,$$

$$= \sqrt{\int_{-\infty}^{+\infty} e^{-x^2/2}\, dx \int_{-\infty}^{+\infty} e^{-y^2/2}\, dy} = \sqrt{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(x^2+y^2)/2}\, dx\, dy},$$

改用 polar form $\because dA = dx \cdot dy = r d\theta \cdot dr \quad \& \quad r^2 = x^2 + y^2$

$$= \sqrt{\int_0^{2\pi}\int_0^{+\infty} e^{-(r^2)/2}\, r\, dr\, d\theta} = \sqrt{\int_0^{2\pi}\left(\int_0^{+\infty} r \cdot e^{-(r^2)/2}\, dr\right) d\theta}$$

$$= \sqrt{\int_0^{2\pi}\left(-e^{-(r^2)/2}\Big|_0^{+\infty}\right) d\theta} = \sqrt{\int_0^{2\pi} -(0-1)\, d\theta} = \sqrt{\int_0^{2\pi} d\theta}$$

$$= \sqrt{\theta\Big|_0^{2\pi}} = \sqrt{2\pi - 0} = \sqrt{2\pi}$$

# Integration 積分

積分運算僅適用於連續的應變數

(continuous dependent variables )。

積分定義式：

$$\boxed{y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau}$$

y 在時間 t 的值為 x 對所有在 t 以前時間的積分。

因此：

$$\frac{d}{dt} y(t) = x(t)$$

經由微分性質，右式可進一步推論：

$$\boxed{\frac{d}{dt}y(t) = x(t)}$$

$$\because FT\{y(t)\} = Y(j\omega)$$

$$\because FT\left\{\frac{d}{dt}y(t)\right\} = j\omega \cdot Y(j\omega)$$

$$\therefore FT\left\{\frac{d}{dt}y(t)\right\} = FT\{x(t)\} = X(j\omega)$$

$$\therefore j\omega \cdot Y(j\omega) = X(j\omega)$$

$$\boxed{Y(j\omega) = \frac{1}{j\omega}X(j\omega)}$$

特例：

未定義 $\omega = 0$ 處

增加項目定義 $\omega = 0$

$\omega = 0$ 此項為零

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau \quad \overset{FT}{\longleftrightarrow} \quad \frac{1}{j\omega} X(j\omega) + \pi X(j0)\delta(\omega)$$

$\omega = 0$ 時出現

驗證積分性質：

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau$$

$$\boxed{\begin{array}{ccc} \delta(t) & \xleftrightarrow{FT} & 1 \\ 1 & \xleftrightarrow{FT} & 2\pi\,\delta(\omega) \end{array}}$$

or

$$\boxed{\begin{array}{l} FT\{\delta(t)\} = 1 \\ FT\{1\} = 2\pi\delta(\omega) \end{array}}$$

$$\therefore U(j\omega) = FT\{u(t)\} = FT\left\{\int_{-\infty}^{t} \delta(\tau)\, d\tau\right\}$$

$$= \frac{1}{j\omega}\cancelto{1}{\Delta(j\omega)} + \pi \cdot \cancelto{1}{\Delta(j0)}\,\delta(\omega)$$

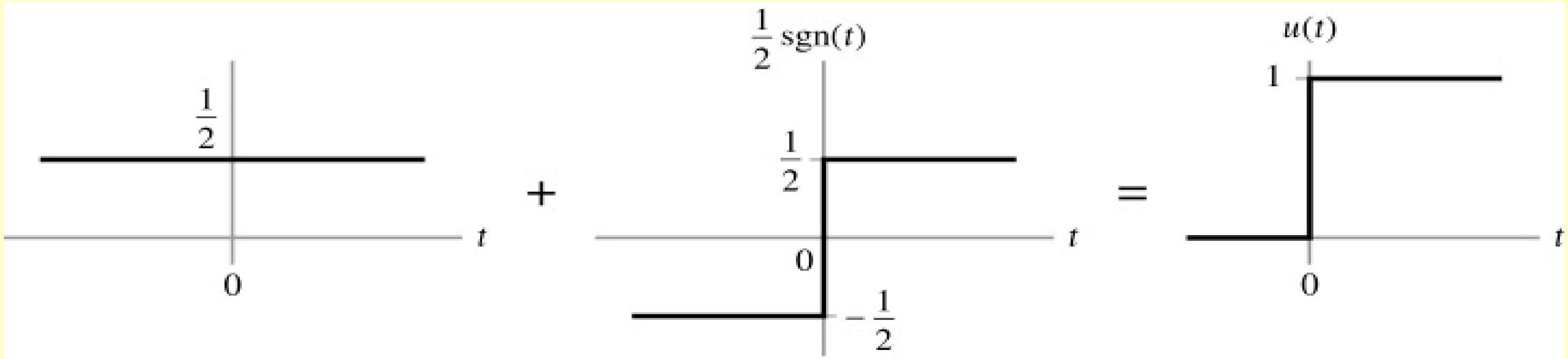
$$\boxed{\delta(t) \xleftrightarrow{FT} \Delta(j\omega)}$$

$$= \frac{1}{j\omega} + \pi \cdot \delta(\omega)$$

A step function can be denoted as the sum of a constant and a signum function.

$$u(t) = \frac{1}{2} + \frac{1}{2}\operatorname{sgn}(t)$$

$$\operatorname{sgn}(t) = \begin{cases} -1, & t < 0 \\ 0, & t = 0 \\ +1, & t > 0 \end{cases}$$



# Another way to derive out U(jω) = ? .

$$\because u(t) = \frac{1}{2} + \frac{1}{2}\operatorname{sgn}(t)$$

$$\therefore U(j\omega) = FT\{u(t)\} = FT\left\{\frac{1}{2}\right\} + FT\left\{\frac{1}{2}\operatorname{sgn}(t)\right\}$$

Considering:

$$\because FT\{1\} = 2\pi\,\delta(\omega)$$

$$\therefore FT\left\{\frac{1}{2}\right\} = \pi\,\delta(\omega)$$

Considering: $FT\{\mathrm{sgn}(t)\} = S(j\omega) = ?$

利用微分性質：

$$\because \frac{d}{dt}\mathrm{sgn}(t) = \frac{d}{dt}\big(-1+2u(t)\big) = 2\delta(t)$$

$\therefore$

$$FT\left\{\frac{d}{dt}\mathrm{sgn}(t)\right\} = j\omega \cdot FT\{\mathrm{sgn}(t)\} = j\omega \cdot S(j\omega)$$

$$= FT\{2\delta(t)\} = 2FT\{\delta(t)\} = 2$$

$$\therefore \quad S(j\omega) = \frac{2}{j\omega}$$

$\because \mathrm{sgn}(t)$ *is an odd function,*
*the average is* $0$.
$\therefore S(j0) = 0$*, is defined.*

$$\therefore U(j\omega) = FT\{u(t)\} = FT\left\{\frac{1}{2}\right\} + \frac{1}{2}FT\{\mathrm{sgn}(t)\}$$

$$= \pi \cdot \delta(\omega) + \frac{1}{j\omega}$$

*or*

$$U(j\omega) = \begin{cases} \dfrac{1}{j\omega}, & \omega \neq 0 \\ \pi\,\delta(\omega), & \omega = 0 \end{cases}$$