Fourier Transform of Signals

訊號的傅立葉轉換

Lecture 3-8

Fourier Representations

DTFS: x[n]: discrete-time and periodic signal

X[k]: discrete and periodic spectrum

FS: x(t): continuous-time and periodic signal

X[k]: discrete-time and non-periodic spectrum

DTFT: x[n]: discrete-time and non-periodic signal

$X(e^{j\Omega})$: continuous and periodic spectrum

FT: x(t): continuous-time and non-periodic signal

$X(j\omega)$: continuous-time and non-periodic spectrum

Time- & Frequency Shifted Properties

時間平移與頻率平移性質

Time-Shift Property :

把訊號 x(t) 做時間平移 t_0 的結果，是把它的FT上 $e^{-j\omega t_0}$ 。

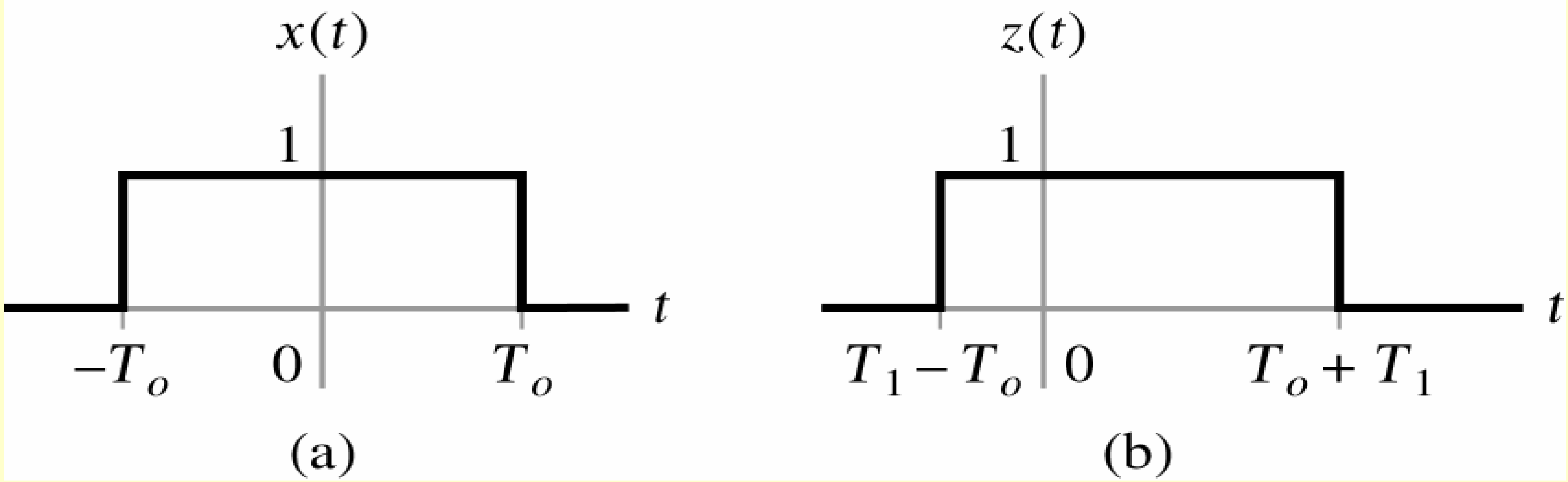
$$x(t-t_0) \xleftrightarrow{FT} e^{-j\omega t_0} X(j\omega)$$

$$x(t-t_0) \xleftrightarrow{FS;\omega_0} e^{-jk\omega_0 t_0} X[k]$$

$$x[n-n_0] \xleftrightarrow{DTFT} e^{-j\Omega n_0} X(e^{j\Omega})$$

$$x[n-n_0] \xleftrightarrow{DTFS;\Omega_0} e^{-jk\Omega_0 n_0} X[k]$$

Verify the time-shift property for Example 3.41, where $z(t) = x(t-T_1)$. Find the FT of $x(t)$ and $z(t)$=?

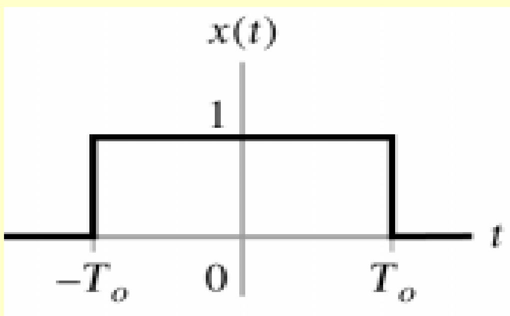


$$X(j\omega) = \int_{-\infty}^{+\infty} x(\tau)\, e^{-j\omega\tau}\, d\tau = \int_{-T_0}^{+T_0} e^{-j\omega\tau}\, d\tau$$

$$= \frac{-1}{j\omega} e^{-j\omega\tau} \Big|_{-T_0}^{T_0} = \frac{-1}{j\omega}\left(e^{-j\omega T_0} - e^{j\omega T_0}\right)$$

$$= \frac{1}{j\omega}\left(e^{j\omega T_0} - e^{-j\omega T_0}\right) = \frac{1}{j\omega}\left(e^{j\omega T_0} - e^{-j\omega T_0}\right)$$

$$= \frac{2}{\omega}\left(\frac{e^{j\omega T_0} - e^{-j\omega T_0}}{j2}\right) = \frac{2}{\omega}\sin\left(\omega T_0\right)$$



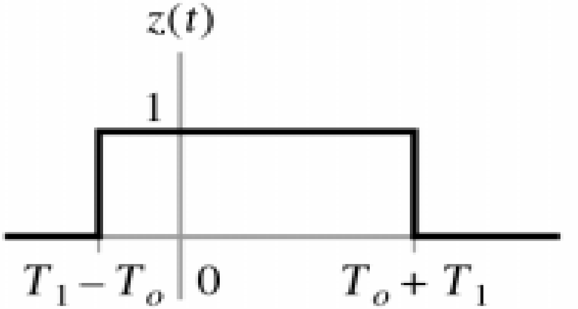
(a)

$$Z(j\omega)=\int_{-\infty}^{+\infty} z(\tau)\, e^{-j\omega\tau}\, d\tau=\int_{-T_0+T_1}^{+T_0+T_1} e^{-j\omega\tau}\, d\tau$$

$$=\frac{-1}{j\omega}e^{-j\omega\tau}\Big|_{-T_0+T_1}^{T_0+T_1}=\frac{-1}{j\omega}\left(e^{-j\omega(T_0+T_1)}-e^{-j\omega(-T_0+T_1)}\right)$$

$$=\frac{1}{j\omega}\left(e^{-j\omega(-T_0+T_1)}-e^{-j\omega(T_0+T_1)}\right)=\frac{1}{j\omega}\left(e^{j\omega T_0}e^{-j\omega T_1}-e^{-j\omega T_0}e^{-j\omega T_1}\right)$$

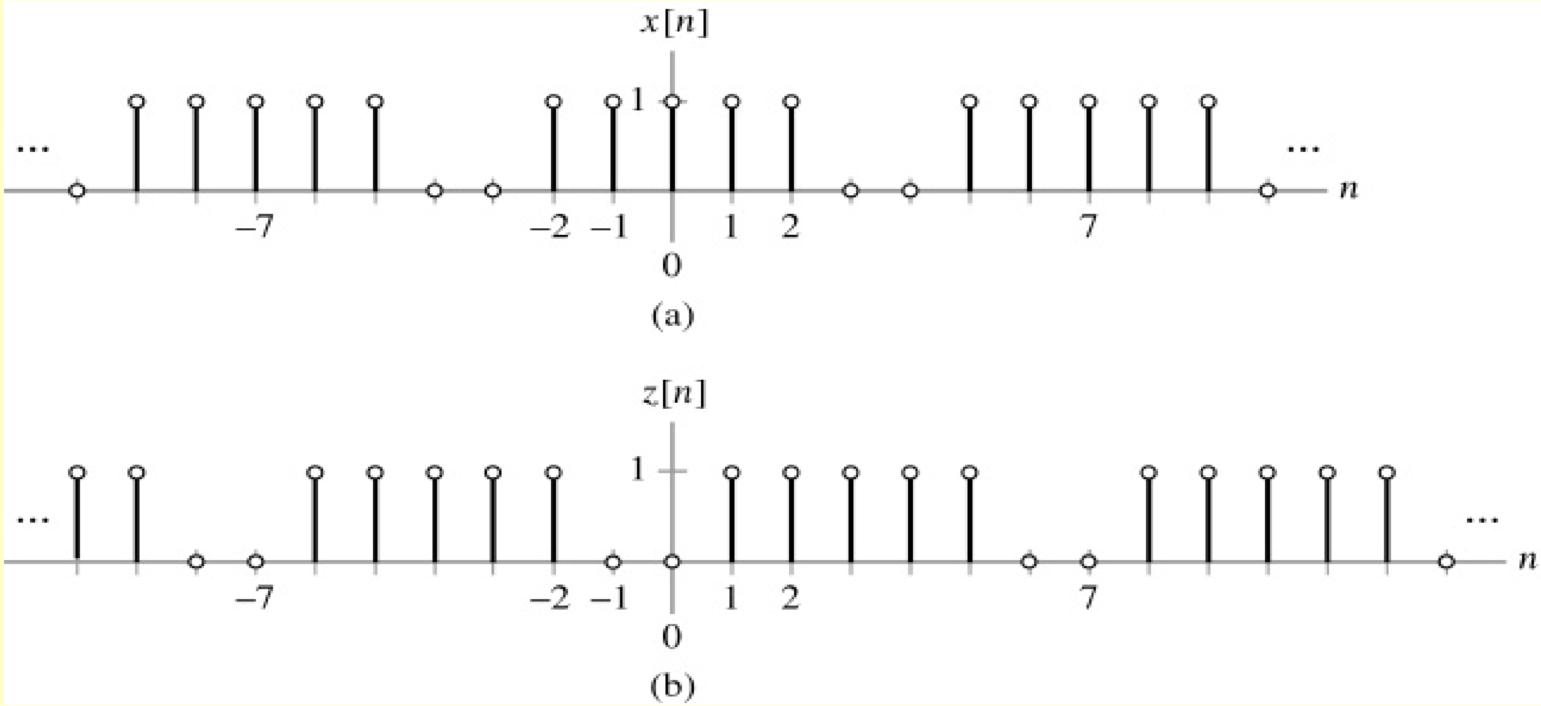
$$=e^{-j\omega T_1}\frac{2}{\omega}\left(\frac{e^{j\omega T_0}-e^{-j\omega T_0}}{j2}\right)=e^{-j\omega T_1}\frac{2}{\omega}\sin(\omega T_0)$$



(b)

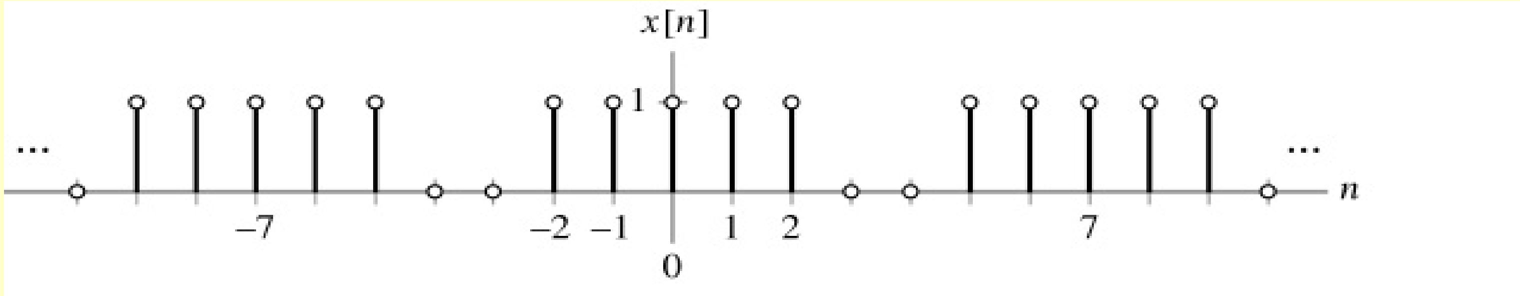
Problem 3.30

利用時間平移特性，使用圖(a)週期性方波的DTFS，尋求圖(b)週期性方波的DTFS。



(a)

(b)



(a)

$$X[k]=\frac{1}{N}\sum_{n=-M}^{N-M-1}x[n]\,e^{-jk\Omega_0 n};<N=7,\ M=2,\ \Omega_0=2\pi/7,\ m=n+2>$$

$$=\frac{1}{7}\sum_{n=-2}^{2}e^{-jk\Omega_0 n}=\frac{1}{7}\sum_{m=0}^{4}e^{-jk\Omega_0(m-2)}=e^{j2k\Omega_0}\frac{1}{7}\sum_{m=0}^{4}e^{-jk\Omega_0 m}$$

$$=e^{j2k\Omega_0}\frac{1}{7}\frac{1-e^{-j5k\Omega_0}}{1-e^{-jk\Omega_0}}=\frac{1}{7}e^{j2k\Omega_0}\frac{e^{-j5k\Omega_0/2}\left(e^{j5k\Omega_0/2}-e^{-j5k\Omega_0/2}\right)}{e^{-jk\Omega_0/2}\left(e^{jk\Omega_0/2}-e^{-jk\Omega_0/2}\right)}$$

$$=\frac{1}{7}\frac{\left(e^{j5k\Omega_0/2}-e^{-j5k\Omega_0/2}\right)/j2}{\left(e^{jk\Omega_0/2}-e^{-jk\Omega_0/2}\right)/j2}=\frac{1}{7}\frac{\sin(5k\Omega_0/2)}{\sin(k\Omega_0/2)}$$

$$=\frac{1}{7}\frac{\sin\left(k\left(\frac{5\pi}{7}\right)\right)}{\sin\left(k\left(\frac{\pi}{7}\right)\right)}$$

利用時間平移特性：

$$Z[k]=e^{-jk\Omega_0 n_0}X[k],\quad \Omega_0=\frac{2\pi}{7}, n_0=3$$

$$Z[k]=e^{-jk\frac{2\pi}{7}(3)}\left[\frac{1}{7}\frac{\sin\left(\frac{k5\pi}{7}\right)}{\sin\left(\frac{k\pi}{7}\right)}\right]$$

** 學生請直接對z[n]做DTFS，並驗證答案是否如上。

Frequency-Shift Property :

頻率平移 γ 相當於在時域中把訊號乘上一個頻率與頻率平移量相等的複數弦波。

$$Z(j\omega) = X(j(\omega-\gamma)) \xleftrightarrow{FT} z(t) = e^{j\gamma t}\, x(t)$$

$$e^{j\gamma t}x(t) \xleftrightarrow{FT} X\big(j(\omega-\gamma)\big)$$

$$e^{jk_0\omega_0 t}x(t) \xleftrightarrow{FS;\omega_0} X[k-k_0]$$

$$e^{j\Gamma n}x[n] \xleftrightarrow{DTFT} X\big(e^{(\Omega-\Gamma)}\big)$$

$$e^{jk_0\Omega_0 n}x[n] \xleftrightarrow{DTFS;\Omega_0} X[k-k_0]$$

Audio & DSP Lab.

Proof:

$$z(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} Z(j\omega)\, e^{j\omega t}\, d\omega, \quad < if\ Z(j\omega) = X\big(j(\omega-\gamma)\big) >$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X\big(j(\omega-\gamma)\big) e^{j\omega t}\, d\omega, \quad < let\ \eta = \omega-\gamma >$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X\big(j\eta\big) e^{j(\eta+\gamma)t}\, d\eta$$

$$= e^{j\gamma t}\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} X\big(j\eta\big) e^{j\eta t}\, d\eta\right]$$

$$= e^{j\gamma t}\ x(t)$$

Example 3.42 利用頻率平移性質求 z(t) 的FT = ?

$$z(t)=\begin{cases} e^{j10t}, & |t|<\pi \\ 0, & |t|>\pi \end{cases}$$

Solution:

$$let\ z(t)=e^{j10t}\ x(t), \quad \therefore x(t)=\begin{cases} 1, & |t|<\pi \\ 0, & |t|>\pi \end{cases}$$

$$\therefore$$

$$Z(j\omega)=X\left(j(\omega-10)\right)$$

Solution: (cont.)

$$X(j\omega)=\int_{-\pi}^{+\pi}e^{-j\omega t}dt=\frac{-1}{j\omega}e^{-j\omega t}\Big|_{-\pi}^{+\pi}=\frac{-1}{j\omega}\left(e^{-j\omega\pi}-e^{j\omega\pi}\right)$$

$$=\frac{1}{j\omega}\left(e^{j\omega\pi}-e^{-j\omega\pi}\right)=\frac{2}{\omega}\left(\frac{e^{j\omega\pi}-e^{-j\omega\pi}}{j2}\right)=\frac{2}{\omega}\sin(\omega\pi)$$

$\therefore$

$$Z(j\omega)=X(j(\omega-10))=\frac{2}{\omega-10}\sin\left((\omega-10)\pi\right)$$

Problem 3.34(b) : $X(j\omega) = \frac{1}{2+j(\omega-3)} + \frac{1}{2+j(\omega+3)}$

Solution: x(t) = ?

$$X(j\omega) = \frac{1}{2+j(\omega-(+3))} + \frac{1}{2+j(\omega-(-3))}$$

$$For\ a\ \ function\ of\ \ Z(j\omega) = \frac{1}{2+j\omega} \quad \overset{FT}{\leftrightarrow} \quad z(t) = e^{-2t}u(t)$$

$$The\ \ frequency\ shifts\ are\ \omega = +3\ and\ \ \omega = -3, \quad hence$$

$$x(t) = e^{j3t}z(t) + e^{-j3t}z(t) = z(t)\left(e^{j3t} + e^{-j3t}\right)$$

$$= 2e^{-2t}u(t)\left(\frac{e^{j3t} + e^{-j3t}}{2}\right) = 2e^{-2t}\cos(3t)u(t)$$

Using Partial-Fraction Expansions to Find the Inverse FT

利用部分分式展開求逆傅立葉轉換

$$X(j\omega)=\frac{b_M(j\omega)^M+b_{M-1}(j\omega)^{M-1}+\ldots+b_1(j\omega)+b_0}{(j\omega)^N+a_{N-1}(j\omega)^{N-1}+\ldots+a_1(j\omega)+a_0}$$

- 因為是 FT 是線性
- 部分分式展開把X(jω) 表示為含已知逆FT轉換項之和

Using Partial-Fraction Expansions to Find the Inverse DTFT

利用部分分式展開求逆DT傅立葉轉換

$$X(e^{j\Omega}) = \frac{\beta_M e^{-j\Omega M} + \beta_{M-1} e^{-j\Omega(M-1)} + \ldots + \beta_1 e^{-j\Omega} + \beta_0}{\alpha_N e^{-j\Omega N} + \alpha_{N-1} e^{-j\Omega(N-1)} + \ldots + \alpha_1 e^{-j\Omega} + 1}$$

- 因為是 DTFT 是線性
- 部分分式展開把$X(e^{j\Omega})$ 表示為含已知逆DTFT轉換項之和

Multiplication Property
乘法性質

乘法性質決定兩個時域訊號乘積的傅立葉表示法

$$y(t)=x(t)z(t) \overset{FT}{\longleftrightarrow} Y(j\omega)=X(j\omega)*Z(j\omega)$$

其中

$$X(j\omega)*Z(j\omega)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\upsilon)\,Z(j(\omega-\upsilon))\,d\upsilon$$

乘法性質決定兩個時域訊號乘積的傅立葉表示法

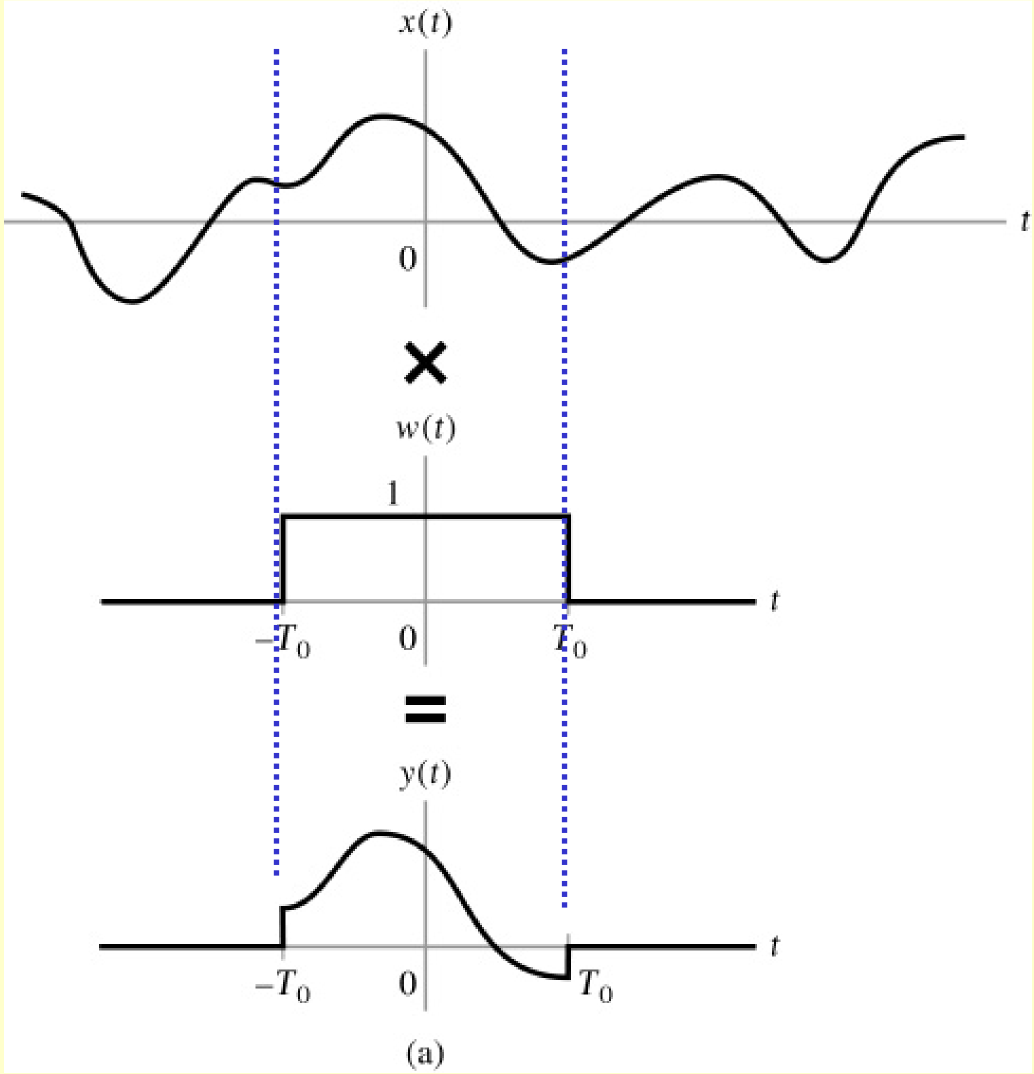
$$y[n] = x[n] \cdot z[n] \xleftrightarrow{DTFT} Y(e^{j\Omega}) = X(e^{j\Omega}) \circledast Z(e^{j\Omega})$$

其中 $X(e^{j\Omega})$ 和 $Z(e^{j\Omega})$ 為 2π 週期性頻譜:

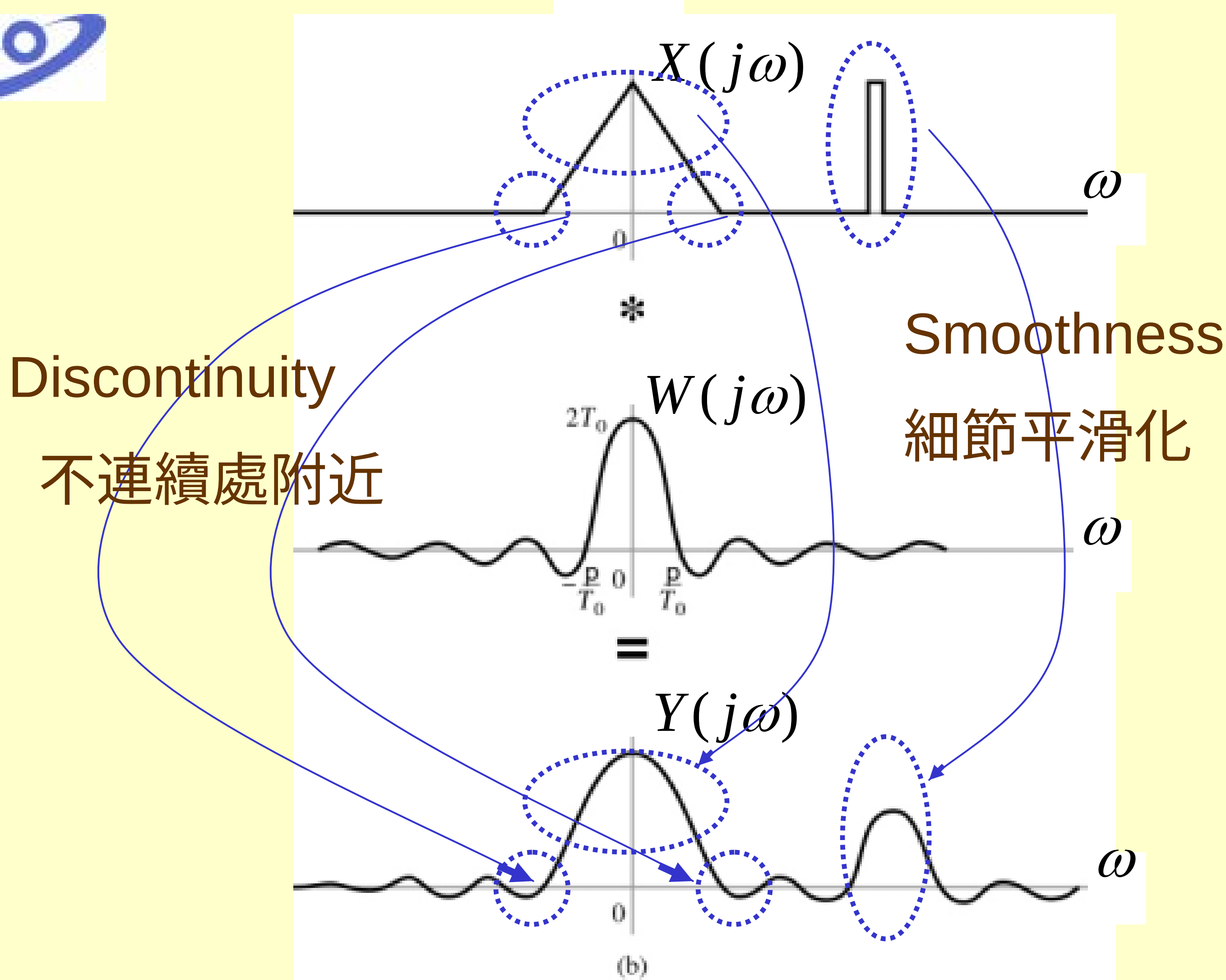
$$X(e^{j\Omega}) \circledast Z(e^{j\Omega}) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} X(e^{j\theta})\, Z(e^{j(\Omega-\theta)})\, d\theta$$

Windowing

- 截斷 (truncate) 一個時域訊號 x(t) 的過程被稱為視窗法 (windowing) 。
- 將一個時域訊號 x(t)乘上一個視窗函數(window function) w(t), y(t) = x(t) w(t)。
- If w(t) is a rectangular shaped window, $w(j\omega)=(2/\omega)\sin(\omega T_0)$, where the windowing is from $-T_0$ to $+T_0$。
- 視窗主瓣(main lobe) $2\pi/T_0$ 把 $X(j\omega)$中細節平滑化，旁瓣(side lobe)把不連續處附近產生振盪。

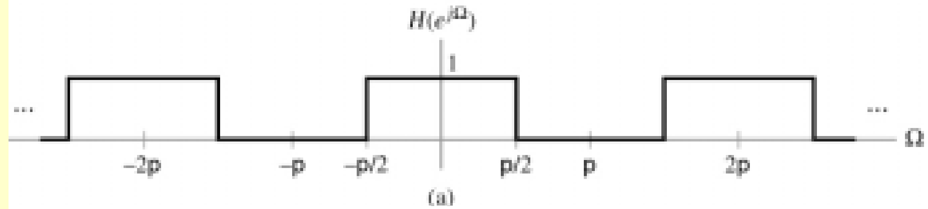


The effect of windowing. (a) Truncating a signal in time by using a window function $w(t)$.



(b) Convolution of the signal and window FT's resulting from truncation in time.

Ex. 3.46:



上圖為一理想離散系統頻率響應，若另一未知系統的脈衝響應為這理想系統的脈衝響應加以截斷於 $-M \le n \le +M$ 區間，請描述其未知系統的頻率響應 = ?

Solution:

這理想系統的<u>脈衝響應</u>可由上圖已知系統頻率響應做逆 DTFT 求出:

$$h[n] = \frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

Solution: (cont.)

未知系統的<u>脈衝響應</u>可由已知系統脈衝響應做截斷：

$$h_t[n] = \begin{cases} h[n], & |n| < M \\ 0, & otherwise \end{cases}$$

截斷的<u>脈衝響應</u> $h_t[n]$ 可由已知系統脈衝響應 $h[n]$ 和一視窗函數 $w[n]$ (window function) 相乘積：$h_t[n] = h[n] \cdot w[n]$

其中

$$w[n] = \begin{cases} 1, & |n| < M \\ 0, & otherwise \end{cases}$$

Solution: (cont.)

應用乘法性質 (Multiplication Property)：

$$h_t[n] = h[n]\cdot w[n] \overset{DTFT}{\longleftrightarrow} H_t(e^{j\Omega}) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} H(e^{j\theta})W(e^{j(\Omega-\theta)})\,d\theta$$

其中 $H(e^{j\theta}) = \begin{cases} 1, & |\theta| \le \pi/2 \\ 0, & \pi/2 < |\theta| < \pi \end{cases}$

參考範例 3.18 可得：

$$X(e^{j(\Omega)}) = \frac{\sin(\Omega(2M+1)/2)}{\sin(\Omega/2)}$$

$$W(e^{j(\Omega-\theta)}) = \frac{\sin((\Omega-\theta)(2M+1)/2)}{\sin((\Omega-\theta)/2)}$$

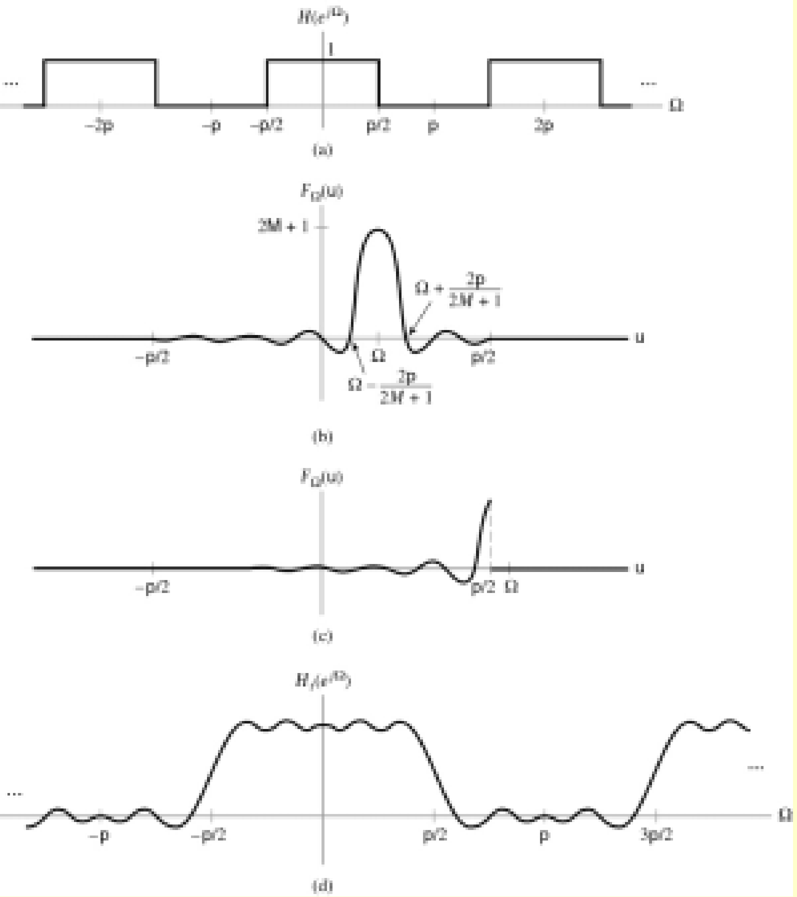
Solution: (cont.)

結果

$$H_t(e^{j\Omega}) = \frac{1}{2\pi}\int_{-\pi/2}^{+\pi/2} F_\Omega(e^{j\theta})\; d\theta$$

其中

$$F_\Omega(e^{j\theta}) = H(e^{j\theta})W(e^{j(\Omega-\theta)}) = \begin{cases} W(e^{j(\Omega-\theta)}), & |\theta| < \frac{\pi}{2} \\ 0, & |\theta| > \frac{\pi}{2} \end{cases}$$



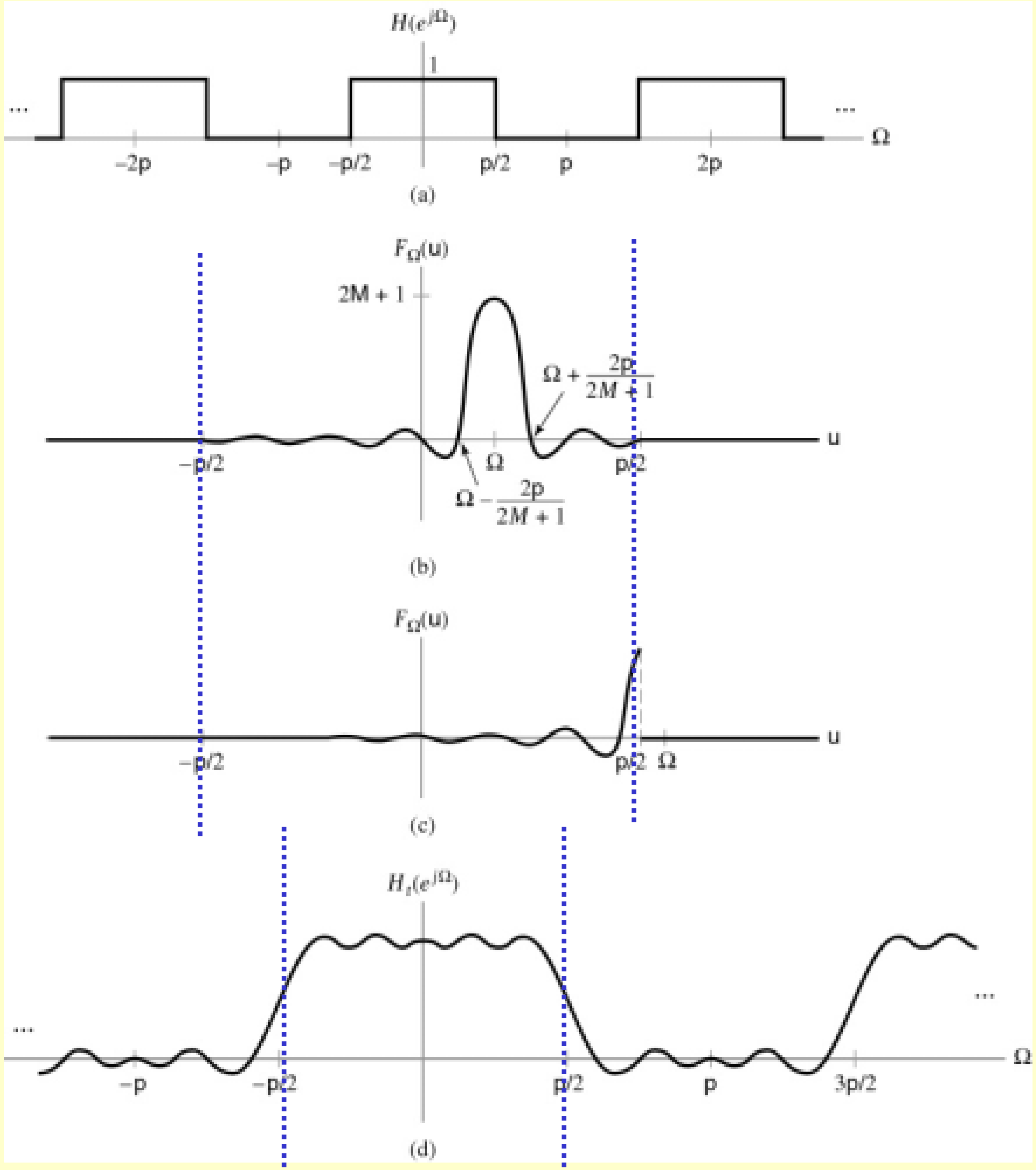
The effect of truncating the impulse response of a discrete-time system.

(a) Frequency response of ideal system.

(b) $F_\Omega(\theta)$ for Ω near zero.

(c) $F_\Omega(\theta)$ for Ω slightly greater than $\pi/2$.

(d) Frequency response of system with truncated impulse response.



H(e^jΩ)
1
...
Ω
−2p
−p
−p/2
p/2
p
2p
(a)
F_Ω(u)
2M + 1
Ω + 2p/(2M + 1)
u
−p/2
Ω
p/2
Ω − 2p/(2M + 1)
(b)
F_Ω(u)
u
−p/2
p/2
Ω
(c)
H_t(e^jΩ)
...
Ω
−p
−p/2
p/2
p
3p/2
(d)

Problem 3.39

應用乘法性質求出訊號 x(t) 的 FT = ?

$$x(t)=\frac{4}{\pi^2 t^2}\sin^2(2t)$$

Solution:

$$x(t)=\frac{2}{\pi t}\sin(2t)\cdot\frac{2}{\pi t}\sin(2t)=w(t)\cdot w(t)$$

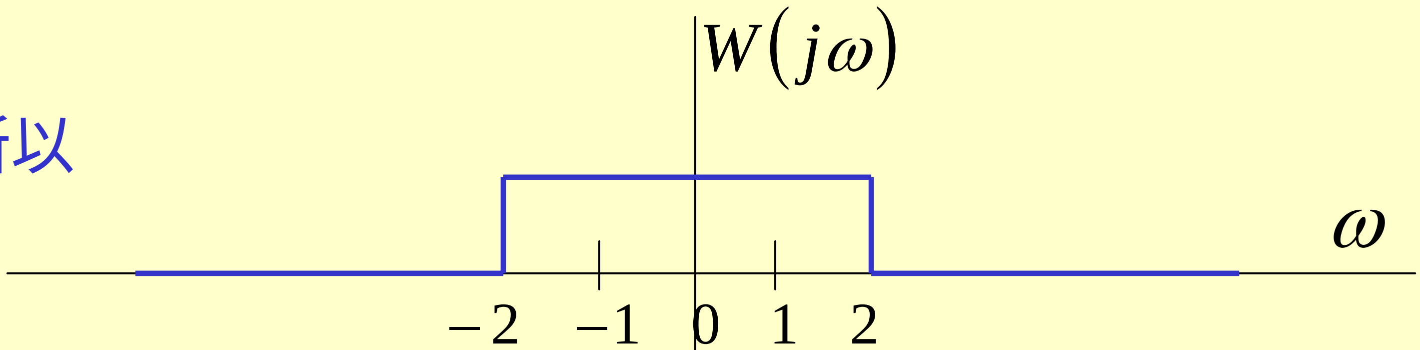
所以

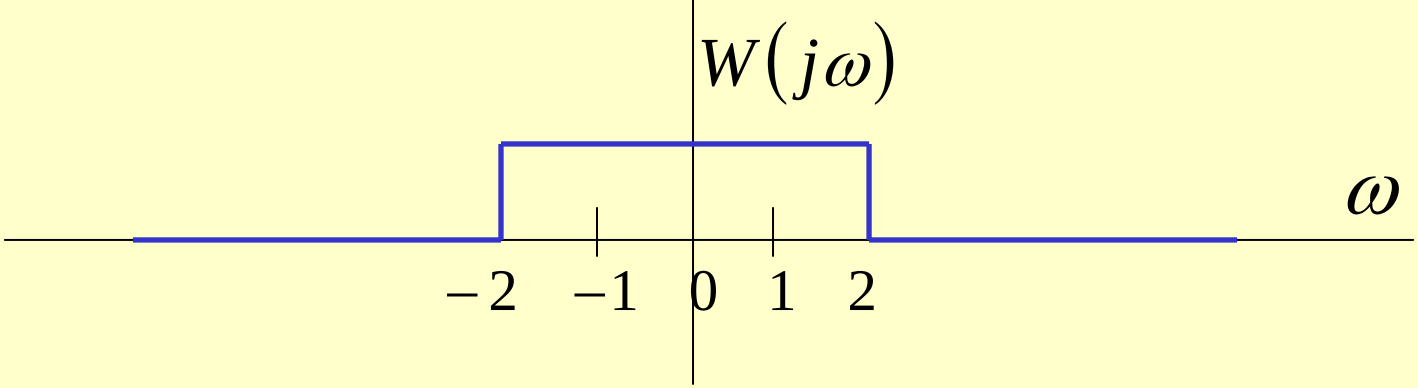
$$X(j\omega)=W(j\omega)*W(j\omega)$$

Solution: (cont.)

$$\because \quad w(t) = \frac{2}{\pi t}\sin(2t)\ ; \quad \therefore \quad W(j\omega) = \begin{cases} 2, & |\omega| \le 2 \\ 0, & |\omega| > 2 \end{cases}$$

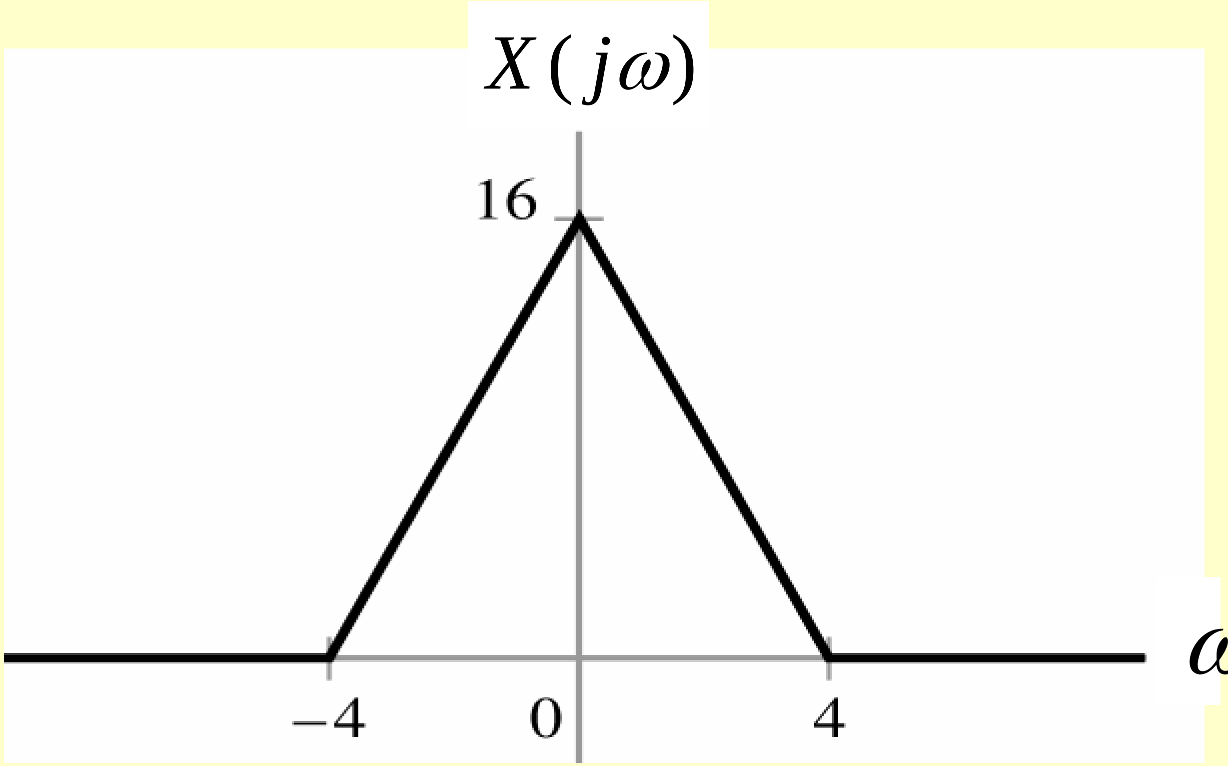
所以





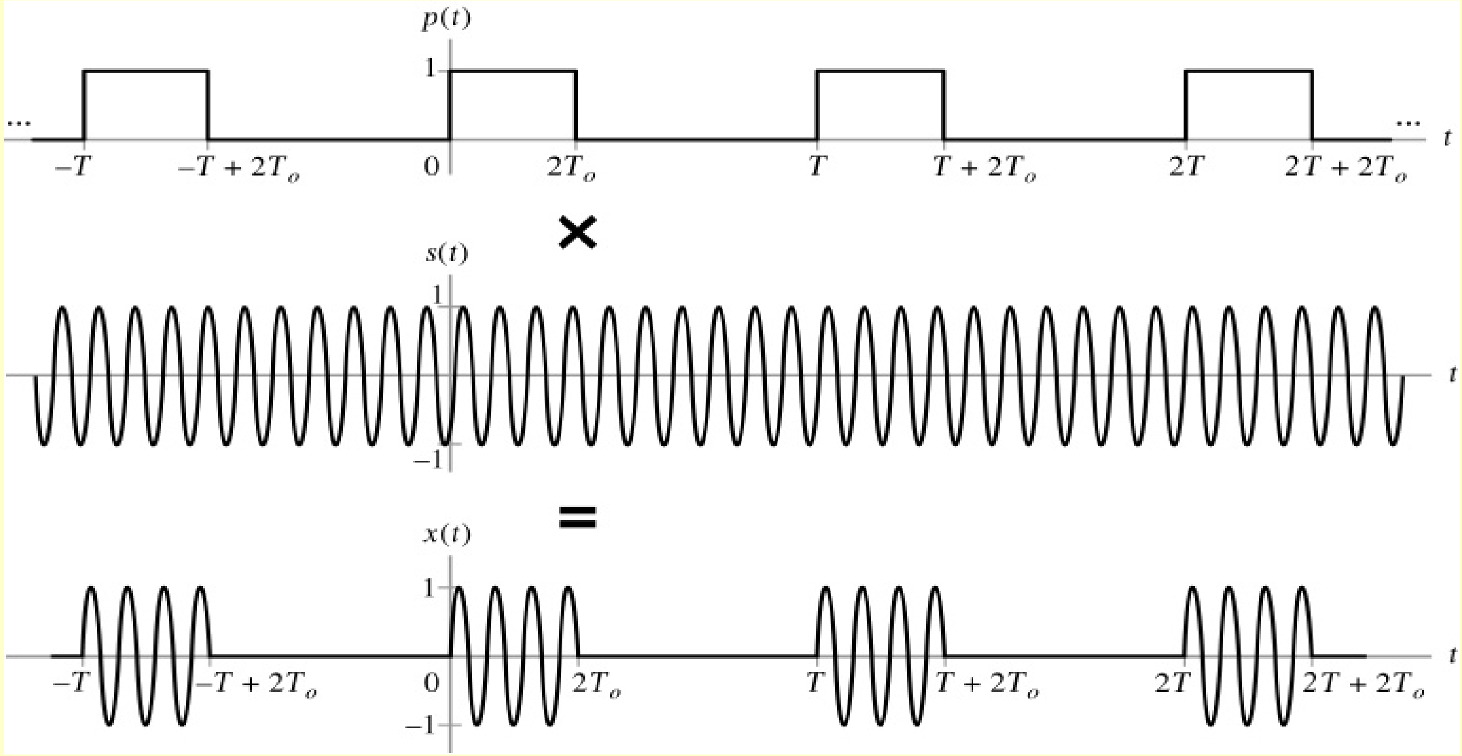
Solution to Problem 3.39:

$$X(j\omega)=W(j\omega)*W(j\omega)$$



Example 3.47: x(t) = s(t)•p(t), s(t)=sin(1000πt/T).

Please find the FT of x(t) = ?



The RF pulse is expressed as the product of a periodic square wave and a sine wave.

Solution: Since s(t) and p(t) are *continuous-time periodic* signals, the spectra are discrete and non-periodic.

The fundamental frequency of p(t) : $\omega_0 = 2\pi/T$,

$s(t) = \sin(1000\ \pi/T) = \sin(\omega t) = \sin(500\ \omega_0 t)$,

$\therefore \omega = 500\ \omega_0 = k\ \omega_0$, $k = 500^{th}$ harmonics.

$$s(t) = \frac{1}{j2} e^{j500\omega_0 t} - \frac{1}{j2} e^{-j500\omega_0 t}$$

$$= S[500]e^{j500\omega_0 t} + S[-500]e^{j-500\omega_0 t}$$

$$\therefore \quad S[k] = \left\{ \begin{array}{ll} \dfrac{1}{j2}, & k = 500 \\ 0, & others \\ \dfrac{-1}{j2}, & k = -500 \end{array} \right\} = \frac{1}{j2}\delta(k-500) - \frac{1}{j2}\delta(k+500)$$

參考範例 3.13 並應用時間平移：

$$P[k] = e^{-jkT_0\omega_0}\,\frac{\sin(k\omega_0 T_0)}{k\pi}$$

$$X[k] = S[k] * P[k]$$

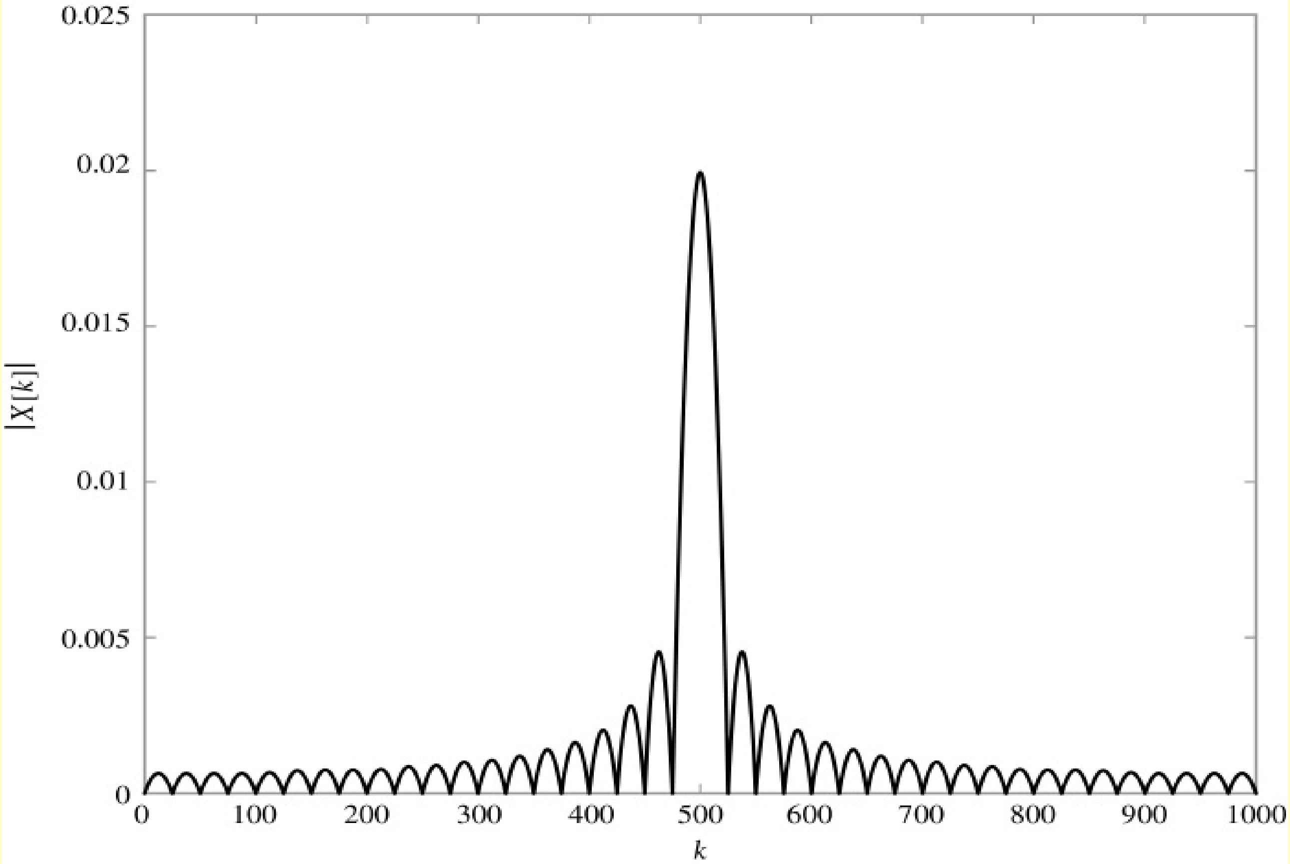
$$= \left(\frac{1}{j2} \delta[k-500] - \frac{1}{j2} \delta[k+500] \right) * \left(e^{-jkT_0\omega_0} \frac{\sin(k\omega_0 T_0)}{k\pi} \right)$$

$$= \frac{1}{j2} e^{-j(k-500)T_0\omega_0} \frac{\sin((k-500)\omega_0 T_0)}{(k-500)\pi}$$

$$- \frac{1}{j2} e^{-j(k+500)T_0\omega_0} \frac{\sin((k+500)\omega_0 T_0)}{(k+500)\pi}$$

應用脈衝函數的篩選性質

FS magnitude spectrum for $0 \le k \le 1000$. The result is depicted as a continuous curve, due to the difficulty of displaying 1000 stems.



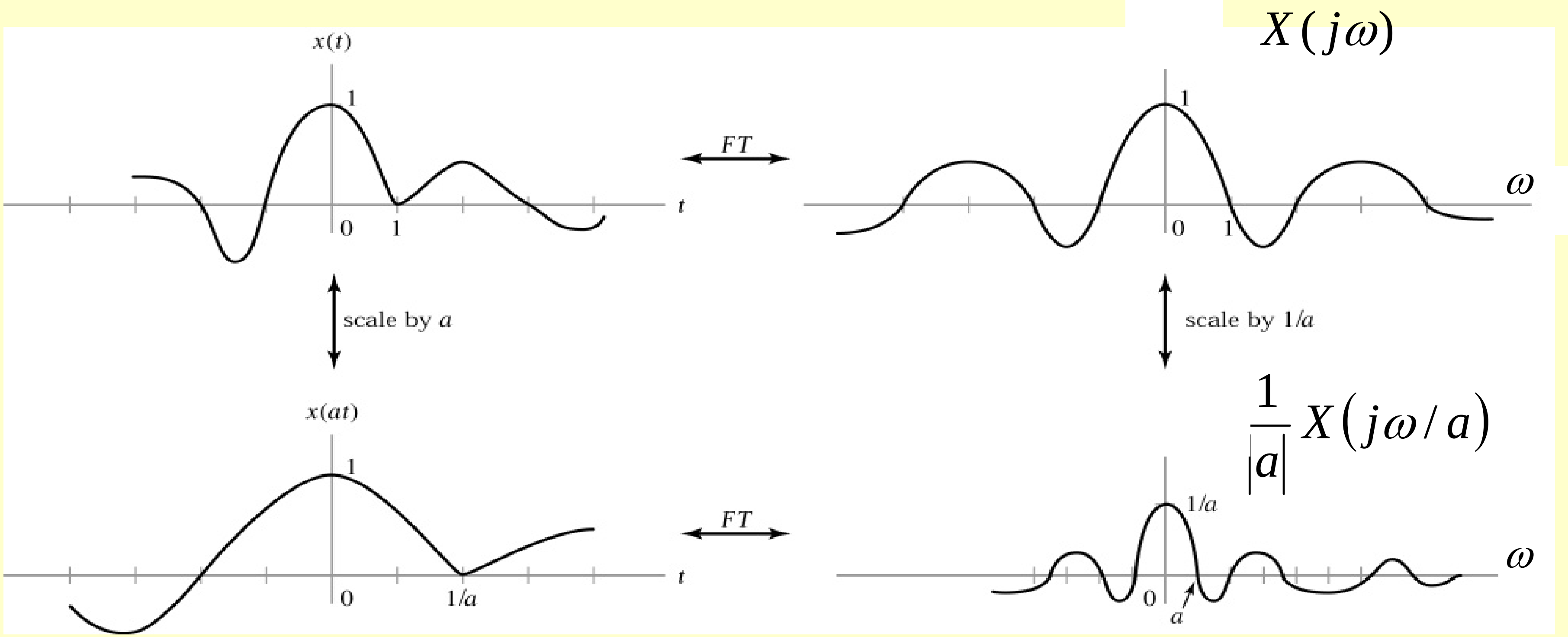
Scaling Properties 比例變換性質

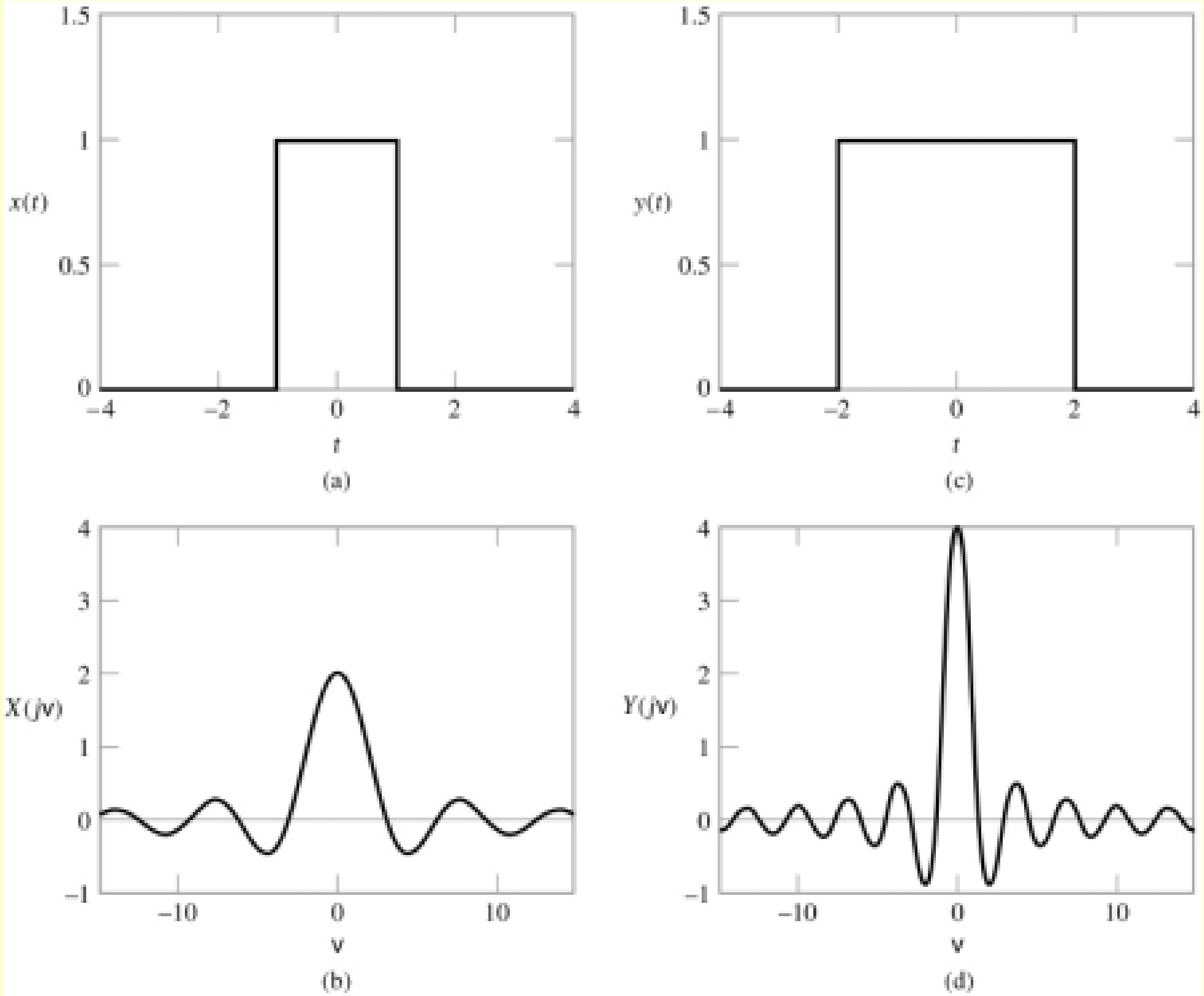
- 顯示出當變換時間變數的比例對訊號的頻域表示

$$z(t) = x(at) \xleftrightarrow{FT} \frac{1}{|a|} X\left(j\frac{\omega}{a} \right)$$

The FT scaling property.

The figure assumes that $0 < a < 1$.





Application of the FT scaling property in Example 3.48.
(a) Original time signal. (b) Original FT.
(c) Scaled time signal $y(t) = x(t/2)$.
(d) Scaled FT $Y(j\omega) = 2X(j2\omega)$.

Parseval Relationship　巴賽瓦關係

訊號時域(time domain)表示法中所包含的能量(energy)或功率(power)等同於它得頻域(frequency domain)表示法中所包含的能量或功率。

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

$$W_x = \int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} x(t) \cdot x^*(t) \quad dt$$

$$= \int_{-\infty}^{+\infty} x(t) \cdot \left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega \right]^* \quad dt$$

$$= \int_{-\infty}^{+\infty} x(t) \cdot \left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} X^*(j\omega) e^{-j\omega t} d\omega \right] \quad dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X^*(j\omega) \left[\int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \right] \quad d\omega$$

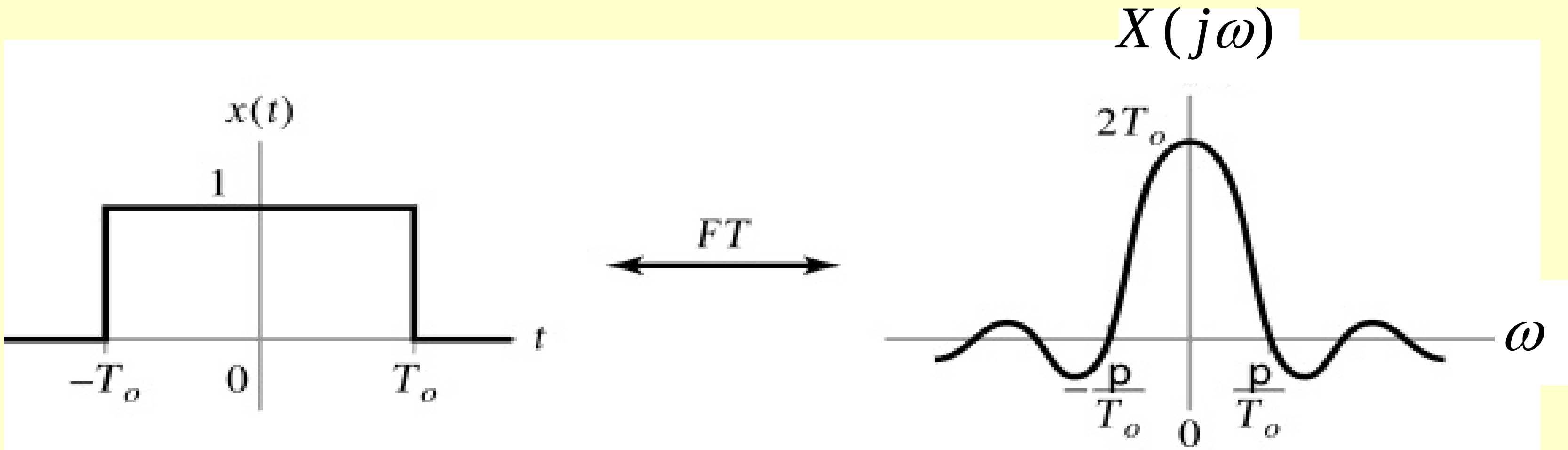
$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X^*(j\omega) [X(j\omega)] \quad d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(j\omega)|^2 \quad d\omega$$

Time-Bandwidth Production 時間頻寬乘積

- 在時域中把訊號壓縮，導致訊號在頻域中擴張。
- 在時域中把訊號擴張，導致訊號在頻域中壓縮。
- “測不準原理” 說明無法精確的決定一個電子的位置和動量。
- 新 “測不準原理” 被用來說明時域解析度和頻域解析度不能同時要求
    - 頻率範圍和時間範圍的乘積永遠有一個常數的下線。

Rectangular pulse illustrating the inverse relationship between the time and frequency extent of a signal.



頻率範圍：$-\infty \sim +\infty$ 但能量集中於主瓣 (main lobe)。

集中於 $|\omega| \le \pi / T_0$ 訊號時間範圍 T_0 和頻率範圍相反。

Duality 對偶性

- 時域/頻域中矩形脈波，在頻域/時域中為 sinc 函數。
- 時域中脈衝函數/常數，在頻域中為常數/脈衝函數。
- 時域中褶積運算/乘法運算，在頻域中為乘法運算/褶積運算。
- 時域/頻域中微分運算，在頻域/時域中為乘 $j\omega$ / $-jt$。
- …
- 上述時間和頻率間互換性質，稱為對偶性。

FT 轉換以一通式表示：

$$y(\upsilon)=\frac{1}{2\pi}\int_{-\infty}^{\infty} z(\eta)\,e^{j\upsilon\eta}\,d\eta$$

若 η = ω ，ν = t ：

$$y(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} z(\omega)\,e^{j\omega t}\,d\omega$$

若 η = t ，ν = - ω ：

$$2\pi\, y(-\omega)=\int_{-\infty}^{\infty} z(t)\,e^{-j\omega t}\,dt$$

通式表示： $y(\upsilon)=\frac{1}{2\pi}\int_{-\infty}^{\infty} z(\eta)e^{j\upsilon\eta}\,d\eta$

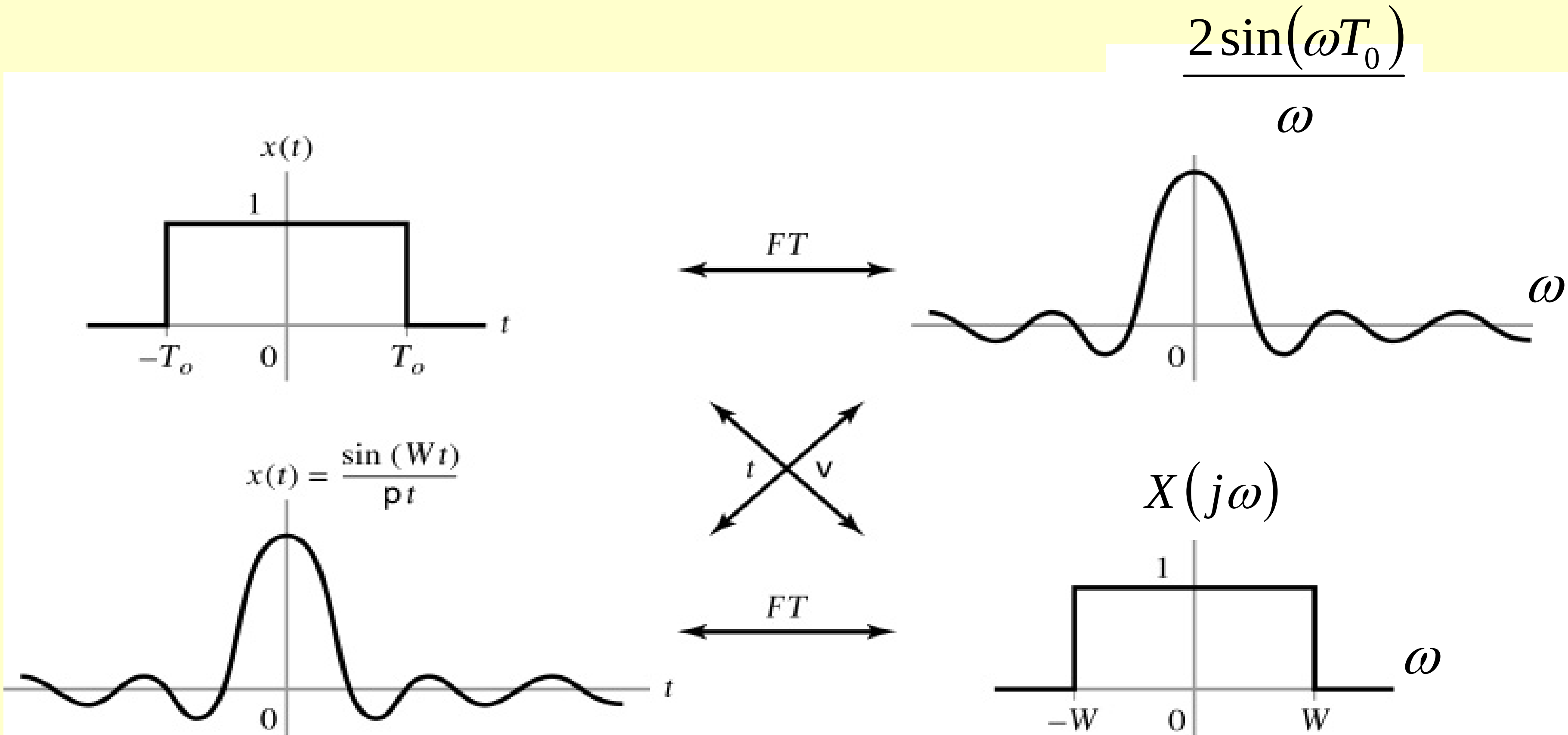
若已知一 FT pair： $f(t) \overset{FT}{\leftrightarrow} F(j\omega)$

η = ω ，ν = t ： $f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}\,d\omega$

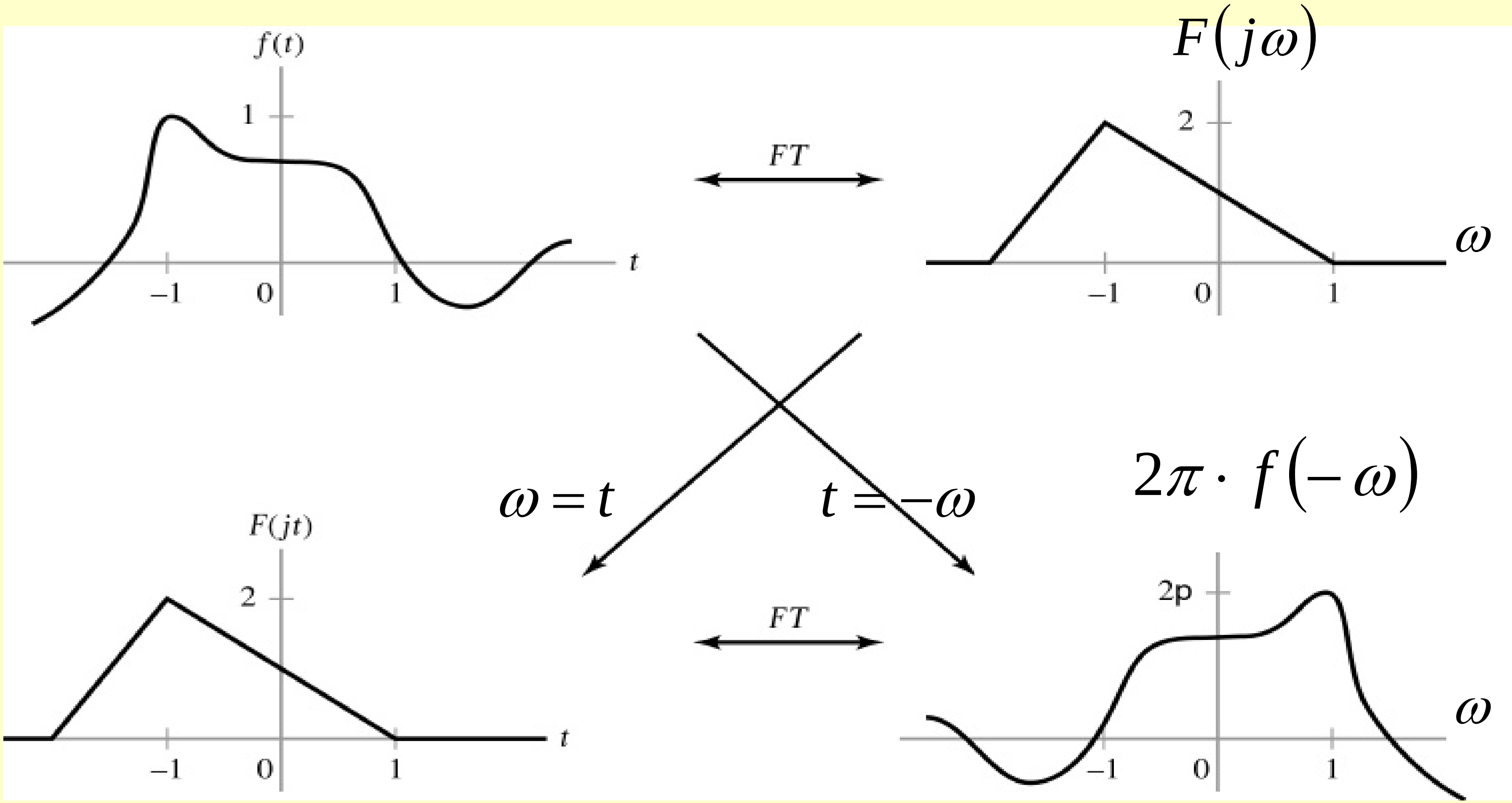
η = t ，ν = - ω ： $2\pi\, f(-\omega)=\int_{-\infty}^{\infty} F(jt)e^{-j\omega t}\,dt$

可對應出另一 FT pair： $F(jt) \overset{FT}{\leftrightarrow} 2\pi\, f(-\omega)$

Duality of rectangular pulses and sinc functions

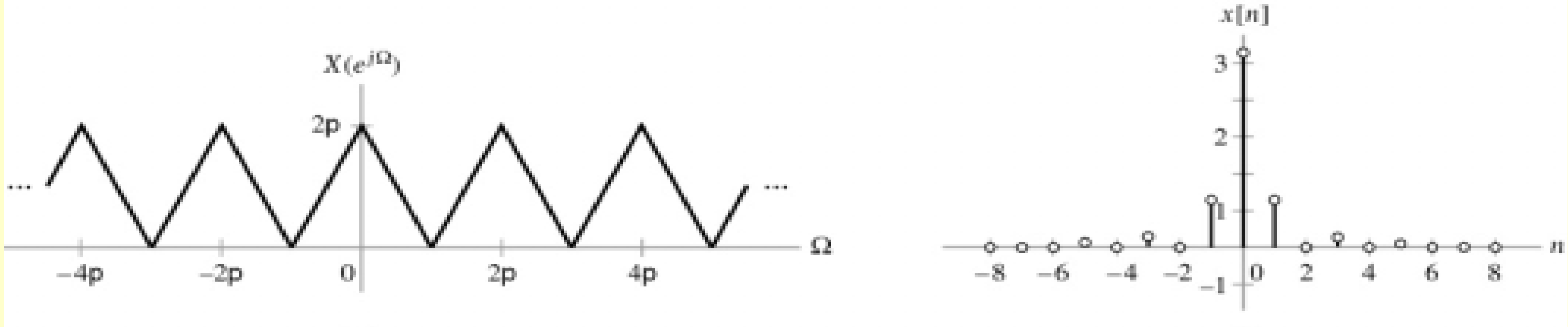


The FT duality property



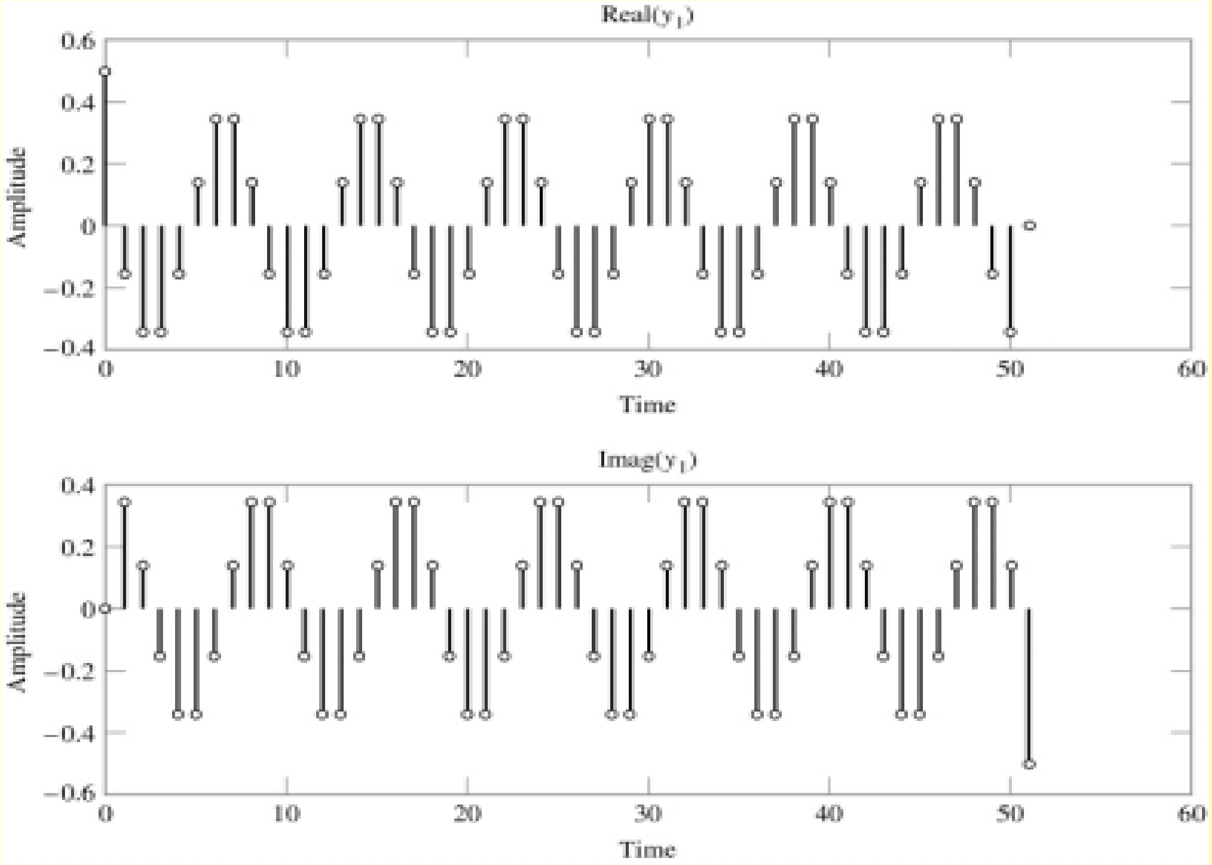
Example 3.53.

(a) Triangular spectrum. (b) Inverse DTFT.



(a) (b)

Sinusoidal steady-state response computed with the use of MATLAB. The values at times 1 through 50 represent the sinusoidal steady-state response.



Solution to Problem 3.46