Fourier Applications

傅立葉表示法對混合訊號的應用

Lecture 4-2

Sampling (取樣)

- 連續時間訊號的取樣
  - 取樣操作從連續時間訊號獲得離散時間訊號
- 離散時間訊號的取樣
  - 取樣操作改變資料傳輸速率與儲存樣本數目

Sampling Continuous-Time Signals

連續時間訊號的取樣

- 取樣操作從連續時間訊號 x(t) 獲得離散時間訊號 x[n]，在取樣間距T_s 的整數倍之處 x(t) 所取的值亦稱為樣本 (sample)。
- x[n] = x(nT_s), n is integer。
- x[n] 的連續時間表示法： $$x_\delta(t) = \sum_{n=-\infty}^{\infty} x[n]\,\delta(t - nT_s)$$

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x[n]\,\delta(t - nT_s) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t - nT_s)$$

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t-nT_s) = \sum_{n=-\infty}^{\infty} x(t)\,\delta(t-nT_s)$$

$$= x(t)\cdot \sum_{n=-\infty}^{\infty} \delta(t-nT_s) = x(t)\cdot p(t),$$

$$where \quad p(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT_s)$$ 單位脈衝列

應用乘積和摺積性質：

$$X_\delta(j\omega) = \frac{1}{2\pi} X(j\omega) * P(j\omega)$$

$$X_\delta(j\omega) = \frac{1}{2\pi} X(j\omega) * \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$$

$$where\ \omega_s = 2\pi / T_s \quad sampling\ frequency$$

$$X_\delta(j\omega) = \frac{1}{2\pi} X(j\omega) * \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

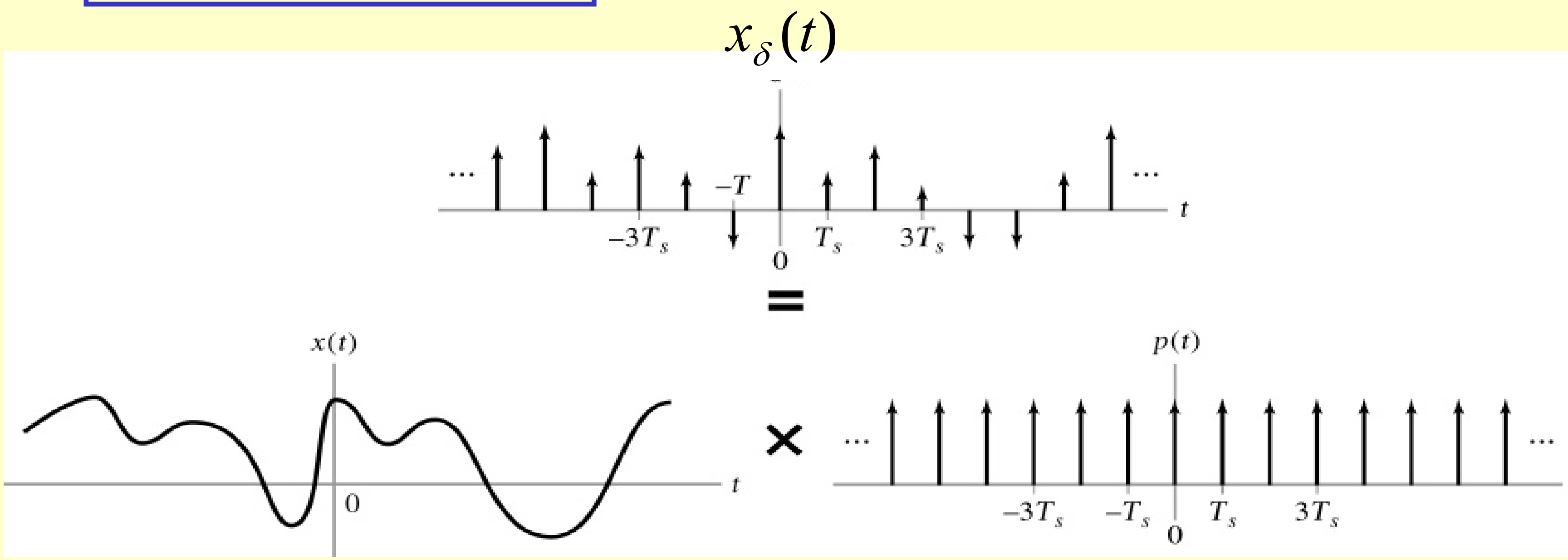
$$X_\delta(j\omega)=\frac{1}{T_s}\sum_{k=-\infty}^{\infty}X(j(\omega-k\omega_s))$$

- 樣本訊號的 FT 是原始訊號的 FT，
  經過不同頻率平移後的無窮和。
- 平移後頻譜(spectrum)的位置距離原來位置是ω_s 的整數倍。
- 若 ω_s 不夠大，平移後的頻譜可能會重疊。
- 原始訊號和其頻率位移的頻譜重疊情形稱為頻疊(aliasing)。

Mathematical Representation of Sampling

- Product of a Given Time Signal and an Impulse Train
- x[n] 的連續時間表示法 $x_d(t)$ or $x_\delta(t)$ 可由取樣程序獲得:

$$x_\delta(t) = x(t) \cdot p(t)$$



Aliasing ?

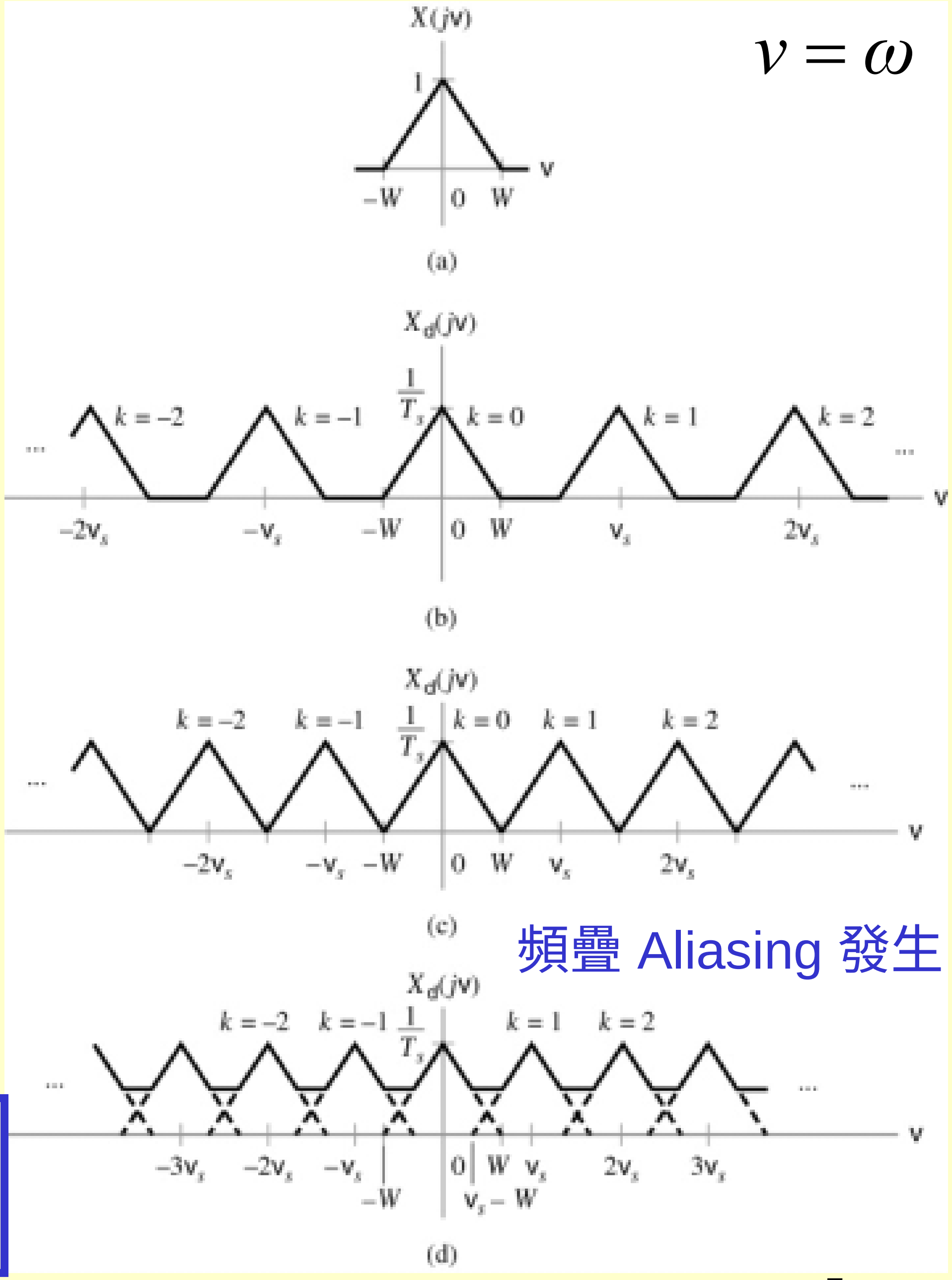
The FT of a sampled signal for different sampling frequencies.

(a) Spectrum of continuous-time signal.

(b) Spectrum of sampled signal when ωs = 3W.
(c) Spectrum of sampled signal when ωs = 2W.
(d) Spectrum of sampled signal when ωs = 1.5W.

$$X_\delta(j\omega)=\frac{1}{T_s}\sum_{k=-\infty}^{\infty}X(j(\omega-k\omega_s))$$

$v=\omega$



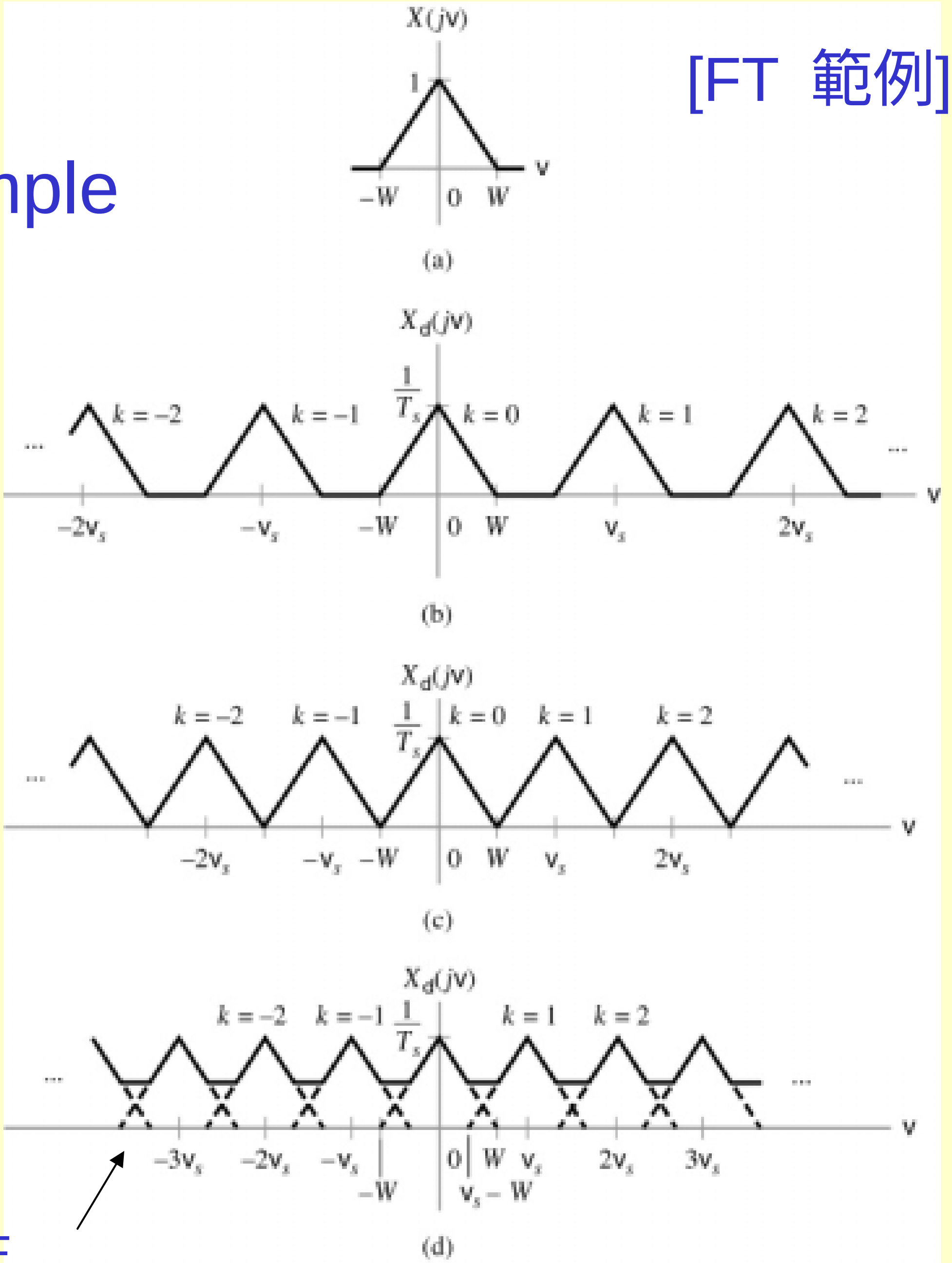
頻疊 Aliasing 發生

[FT 範例]

Aliasing in FT Example

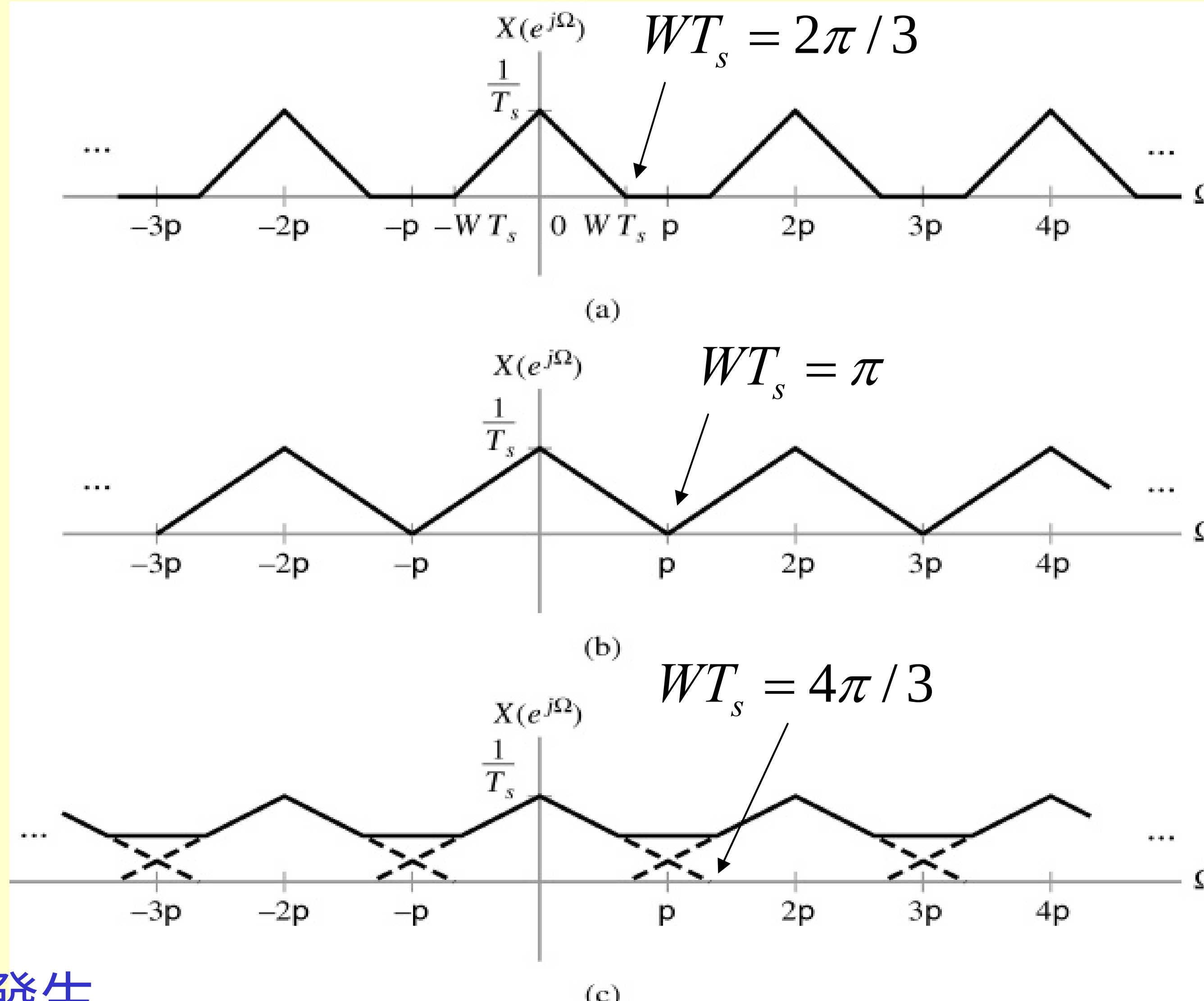
$$v = \omega$$

$$v_s = \omega_s$$



頻疊 Aliasing 發生

Aliasing in DTFT Example



$$\because \Omega = \omega T_s$$

$$\therefore \Omega_s = \omega_s T_s = \frac{2\pi}{T_s} T_s = 2\pi$$

<u>頻疊 Aliasing 發生</u>

The DTFTs corresponding to the FTs, [DTFT 範例]
(a) ωs = 3W. (b) ωs = 2W. (c) ωs = 1.5W.

The DTFTs corresponding to the FTs, (a) $\omega_s = 3W$. (b) $\omega_s = 2W$. (c) $\omega_s = 1.5W$.

$$\because \Omega = \omega T_s, \quad \therefore \Omega_s = \omega_s T_s = \frac{2\pi}{T_s} T_s = 2\pi$$

$$\because \text{FT spectrum bandwidth,} \quad \omega_m = W$$

$$\therefore \text{DTFT spectrum bandwidth,} \ \Omega_{\mathrm{m}} = \omega_m T_s = W T_s$$

$$(a) \quad \omega_s = 2\pi / T_s = 3W, \quad \therefore T_s = 2\pi / 3W$$

$$\therefore \quad \underline{W T_s = 2\pi / 3}$$

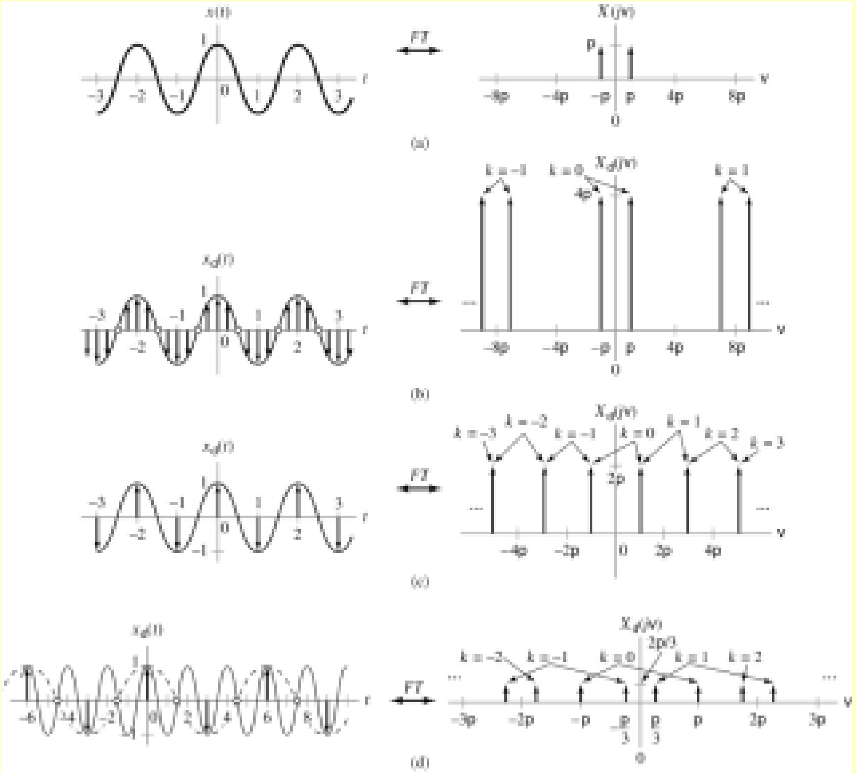
$$(b) \quad \omega_s = 2\pi / T_s = 2W, \quad \therefore T_s = \pi / W$$

$$\therefore \quad \underline{W T_s = \pi}$$

$$(c) \quad \omega_s = 2\pi / T_s = 3W / 2, \quad \therefore T_s = 4\pi / 3W$$

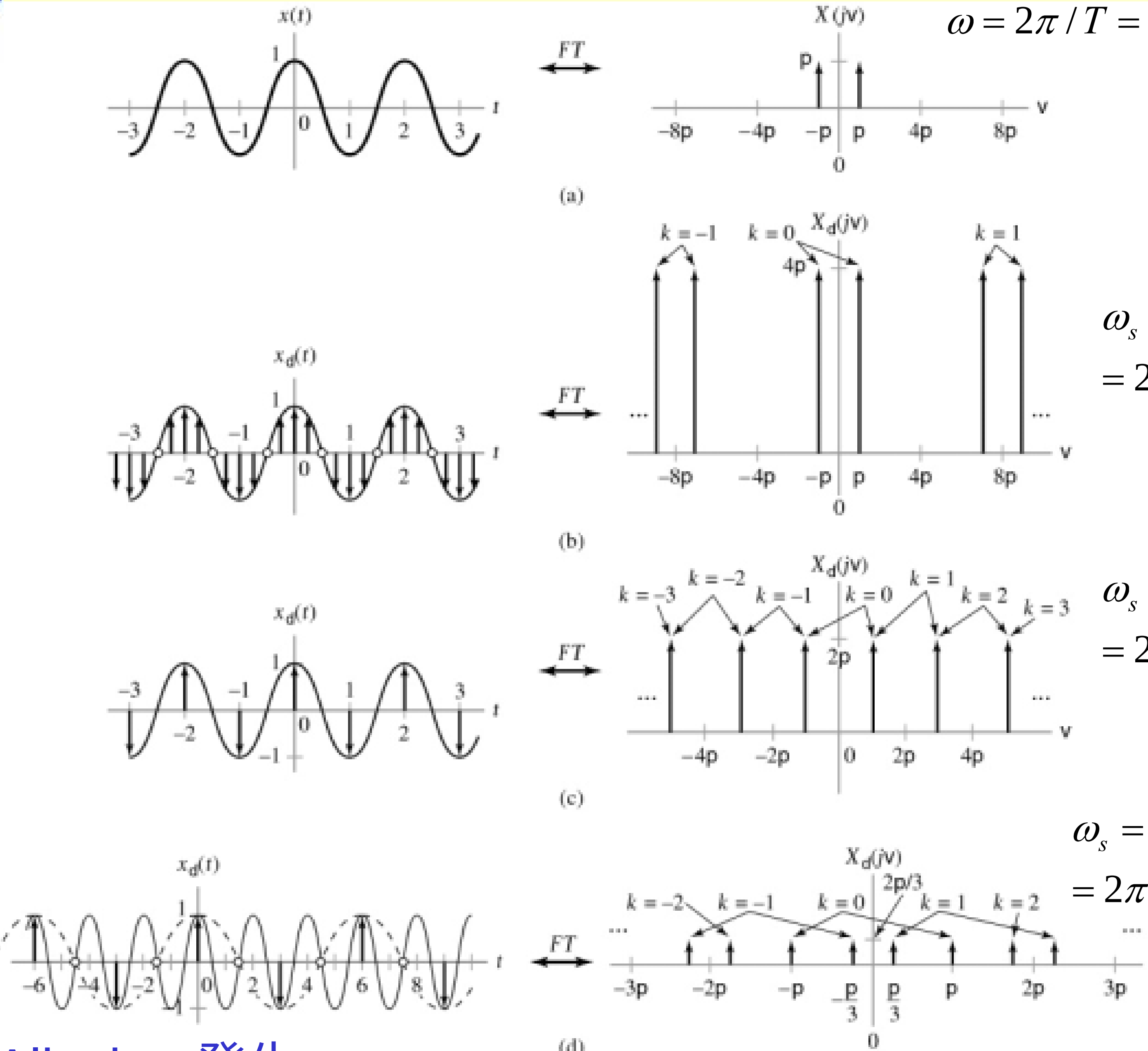
$$\therefore \quad \underline{W T_s = 4\pi / 3}$$

Sampling a Sinusoid at Different Rates



The effect of sampling a sinusoid at different rates (Example 4.9). 對弦波取樣範例
(a) Original signal and FT.
(b) Original signal, impulse sampled representation and FT for $T_s = 1/4$.
(c) Original signal, impulse sampled representation and FT for $T_s = 1$.
(d) Original signal, impulse sampled representation and FT for $T_s = 3/2$.
A cosine of frequency $\pi/3$ is shown as the dashed line.

$$\omega = 2\pi / T = 2\pi / 2 = \pi$$



$T_s = 1/4$

$$\omega_s = 2\pi / T_s = 2\pi / (1/4) = 8\pi$$

$T_s = 1$

$$\omega_s = 2\pi / T_s = 2\pi / 1 = 2\pi$$

$T_s = 3/2$

$$\omega_s = 2\pi / T_s = 2\pi / (3/2) = 4\pi / 3$$

頻疊 Aliasing 發生

Aliasing in a Movie

(a) Wheel rotating at ω (radians per second) and moving from right to left at ν (meters per second).

(b) Sequence of movie frames, assuming that the wheel rotates less than one-half turn between frames.

(c) Sequences of movie frames, assuming that the wheel rotates between one-half and one turn between frames.

(d) Sequence of movie frames, assuming that the wheel rotates one turn between frames.

電影拍攝速度：30 frames/sec., T_s = 1/30 秒

(a) 輪子轉動 ω radians/sec. (r = 1/4m)，
由右向左移動 ν meters/sec. = ωr = ω/4 (m/sec.)

(b) 電影拍攝取樣所拍到角度變化: ωT_s < 半圈 (i.e. < π).
→ ω < 30π (輪子轉動速率 ω < 電影拍速率的一半)
→ 正常

(c) 電影拍攝取樣所拍到角度變化: 半圈< ωT_s < 一圈 (i.e. π < ωT_s < 2π).
→ 30π < ω < 60π → 30π/4 < ω/4 < 60π/4
→ 23.5 m/sec. < ν < 47.12 m/sec.
→ 輪子有反轉錯覺(頻疊產生)

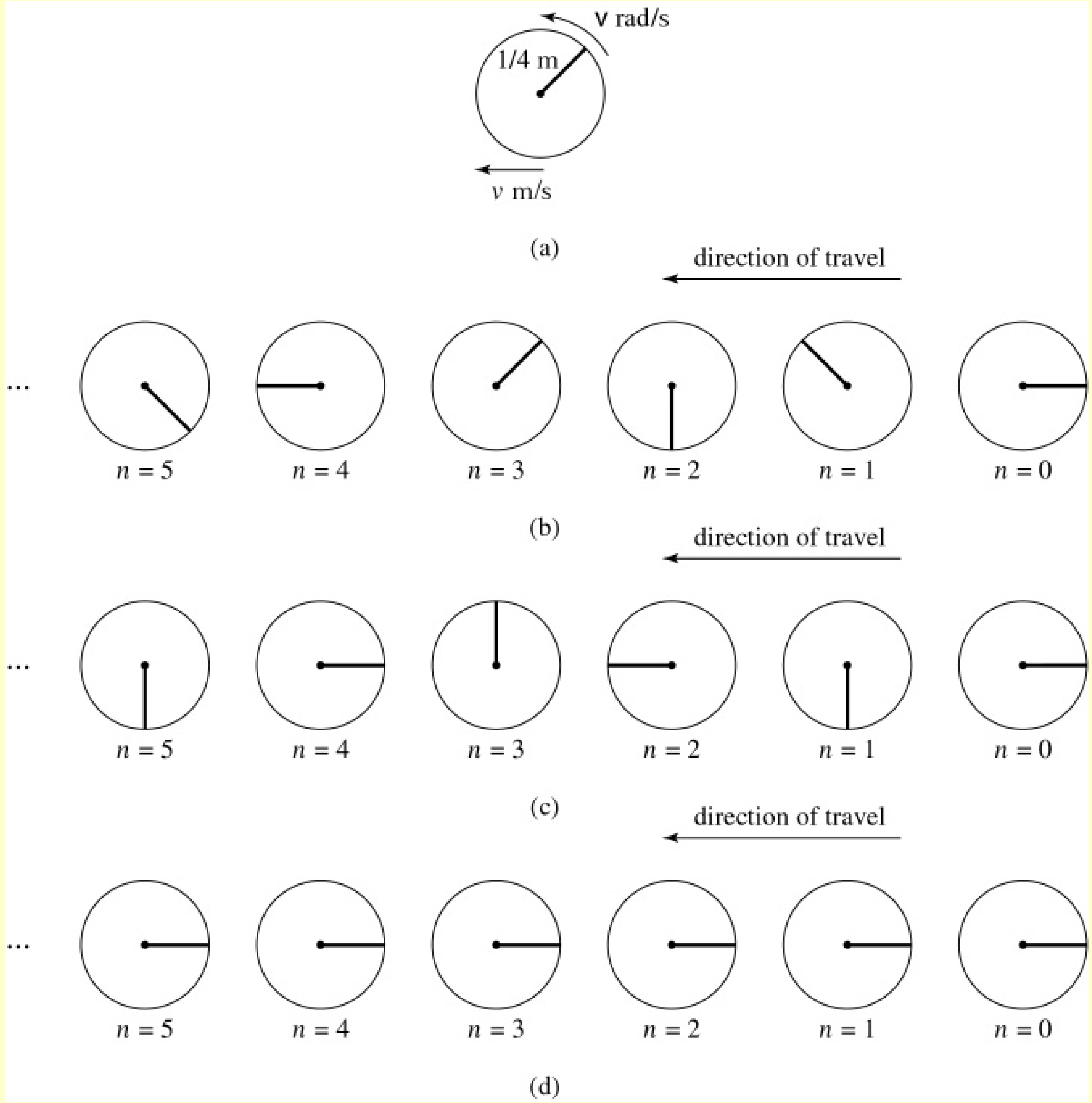
(d) 電影拍攝取樣所拍到角度變化: ωT_s = 一圈 (i.e. ωT_s = 2π).
→ ω = 60π → ν = 47.12 m/sec.
→ 輪子有靜止錯覺(頻疊產生)

(a)

(b)

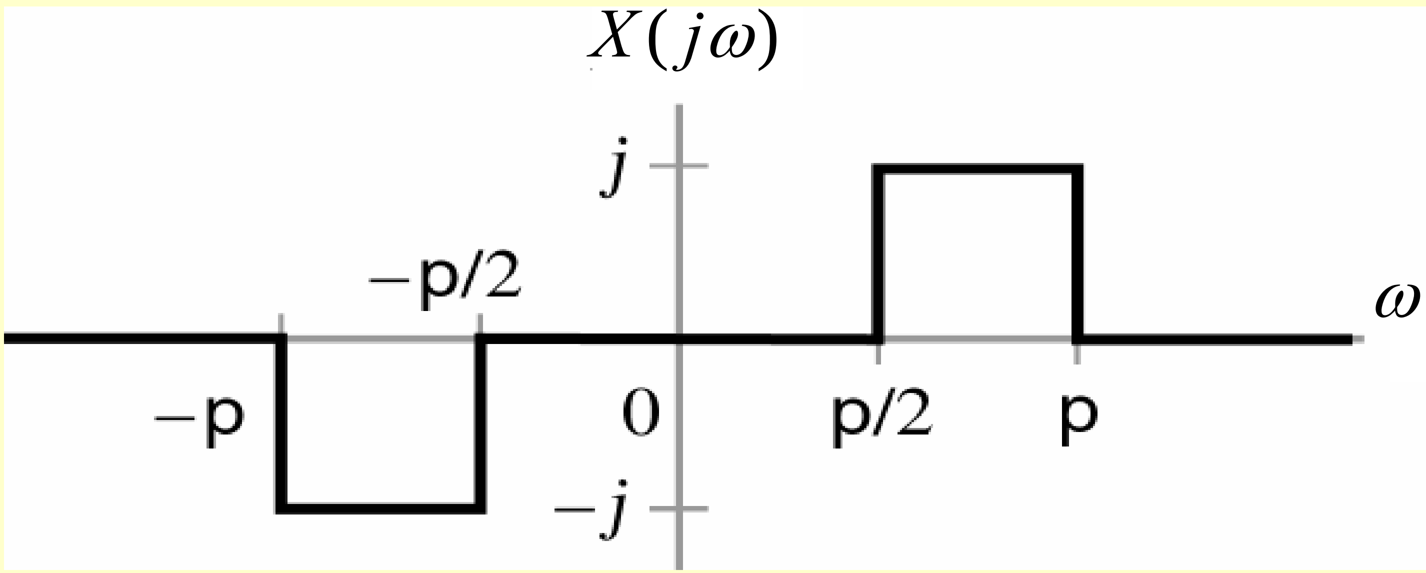
(c)

(d)



ν rad/s
1/4 m
ν m/s
(a)
direction of travel
...
n = 5
n = 4
n = 3
n = 2
n = 1
n = 0
(b)
direction of travel
...
n = 5
n = 4
n = 3
n = 2
n = 1
n = 0
(c)
direction of travel
...
n = 5
n = 4
n = 3
n = 2
n = 1
n = 0
(d)

Problem 4.10 下圖為連續時間訊號的 FT 頻譜，試分別對 (a) $T_s = 1/2$ (b) $T_s = 2$ 求取樣後的 FT 並繪出。



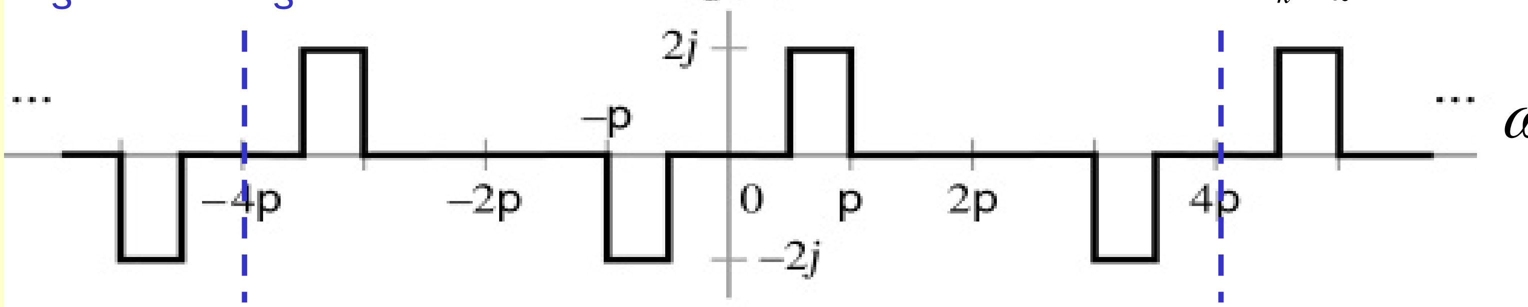
Sampled Spectra: $$X_\delta(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X\big(j(\omega - k\omega_s)\big)$$

Solution to Problem 4.10

$T_s = ½$

$\omega_s = 2\pi/T_s = 4\pi$

$$X_\delta(j\omega) = 2\sum_{k=-\infty}^{\infty} X(j(\omega - k4\pi))$$

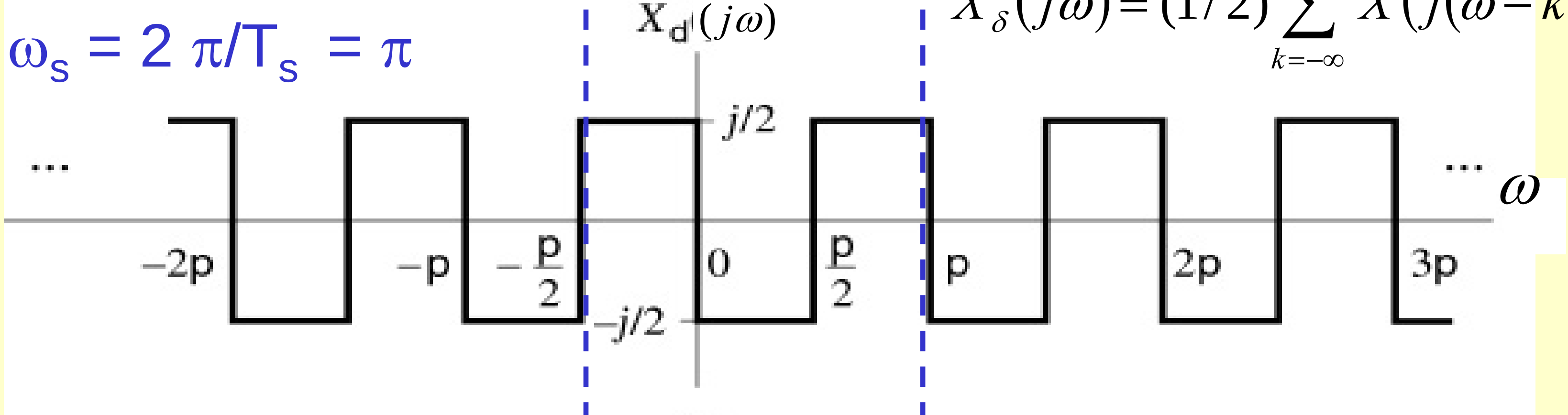


(a)

$T_s = 2$

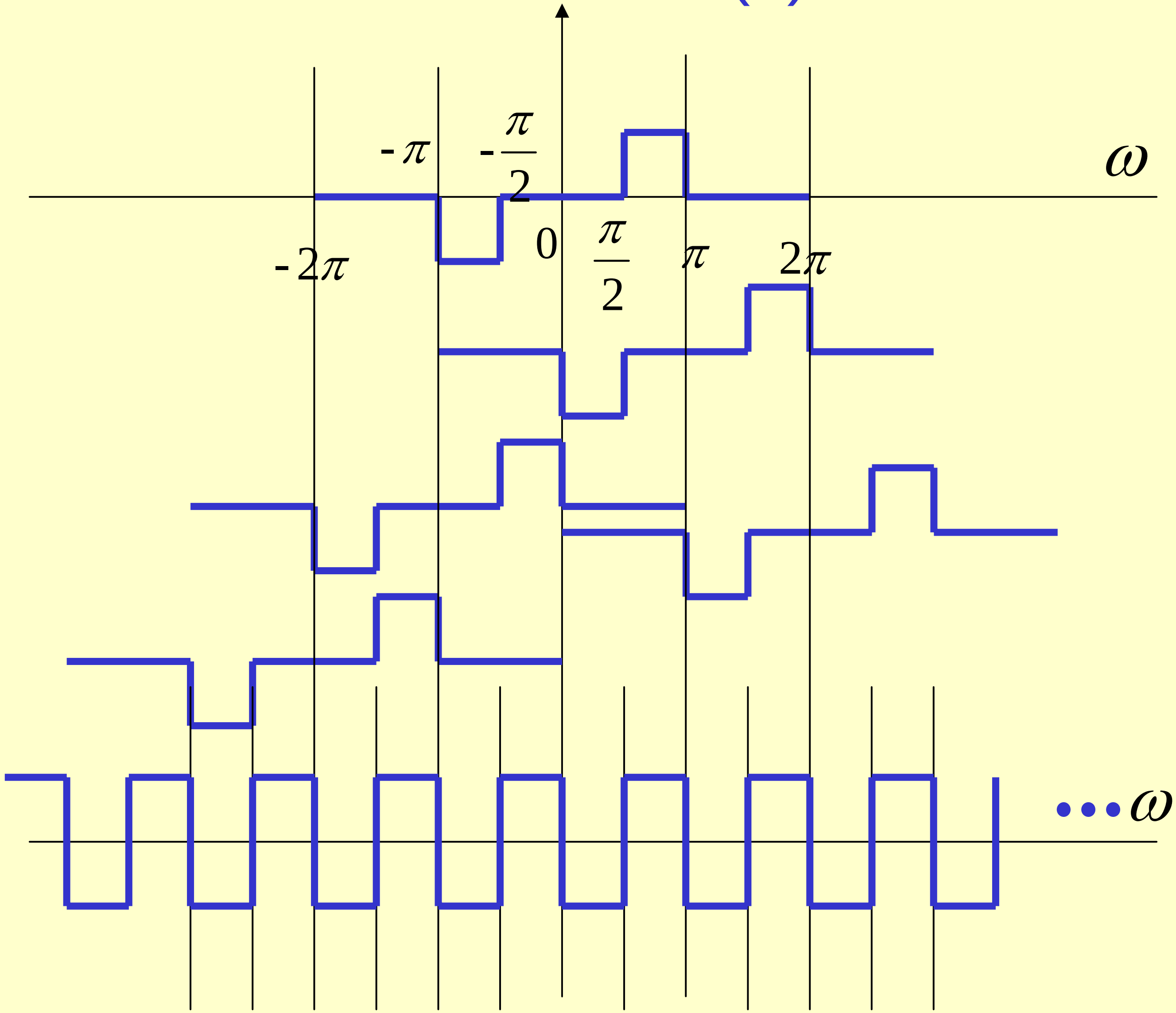
$\omega_s = 2\pi/T_s = \pi$

$$X_\delta(j\omega) = (1/2)\sum_{k=-\infty}^{\infty} X(j(\omega - k\pi))$$



(b)

Problem 4.10(b)



Sub-Sampling: Sampling Discrete-Time Signals

次取樣：離散時間訊號的取樣

- y[n] = x[q n] 為經過次取樣的樣本，q 需為正整數。
- x[n] 原表示為連續性時間訊號 x(t) 的離散樣本。
- y[n] 也可表示為連續性時間訊號 x(t) 的離散樣本，取樣間距為原取樣的 q 倍，仍需注意 aliasing 的發生與否。

$$x[n] = x(nT_s)$$

$$y[n] = x[nq] = x(nqT_s)$$

Problem 4.11

若 q = 2, or q = 5. 繪出經過次取樣後訊號 y[n] = x[qn] = ?

$$x[n] = 2\cos\left(\frac{\pi}{3}n\right) = e^{j\frac{\pi}{3}n} + e^{-j\frac{\pi}{3}n}$$

$$\Leftrightarrow X(e^{j\Omega}) = 2\pi\delta\left(\Omega - \frac{\pi}{3}\right) + 2\pi\delta\left(\Omega + \frac{\pi}{3}\right)$$

$$< q = 2\ case >$$

$$y[n] = x[2n] = 2\cos\left(\frac{\pi}{3}2n\right) = e^{j\frac{\pi}{3}2n} + e^{-j\frac{\pi}{3}2n}$$

$$\Leftrightarrow Y(e^{j\Omega}) = \pi\,\delta\left(\Omega - \frac{2\pi}{3}\right) + \pi\,\delta\left(\Omega + \frac{2\pi}{3}\right), \quad -\pi < \Omega < +\pi$$

P 4.11 (cont.) y[n] = x[5n]

$$x[n] = 2\cos\left(\frac{\pi}{3}n\right) = e^{j\frac{\pi}{3}n} + e^{-j\frac{\pi}{3}n}$$

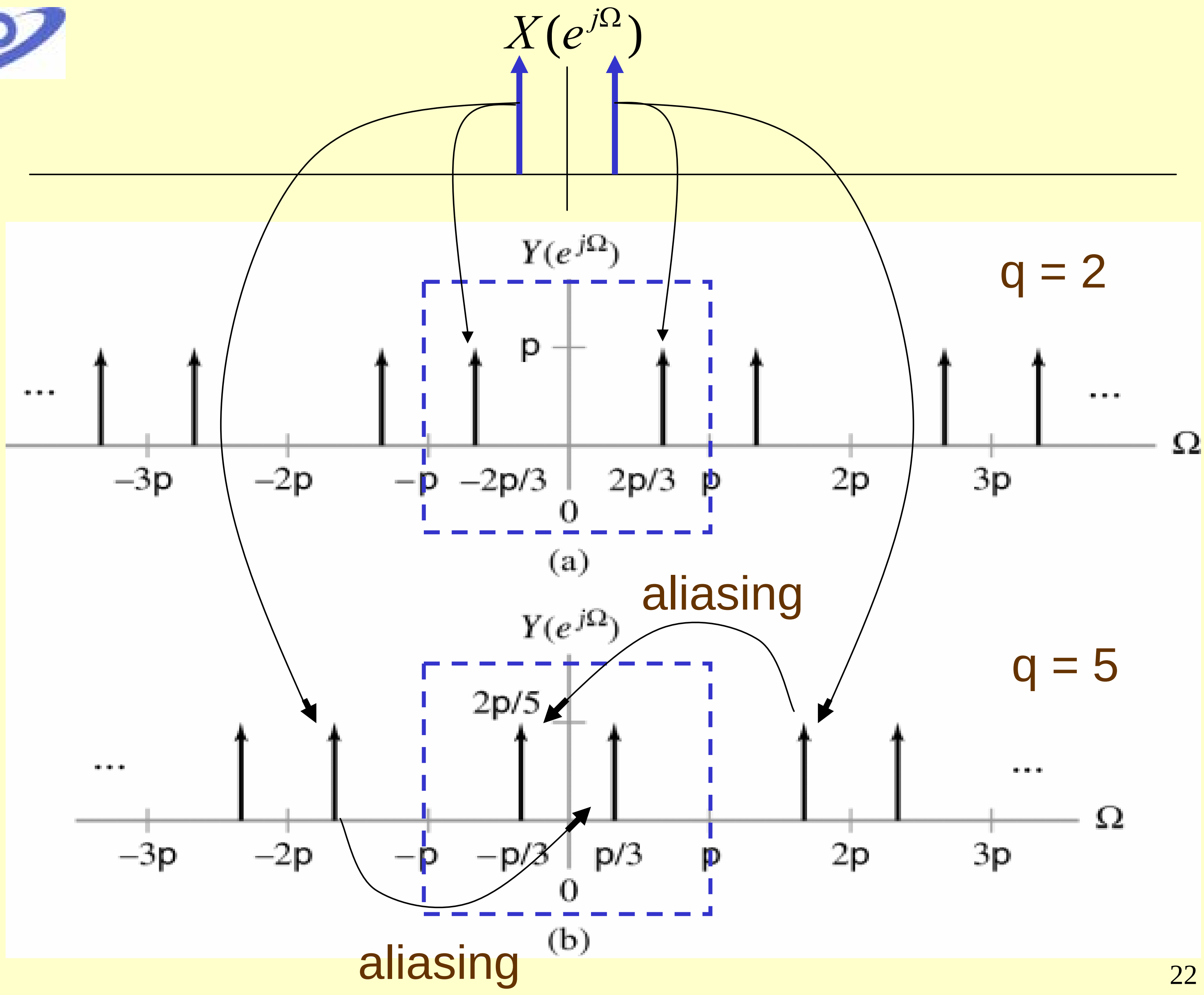
$$\Leftrightarrow X(e^{j\Omega}) = 2\pi\delta\left(\Omega - \frac{\pi}{3}\right) + 2\pi\delta\left(\Omega + \frac{\pi}{3}\right)$$

$$< q = 5 \quad case >$$

$$y[n] = x[5n] = 2\cos\left(\frac{\pi}{3}5n\right) = e^{j\frac{\pi}{3}5n} + e^{-j\frac{\pi}{3}5n}$$

$$\Leftrightarrow Y(e^{j\Omega}) = \frac{2\pi}{5}\delta\left(\Omega - \frac{5\pi}{3} \pm 2\pi\right) + \frac{2\pi}{5}\delta\left(\Omega + \frac{5\pi}{3} \pm 2\pi\right)$$

$$= \frac{2\pi}{5}\delta\left(\Omega + \frac{\pi}{3}\right) + \frac{2\pi}{5}\delta\left(\Omega - \frac{\pi}{3}\right), \quad -\pi < \Omega < \pi$$

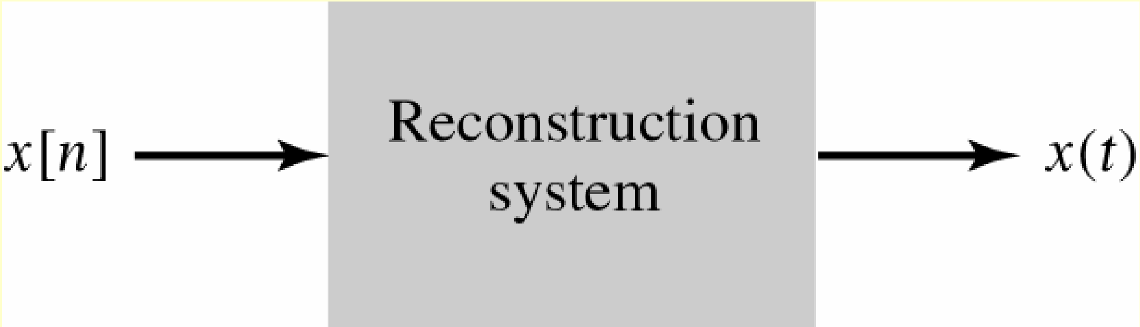

X(e^{jΩ})
Y(e^{jΩ})
q = 2
p
...
...
Ω
–3p
–2p
–p
–2p/3
2p/3
p
2p
3p
0
(a)
aliasing
Y(e^{jΩ})
q = 5
2p/5
...
...
Ω
–3p
–2p
–p
–p/3
p/3
p
2p
3p
0
(b)
aliasing

從離散樣本重建連續時間性訊號

取樣定理 (Sampling Theorem)

理想的訊號重建 (Ideal Reconstruction)

重建基本方法-零階保持器 (Zero-Order Hold)



Sampling Theorem

取樣定理:

令 $x(t) \overset{FT}{\leftrightarrow} X(j\omega)$ 表一頻寬受限的訊號，頻寬 ω_m

其頻譜 $X(j\omega) = 0, \quad \forall\ |\omega| > \omega_m$

取樣頻率 $\omega_s = 2\pi/T_s \geq 2\ \omega_m$

(or $f_s \geq 2 f_m$)

奈奎斯特頻率(Nyquist frequency) : （取樣頻率的底線）

$\omega_n = 2\pi/T_s = 2\ \omega_m$ (or $f_n = 2 f_m$)

Not One to One Mapping

離散樣本 x[n] (火柴棒形狀) 有可能是來自於下述兩組連續時間訊號的任一信號。

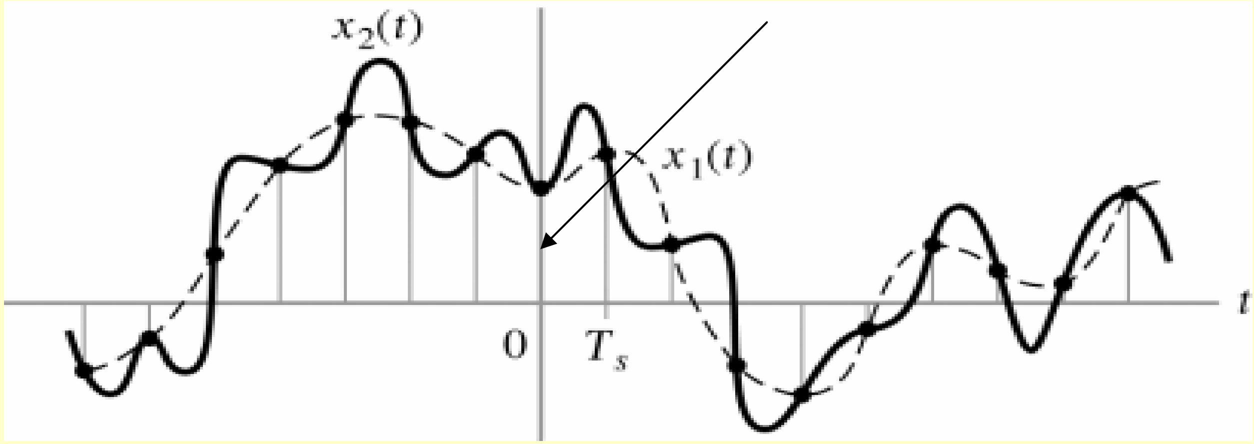
$$x[n] = x_1(nT_s) = x_2(nT_s)$$

連續時間與離散時間訊號無法一對一對應，頻疊 (aliasing) 因而產生。

$x_1(t)$ - - - - - - - - -

$x_2(t)$ ————

$x[n]$



Example 4.12

假設 x(t) = sin(πt)/ πt，試求取樣間距應滿足的條件 T_s= ?，使得 x[n] = x(nT_s) 能唯一的表示。

Solution:

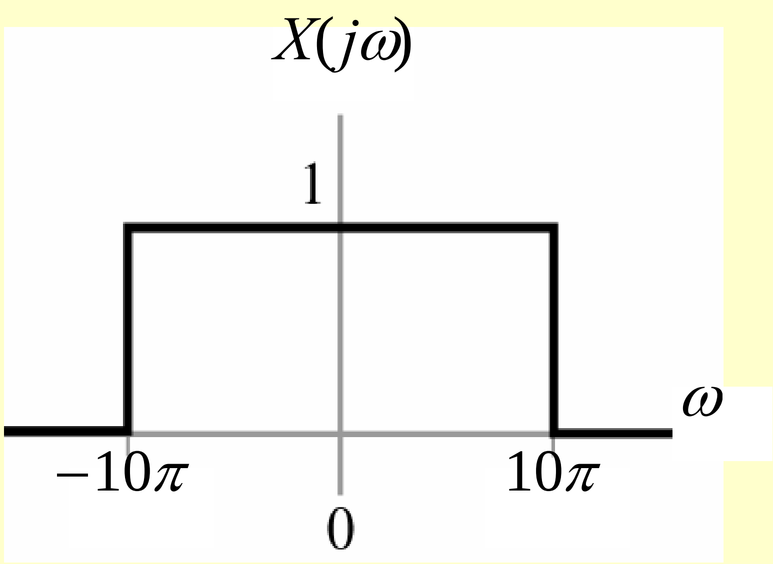
$$\because \quad x(t)=\frac{\sin(\pi t)}{\pi t} \overset{FT}{\leftrightarrow} X(j\omega)=\begin{cases}1, & |\omega|\le 10\pi \\ 0, & |\omega|>10\pi\end{cases}$$

$$\therefore \quad \omega_{\mathrm{m}}=10\pi$$

$$\because \quad \frac{2\pi}{T_s}\ge 2\omega_{\mathrm{m}}=20\pi,$$

$$\therefore \frac{1}{T_s}\ge 10,$$

$$\therefore T_s<0.1$$



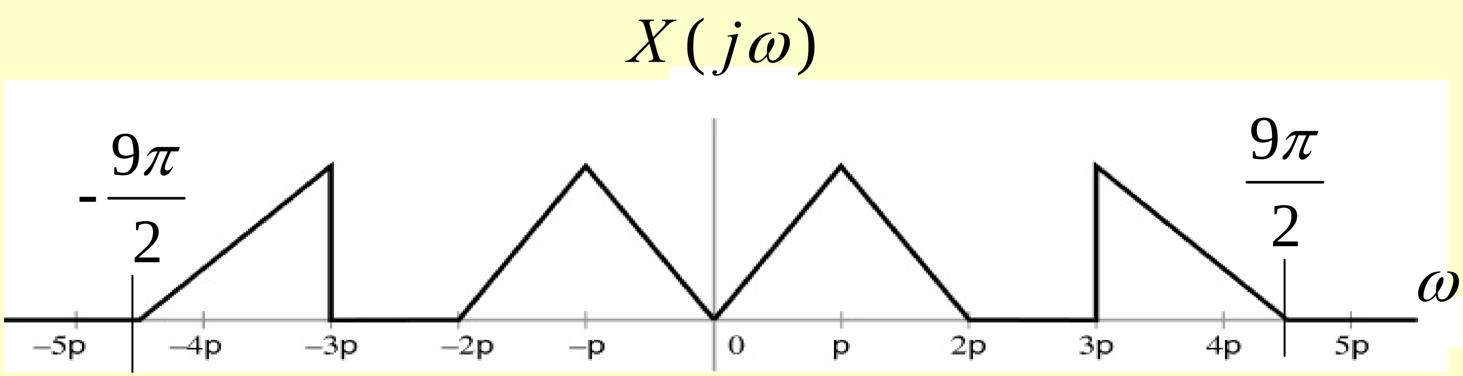
Problem 4.12

試求取樣間距應滿足的條件 T_s = ?，使得 $x[n] = x(nT_s)$ 能唯一的表示：亦即無頻疊發生 (下圖為 $x(t)$ 的 FT)

Solution:

$$\because \quad \omega_m = \frac{9\pi}{2}, \quad \frac{2\pi}{T_s} \geq 2\omega_m = 9\pi,$$

$$\therefore \quad \frac{2}{T_s} \geq 9, \quad \therefore \quad T_s < \frac{9}{2}.$$



理想的訊號重建 (Ideal Reconstruction)

取樣定理指出我們須以多快的速率取樣，才可以使樣本唯一代表連續時間訊號。

時域取樣過程： (原始連續訊號與脈衝串列的乘積)

$$x_\delta(t) = x(t)\cdot p(t) = x(t)\cdot\sum_{n=-\infty}^{\infty}\delta(t - nT_s)$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t - nT_s)$$

$$= \sum_{n=-\infty}^{\infty} x[n]\,\delta(t - nT_s)$$

理想的訊號重建 (cont.)

頻域取樣過程：

樣本訊號的 FT 是原來訊號的 FT 經過不同頻率平移之後的無窮和。(若 ω_s 不夠大，頻率平移版本有可能會重疊)

$$X_\delta(j\omega) = \frac{1}{2\pi} X(j\omega) * P(j\omega)$$

$$= \frac{1}{2\pi} X(j\omega) * \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$$

$$= \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

重建的目標是對 X$_\delta$(jω) 做運算以便轉換成 X(jω)，這樣運算用來刪除下式中的 kω_s 項。

$$X_\delta(j\omega)=\frac{1}{T_s}\sum_{k=-\infty}^{\infty}X\big(j(\omega-k\omega_s)\big)$$

欲轉換成 X(jω)，需做下述乘積： Low Pass Filtering

$$X(j\omega)=X_\delta(j\omega)\cdot H_r(j\omega)$$

其中

$$H_r(j\omega)=\begin{cases}T_s, & |\omega|\le\omega_s/2\\ 0, & |\omega|>\omega_s/2\end{cases}$$

Time Signal Reconstruction

時域重建運算：

$$x(t) = x_\delta(t) * h_r(t) = h_r(t) * x_\delta(t)$$

$$= h_r(t) * \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT_s) = \sum_{n=-\infty}^{\infty} x[n]\ h_r(t - nT_s)$$

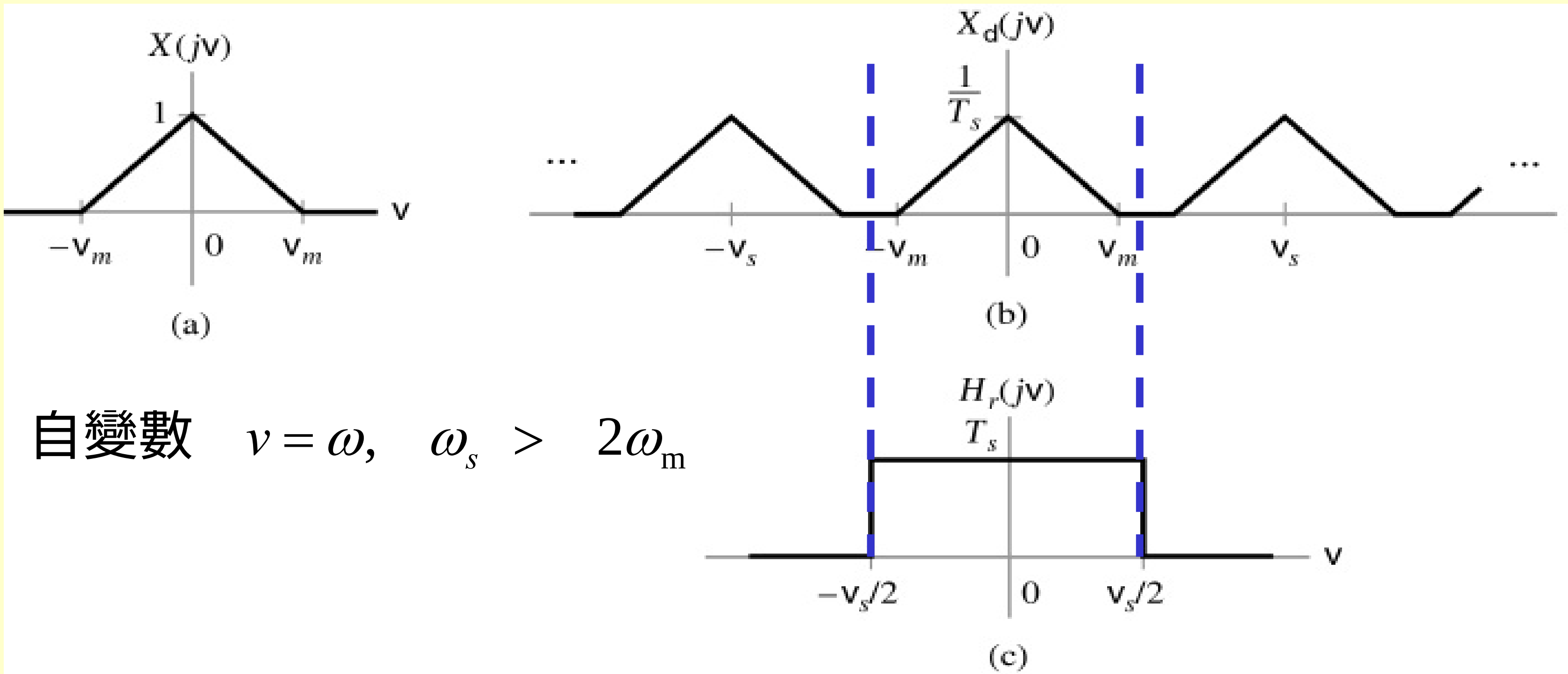
$$= \sum_{n=-\infty}^{\infty} x[n]\,\mathrm{sinc}(\omega_s (t - nT_s)/(2\pi))$$

其中

$$\mathrm{sinc(x)} = \frac{\sin(\pi\,\mathrm{x})}{\pi\,\mathrm{x}}$$

$$h_r(t) = \frac{T_s \sin(\frac{\omega_s}{2} t)}{\pi\, t}$$

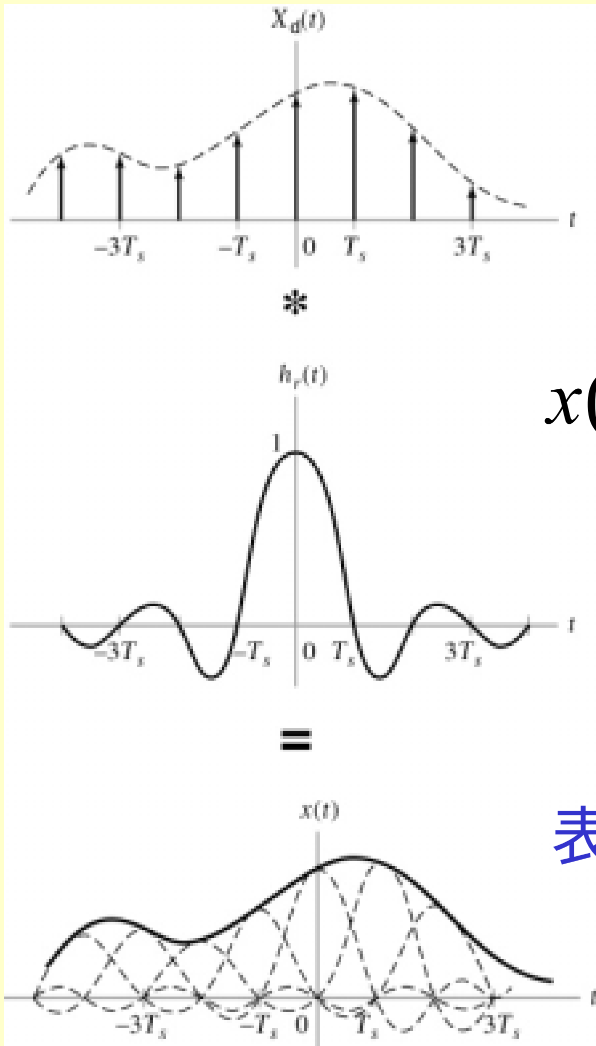
Ideal Reconstruction
原訊號理想重建



自變數 $v = \omega, \quad \omega_s > 2\omega_m$

(a) Spectrum of original signal.
(b) Spectrum of sampled signal.
(c) Frequency response of reconstruction filter.

Ideal Reconstruction in the Time-Domain



$$x(t) = x_\delta(t) * h_r(t) = \sum_{n=-\infty}^{\infty} x[n]\ h_r(t - nT_s)$$

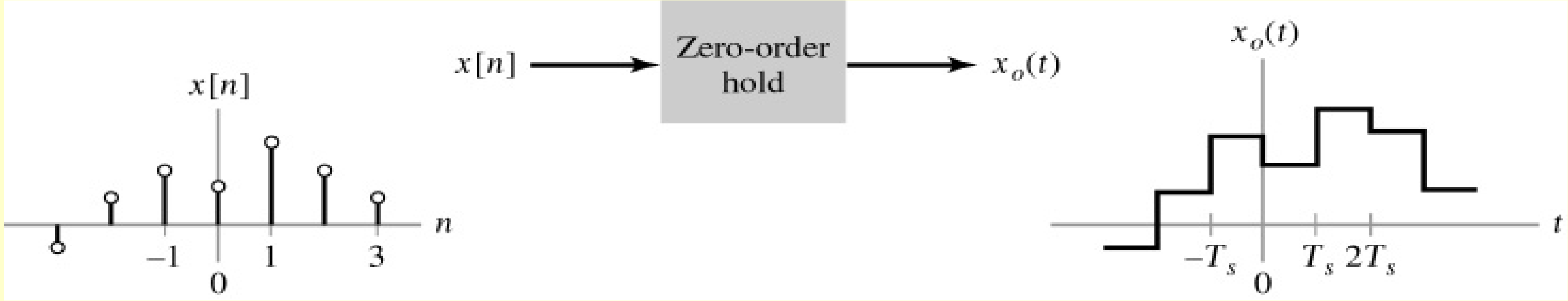
$$= \sum_{n=-\infty}^{\infty} x[n]\,\mathrm{sinc}(\omega_s(t - nT_s)/(2\pi))$$

表示為許多 sinc 函數的時間平移加權疊加

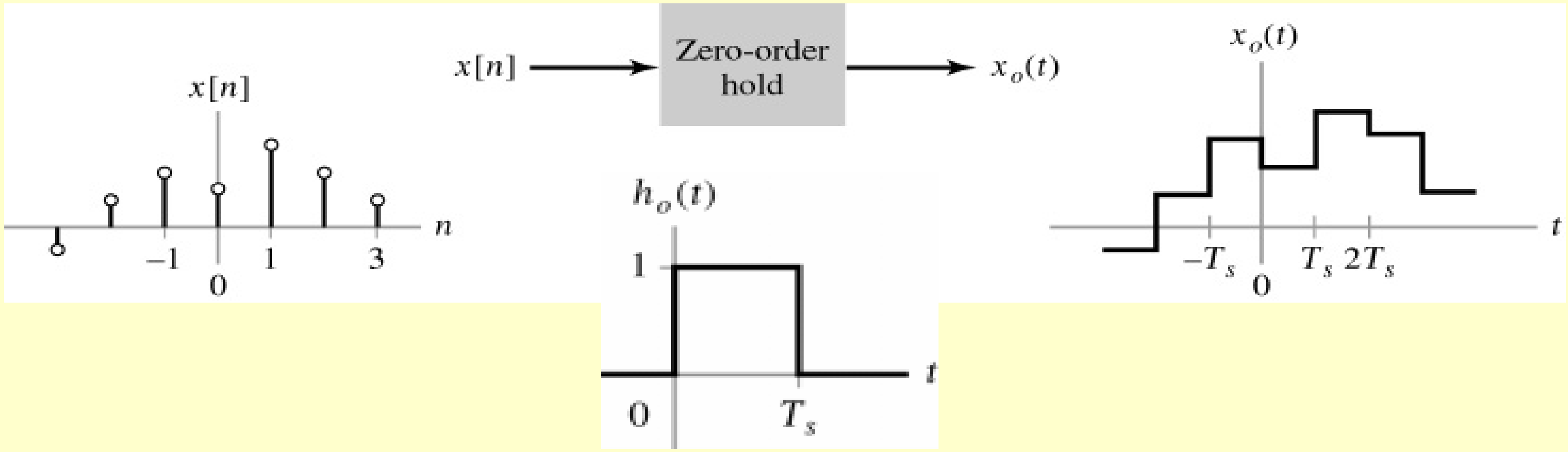
Zero- Order Holder

重建基本方法-零階保持器 (Zero-Order Hold)

零階保持器將數值 x[n] 維持或保留 T_s 秒的時間，產生一個階梯式近似連續訊號



Reconstruction via a Zero-Order Holder



重建的訊號：$h_0(t)$ 脈波的時間平移與加權疊加

$$x_0(t) = x_\delta(t) * h_0(t) = \sum_{n=-\infty}^{\infty} x[n]\, h_0(t - nT_s)$$

Reconstruction in Frequency

頻域考量：

$$X_0(j\omega)=H_0(j\omega)\cdot X_\delta(j\omega)$$

$$=\left[2e^{-j\omega T_s/2}\frac{\sin(\omega T_s/2)}{\omega}\right]\cdot X_\delta(j\omega)$$

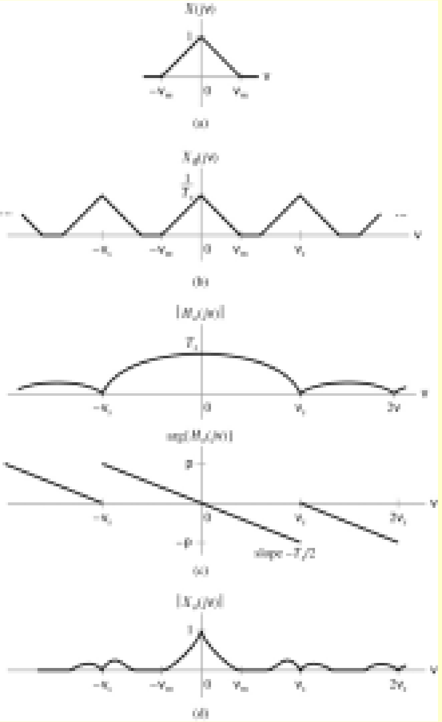
H(jω) 頻率響應考量：

振幅響應主瓣(main lobe) 曲率造成非線性失真

H(jω) 相位響應考量：

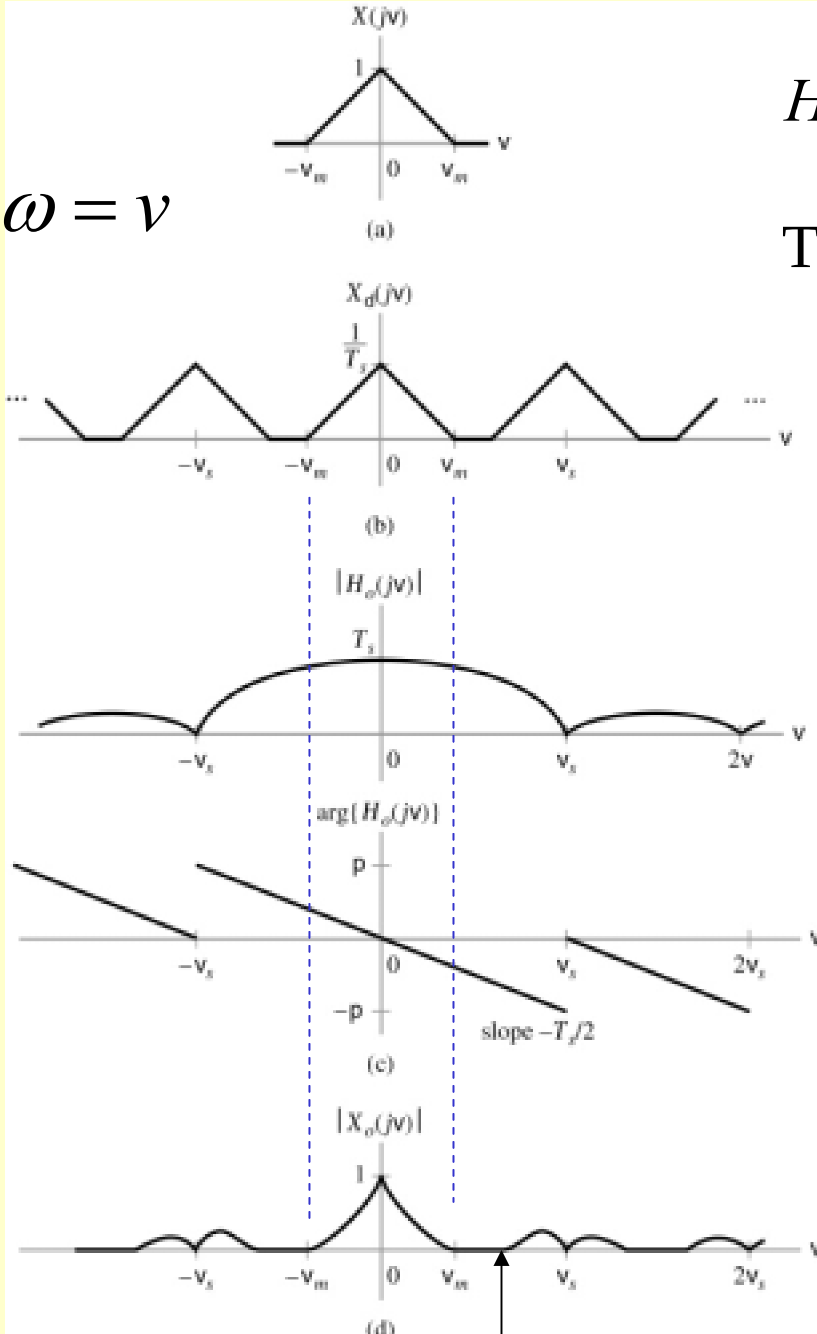
線性相位位移相當於 $T_s/2$ 秒時間延遲

Effect of the Zero-Order Hold in the Frequency Domain



(a) Spectrum of original continuous-time signal.
(b) FT of sampled signal.
(c) Magnitude and phase of Ho($j\omega$).
(d) Magnitude spectrum of signal reconstructed using zero-order hold.

$$\omega = v$$



$$H_0(j\omega) = \left[2e^{-j\omega T_s/2} \frac{\sin(\omega T_s/2)}{\omega} \right]$$

The 1st zero crossing: $\omega = \pm\, \omega_s$

$\omega_s - \omega_m$

Zero-Order Holder Disadvantages

零階保持器重建連續時間訊號引入三項被修改處：

1. 線性相位位移相當於 $T_s/2$ 秒時間延遲
2. 在 $-\omega_m$ 和 ω_m 之間有一段 $X_\delta(j\omega)$ 的失真 (因main lobe 曲率所造成)
3. 其它部份 ω_s 倍數處有失真與衰減

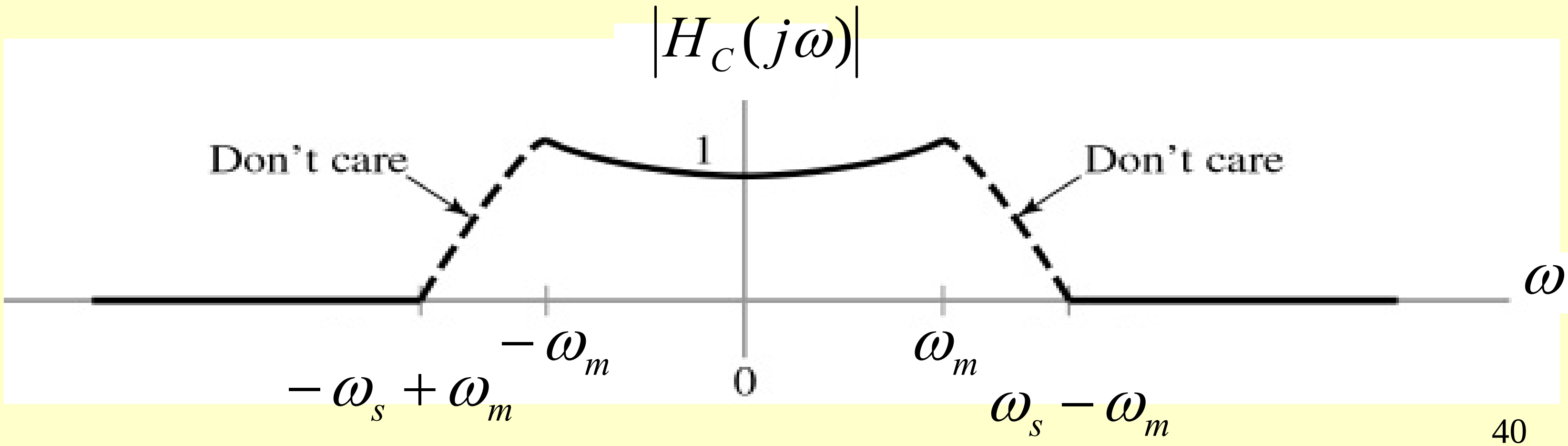
上述問題可經由通過一個補償濾波器來改善

Compensation Filter (or Anti-imaging Filter)

The Compensation Filter (anti-imaging filter)

The Compensation Filter (anti-imaging filter) used to eliminate some of the distortion introduced by the zero-order hold.

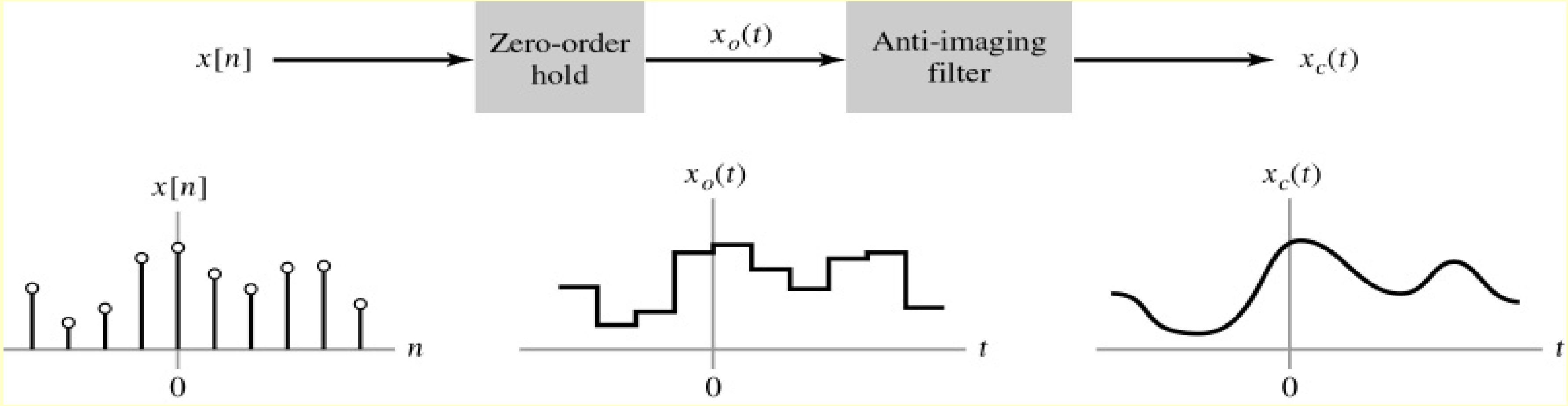
$$H_c(j\omega) = \begin{cases} \dfrac{\omega T_s}{2\sin(\omega T_s/2)}, & |\omega| < \omega_m \\ 0, & |\omega| > \omega_s - \omega_m \end{cases}$$



Block Diagram of a Practical Reconstruction System (重建系統)

- 零階保持器 重建連續訊號時，產生三項失真
- 抗映像濾波器 去除高頻映像頻譜並改善非線性失真

三項失真　　　　失真處被改善



Anti-Imaging Filter Design With and Without Over-Sampling

Anti-imaging filter design with and without over-sampling.
找出 抗映像濾波器限制

(a) Magnitude of *H*o(*jf*) for 44.1-kHz sampling rate. Dashed lines denote signal pass-band and images.

(b) Magnitude of *H*o(*jf*) for eight-times over-sampling (352.8-kHz sampling rate. Dashed lines denote signal pass-band and images.

(c) Normalized constraints on pass-band response of anti-imaging filter. Solid lines assume a 44.1-kHz sampling rate; dashed lines assume eight-times over-sampling. The normalized filter response must lie between each pair of lines.

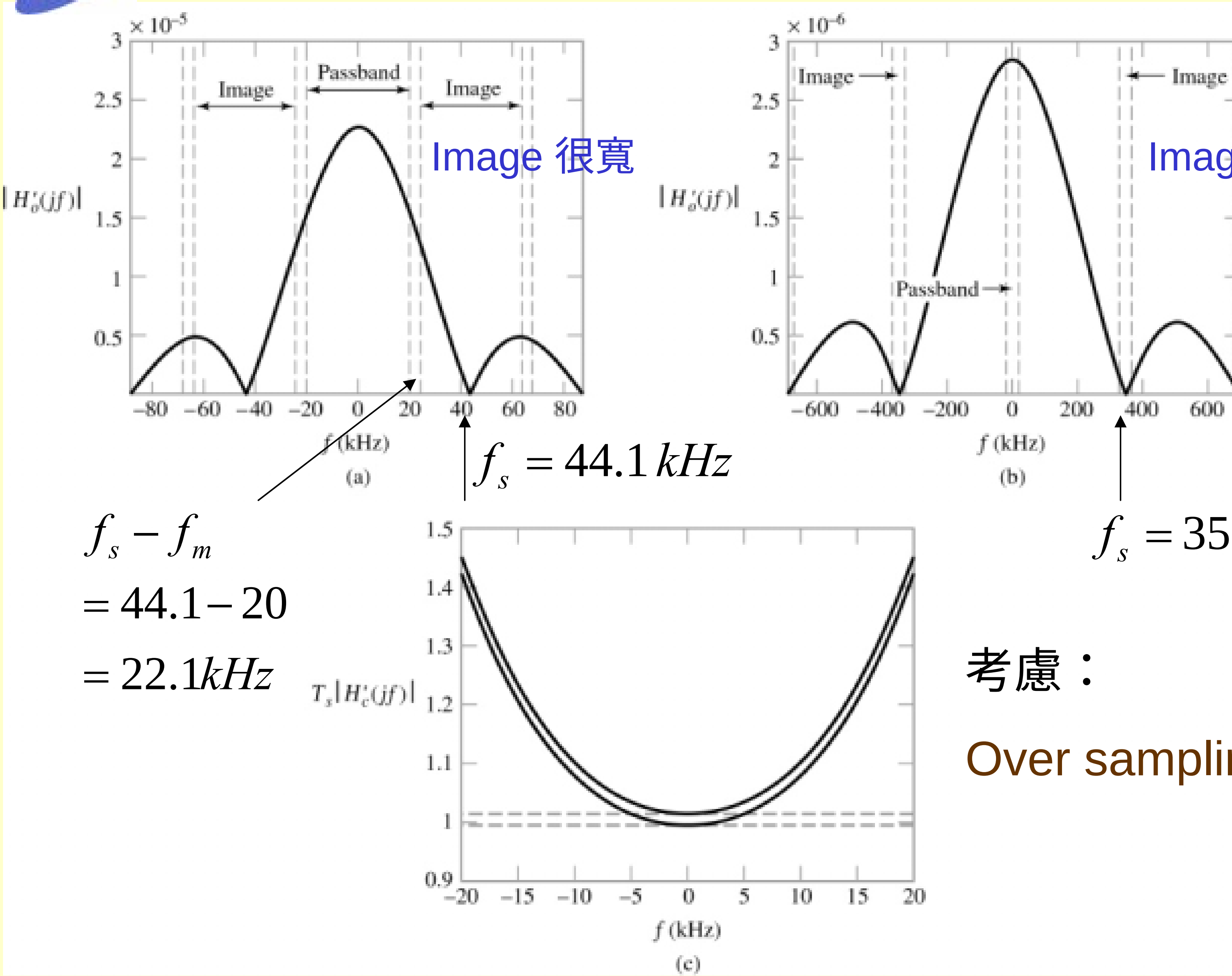


Image 很寬

Image 很窄

$f_s = 44.1\,kHz$

$f_s = 352.8\,kHz$

$$f_s - f_m = 44.1 - 20 = 22.1kHz$$

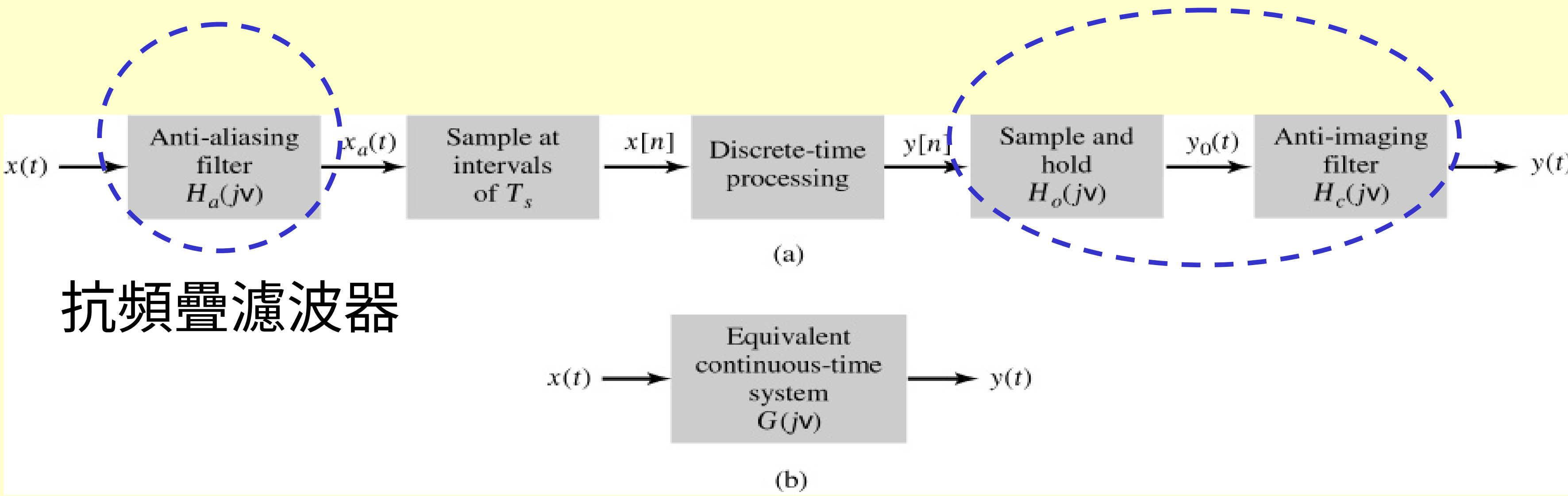
考慮：

Over sampling

Block Diagram for Discrete-Time Processing of Continuous-Time Signals

(a) A basic system.

(b) Equivalent continuous-time system.



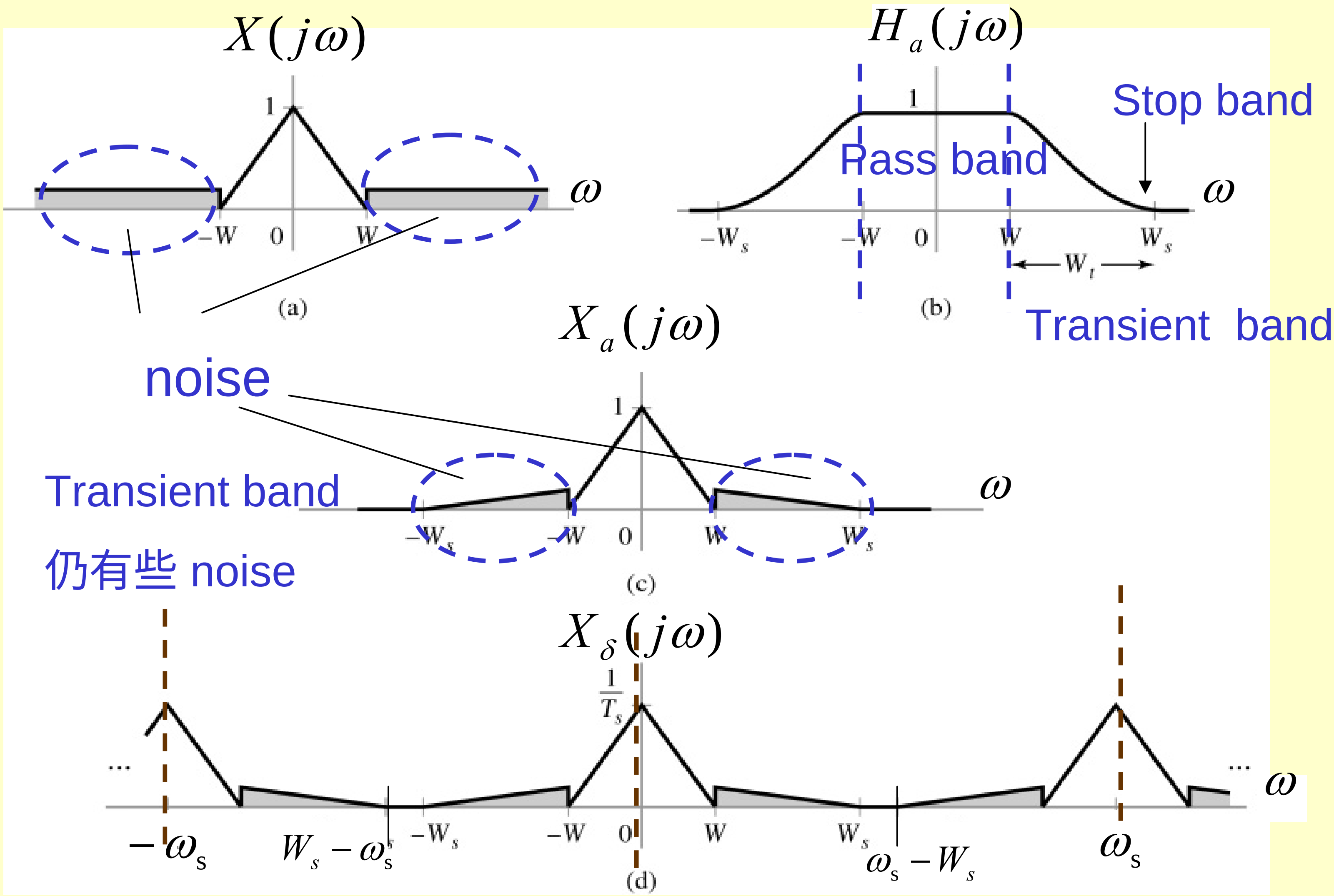
Effect of Over-Sampling on Anti-Aliasing Filter Specifications (抗頻疊濾波器)

Effect of over-sampling on Anti-aliasing filter specifications:
(抗頻疊濾波器)

(a) Spectrum of original signal.
(b) Anti-aliasing filter frequency response magnitude.
(c) Spectrum of signal at the anti-aliasing filter output.
(d) Spectrum of the anti-aliasing filter output after sampling. The graph depicts the case of $\omega_s > 2W_s$.

*抗頻疊濾波器*作用在於使非限頻訊號成為限頻訊號

Anti-aliasing Filter可限制頻寬



抗頻疊濾波器(anti-aliasing filter) 在訊號取樣前，限制訊號頻寬，才可使用取樣理論。

為避免雜訊和訊號本身產生頻疊，要求： $\omega_s - W_s > W_s$

避免雜訊和訊號 pass band 產生頻疊，更嚴格要求：

$$\omega_s - W_s > W$$

$$\because \quad W_s = W + W_t$$

$$\therefore \quad \omega_s - (W + W_t) > W$$

$$\omega_s - W_t > 2W \quad or \quad W_t < \omega_s - 2W$$

- 因而形成過渡頻帶 W_t 的限制條件 (限制嚴格則成本高)
- 提高 ω_s 與 降低 W 可降低過渡頻帶的限制，進而節省成本
- Decimation & Interpolation 則可改變 W 頻寬

Decimation (十分法)

若以 取樣間距 T_{s1} 取樣 x(t) 訊號 獲得 $x_1[n]$

若以 取樣間距 T_{s2} 取樣 x(t) 訊號 獲得 $x_2[n]$

其中 取樣間距 $T_{s1} = q\, T_{s2}$， q 是整數

效果如下：

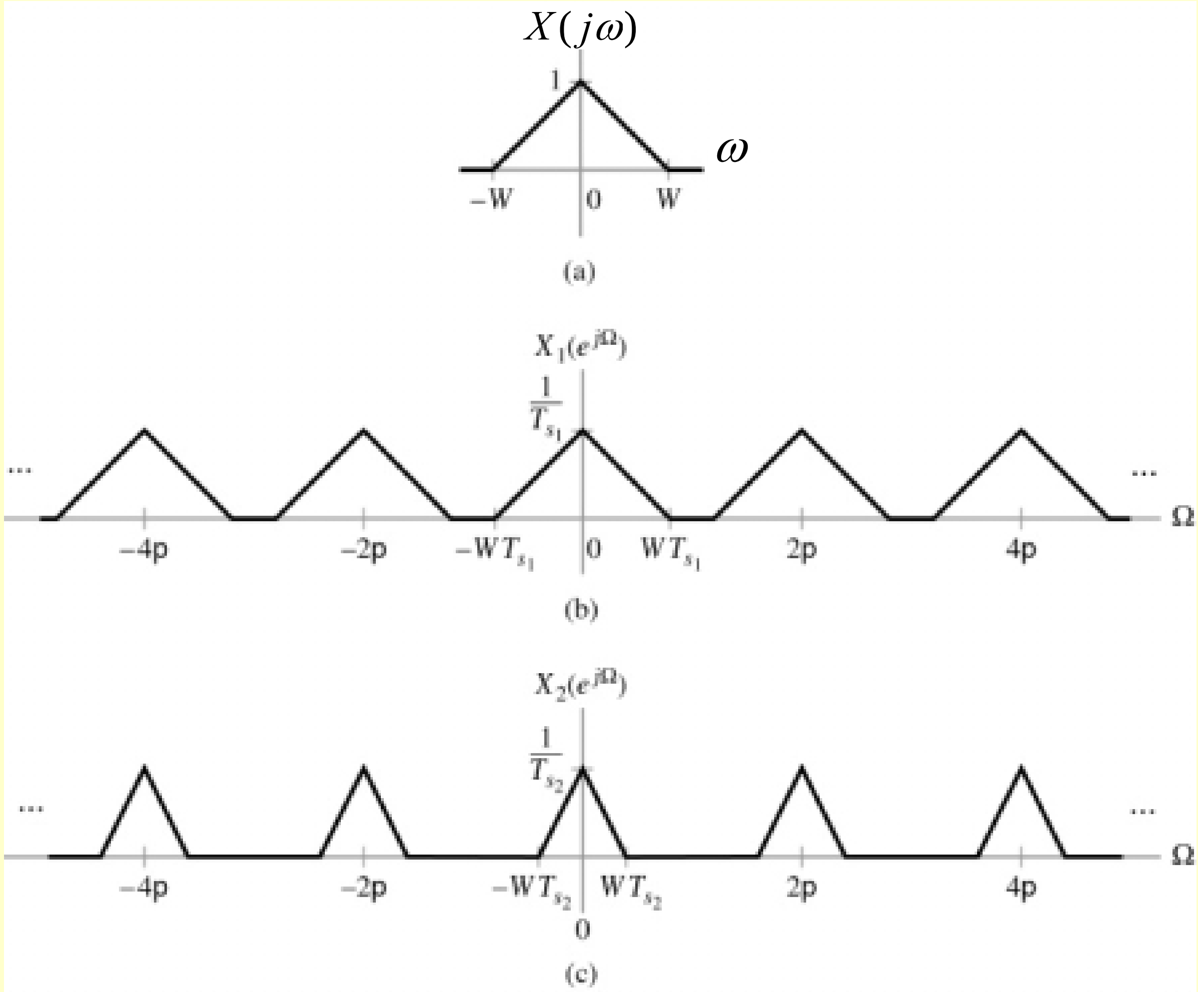
Decimation 將取樣間距 T_{s2} 增加為 T_{s1} (間距擴大)

Decimation 減少取樣比率 (WT_s 上升)

Decimation 取樣將頻譜 $X_2(e^{j\Omega})$ 改成 $X_1(e^{j\Omega})$

Decimation 取樣將訊號頻譜的頻寬 wT_{s2} 增加為 wT_{s1}

Decimation 取樣將原始訊號頻率增加 q 倍



Effect of changing the sampling rate.

(a) Underlying continuous-time signal FT.

(b) DTFT of sampled data at sampling interval *Ts*1.

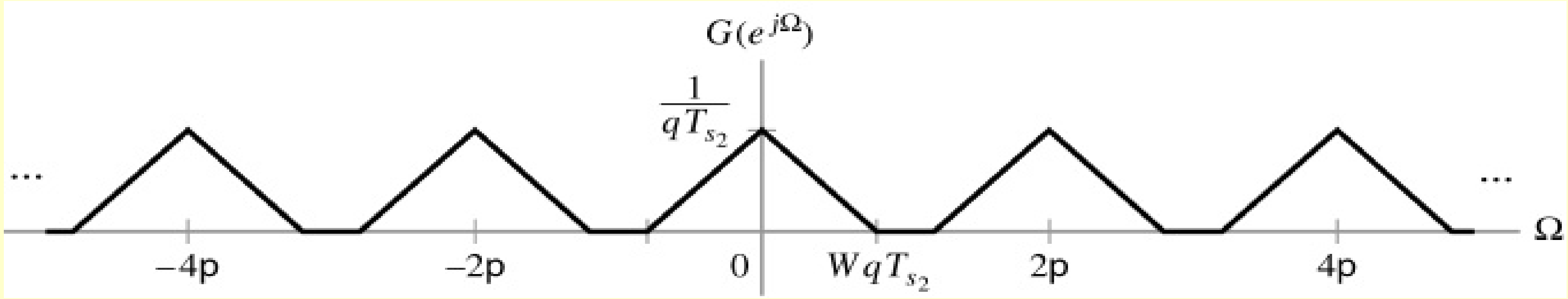
(c) DTFT of sampled data at sampling interval *Ts*2.

The Spectrum Results from Sub-Sampling the DTFT $X_2(e^{j\Omega})$ by a Factor of q

$g[n] = x(qT_s\, n)$ 訊號可由取樣 $x_2[n] = x(nT_s)$ 獲得

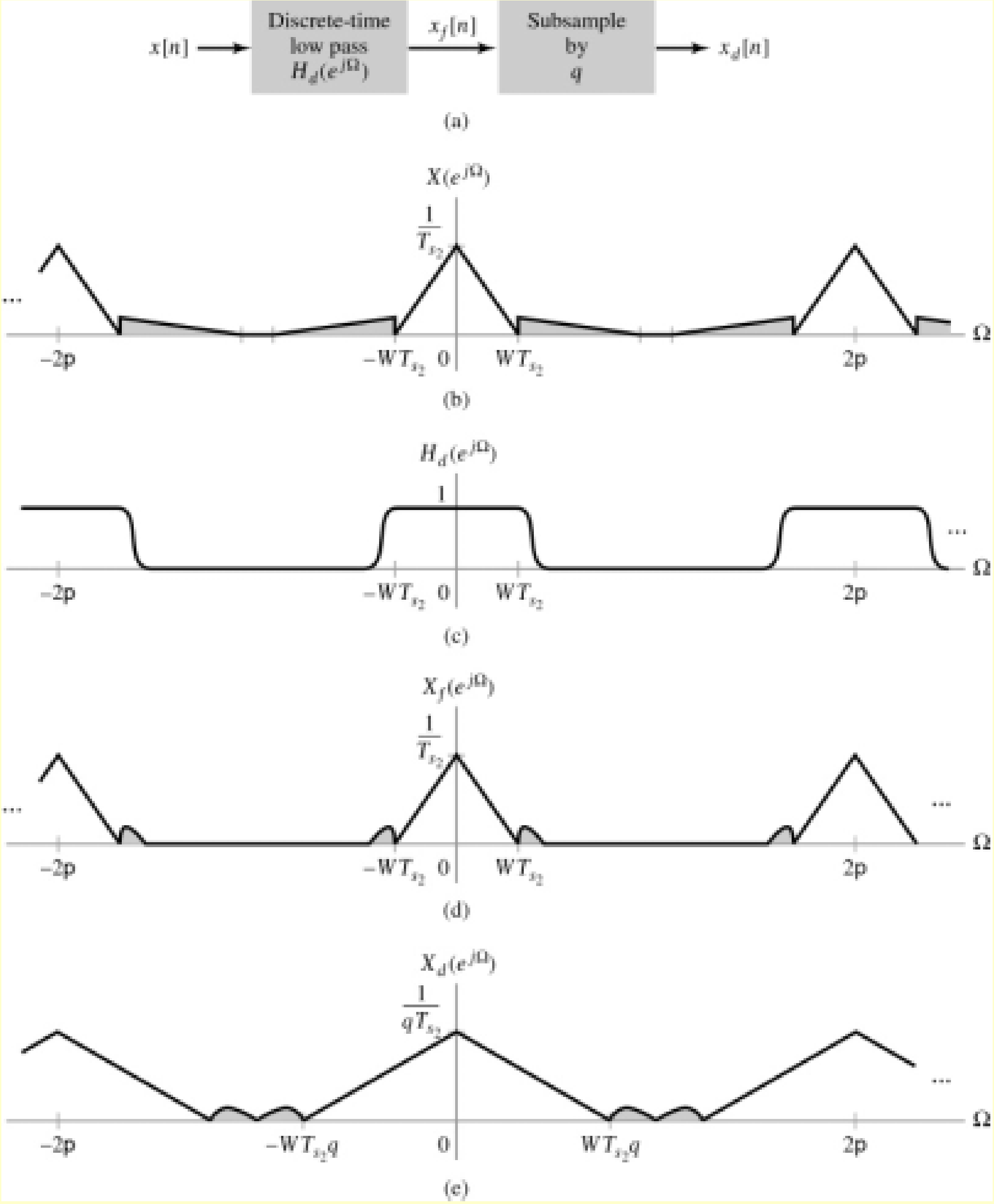
$$x_2[n] = x(nT_s), \quad g[n] = x_2[q\,n]$$

$$G(e^{j\Omega}) = \frac{1}{q}\sum_{m=0}^{q-1} X_2\left(e^{j((\Omega - m2\pi)/q)}\right)$$

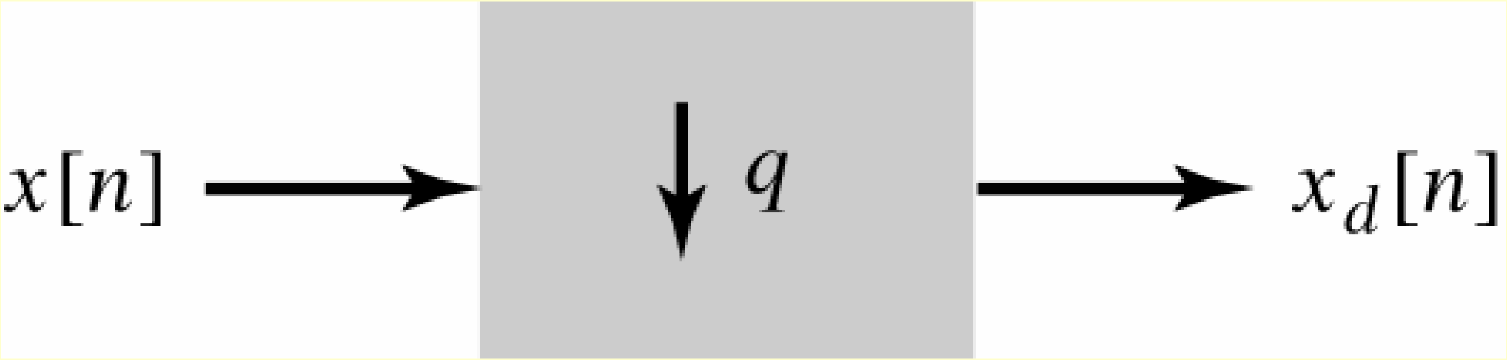


Frequency-domain interpretation of decimation

(a) Block diagram of decimation system.
(b) Spectrum of over-sampled input signal. Noise is depicted as the shaded portions of the spectrum.
(c) Filter frequency response.
(d) Spectrum of filter output.
(e) Spectrum after sub-sampling.


x[n]
Discrete-time low pass $H_d(e^{j\Omega})$
$x_f[n]$
Subsample by q
$x_d[n]$
(a)
$X(e^{j\Omega})$
$\frac{1}{T_{s_2}}$
−2p
$-WT_{s_2}$
0
WT_{s_2}
2p
Ω
(b)
$H_d(e^{j\Omega})$
1
(c)
$X_f(e^{j\Omega})$
(d)
$X_d(e^{j\Omega})$
$\frac{1}{qT_{s_2}}$
$-WT_{s_2}q$
$WT_{s_2}q$
(e)

Symbol for Decimation by a Factor of *q*



內插法取樣 (Interpolation)

以 取樣間距 T_{s1} 取樣 x(t) 訊號 獲得 $x_1[n]$

以 取樣間距 T_{s2} 取樣 x(t) 訊號 獲得 $x_2[n]$

取樣間距 $T_{s1} = q\,T_{s2}$， q 是整數

內插 取樣將取樣間距 T_{s1} 減少為 T_{s2}

內插 取樣增加取樣比率

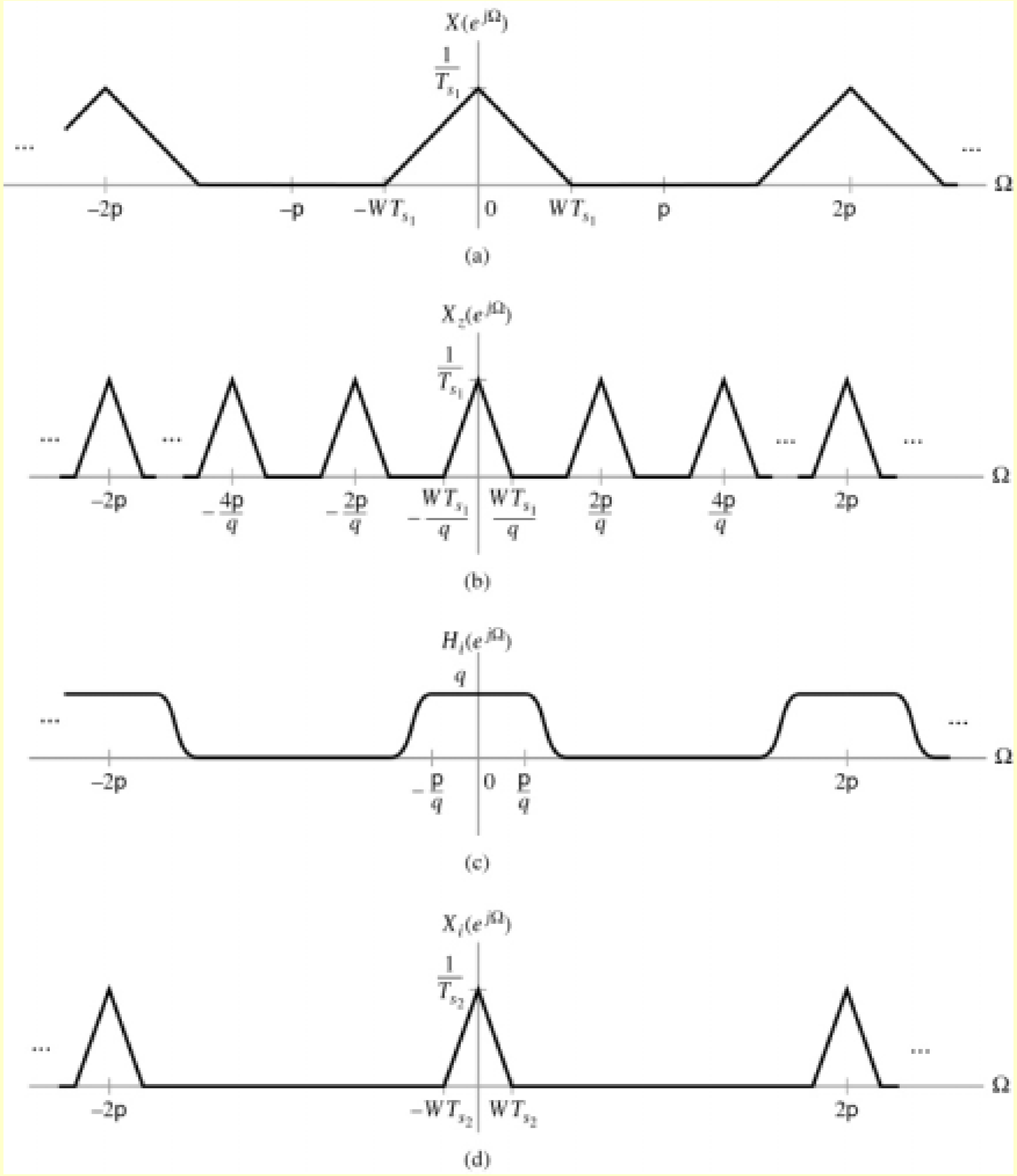
內插 取樣將頻譜 $X_1(e^{j\Omega})$ 改成 $X_2(e^{j\Omega})$

內插 取樣將頻譜頻寬 wT_{s1} 減少為 $wT_{s2} = wT_{s1}/q$

內插 取樣將原始訊號頻率減少 q 倍

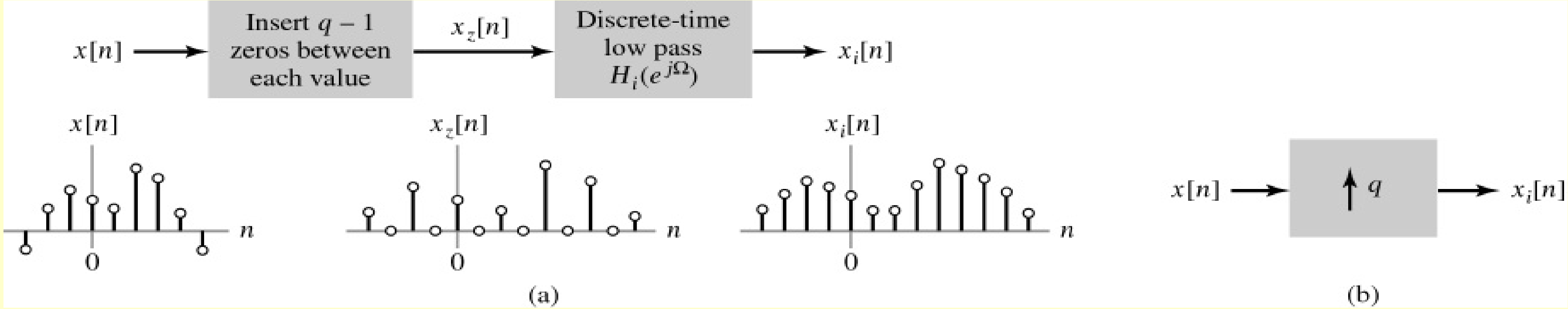
Frequency-Domain Interpretation of Interpolation

(a) Spectrum of original sequence.
(b) Spectrum after inserting $q - 1$ zeros in between every value of the original sequence.
(c) Frequency response of a filter for removing undesired replicates located at $\pm 2\pi/q$, $\pm 4\pi/q$, …, $\pm (q-1)2\pi/q$.
(d) Spectrum of interpolated sequence.



X(e^jΩ)
1/T_s1
–2p
–p
–WT_s1
0
WT_s1
p
2p
Ω
(a)
X_z(e^jΩ)
1/T_s1
–2p
– 4p/q
– 2p/q
– WT_s1/q
WT_s1/q
2p/q
4p/q
2p
Ω
(b)
H_i(e^jΩ)
q
–2p
– p/q
0
p/q
2p
Ω
(c)
X_i(e^jΩ)
1/T_s2
–2p
–WT_s2
WT_s2
2p
Ω
(d)

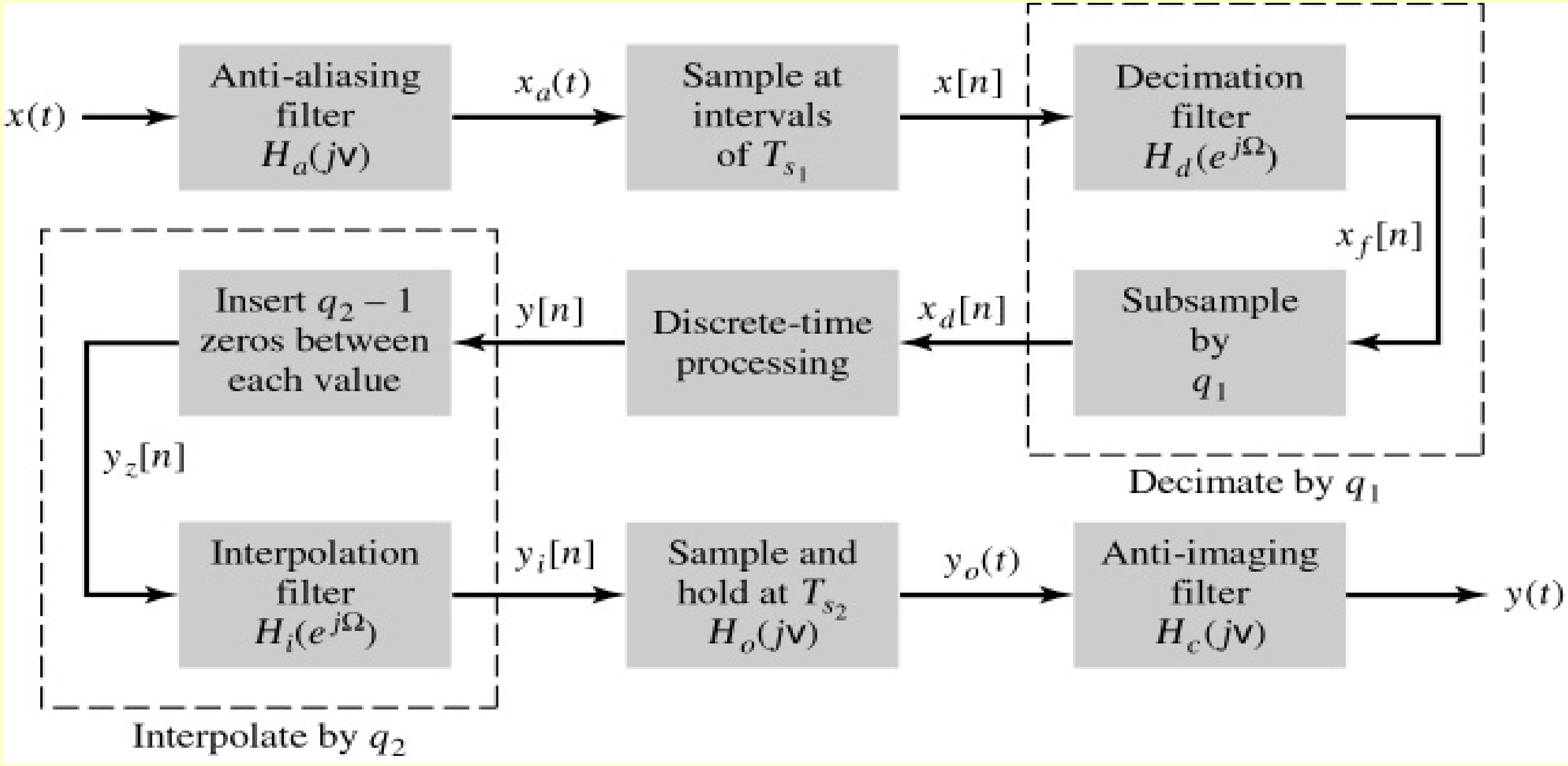
Interpolation System Diagram



(a) Block diagram of an interpolation system.

(b) Symbol denoting interpolation by a factor of *q*.

Block Diagram of a System for Discrete-Time Processing of Continuous-Time Signals Including Decimation and Interpolation.



FS for Finite-Duration Non-periodic Signals

Relating the DTFS to the DTFT:

x[n] 為一個長度為 M 的有限時間訊號：

$$x[n] = 0, \quad n < 0 \quad or \quad n \geq M$$

x[n] 的 DTFT：

$$X(e^{j\Omega}) = \sum_{n=0}^{M-1} x[n]\, e^{-j\Omega n}$$

擴充 x[n] 為一個週期為 N ≥ M 的週期訊號：$\tilde{x}[n]$

$$\tilde{x}[n] = \sum_{m=-\infty}^{\infty} x[n+mN]$$

$\tilde{x}[n]$ 的 DTFS：

$$\tilde{X}[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk\Omega_0 n} = \frac{1}{N}\sum_{n=0}^{M-1} x[n]\, e^{-jk\Omega_0 n}$$

比較 $X(e^{j\Omega})$ 和 $\tilde{X}[k]$ ：

$$X(e^{j\Omega}) = \sum_{n=0}^{M-1} x[n]\, e^{-j\Omega n}$$

DTFT 頻譜

$$\tilde{X}[k] = \frac{1}{N} \sum_{n=0}^{\mathrm{M}-1} x[n]\, e^{-jk\Omega_0 n}$$

DTFS 頻譜

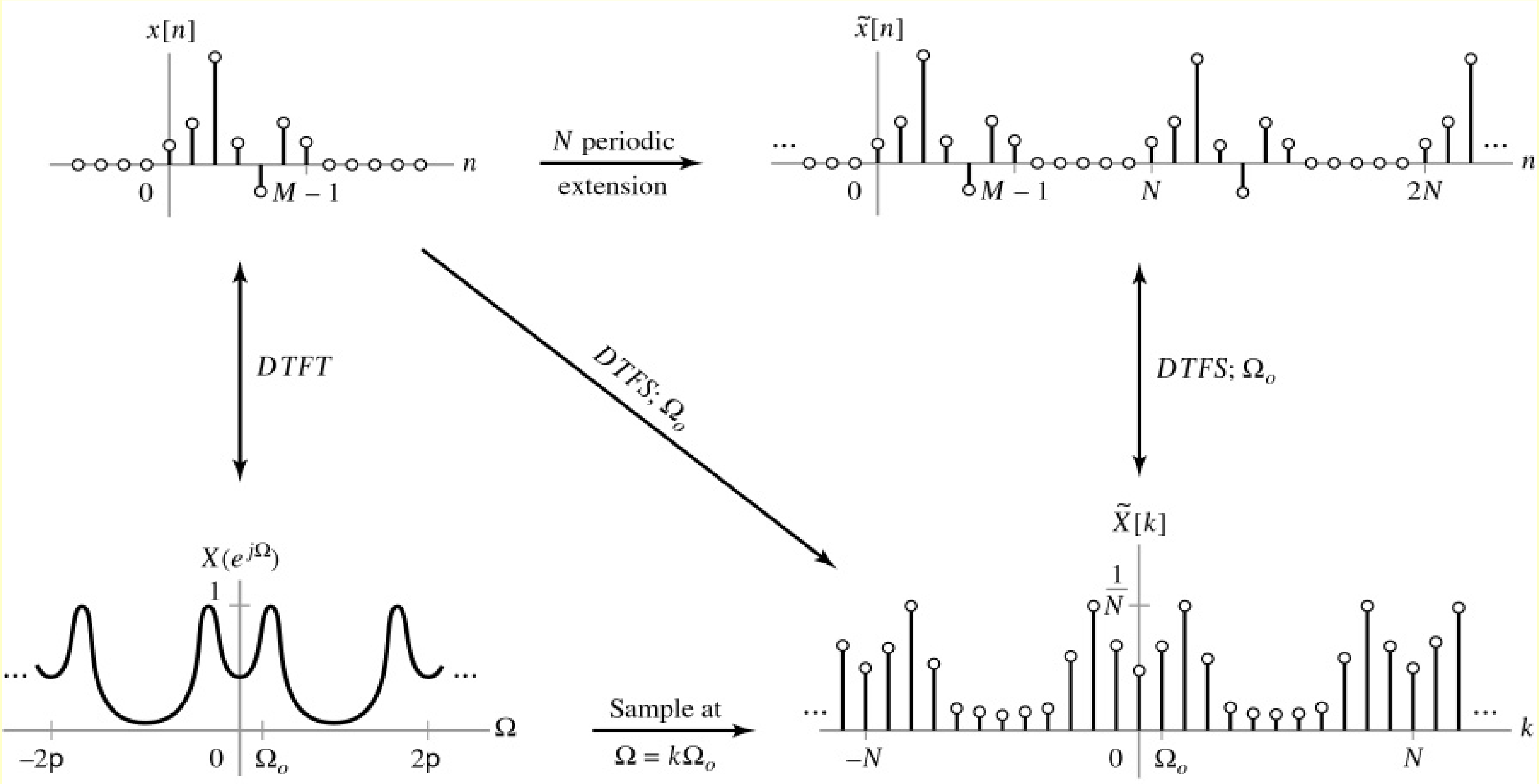
可得:

$$\tilde{X}[k] = \frac{1}{N} X(e^{j\Omega}) \Big|_{\Omega = k\Omega_0}$$

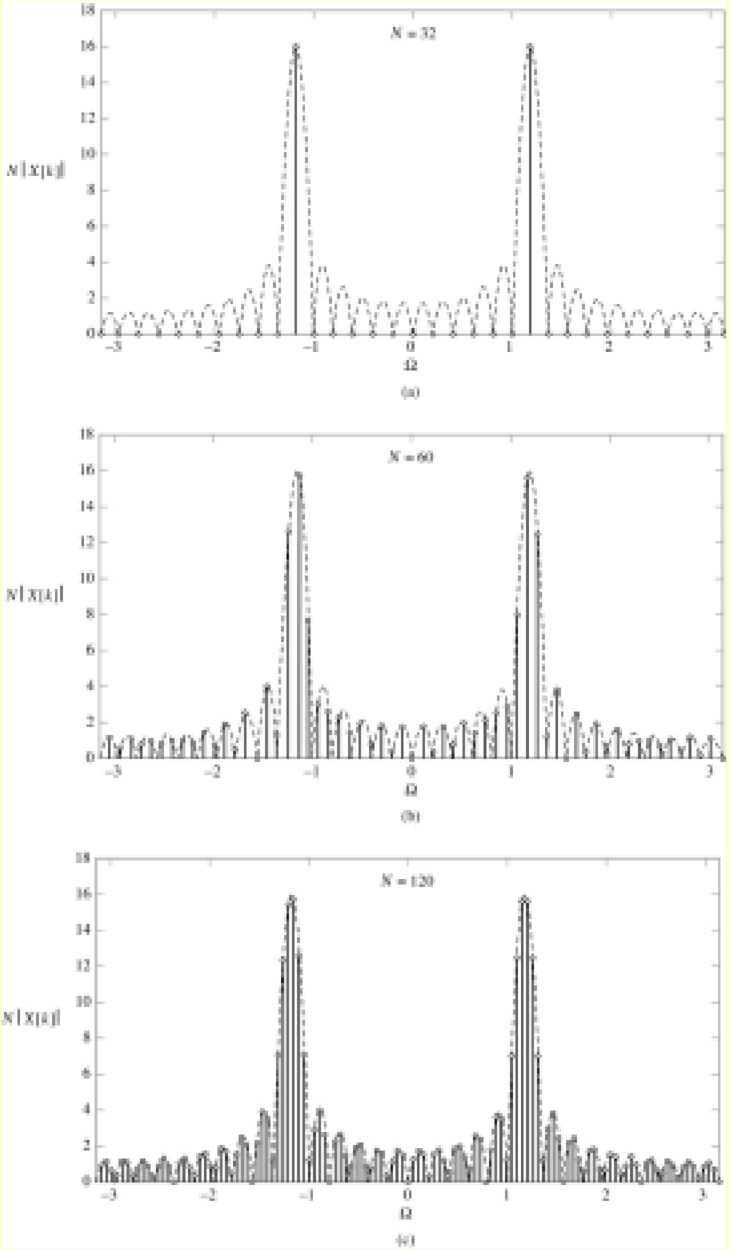
The DTFS of a Finite-Duration Non-Periodic Signal

Duration = M

Expanded to Duration N



The DTFT and length-*N* DTFS of a 32-point cosine.



The dashed line denotes $|X(e^{j\Omega})|$, while the stems represent $N|X[k]|$.

(a) $N = 32$,
(b) $N = 60$,
(c) $N = 120$.

Fourier Series Representations of Finite-Duration Non-periodic Signals

Relating the FS to the FT:

x(t) 為一個長度為 T_0 的有限時間訊號：

$$x(t) = 0, \quad t < 0 \quad or \quad t \geq T_0$$

x(t) 的 FT：

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

Relating the FS to the FT (cont.)

擴充 x(t) 為一個週期為 T ≥ T_0 的週期訊號：

$$\tilde{x}(t) = \sum_{m=-\infty}^{\infty} x(t + mT)$$

$\tilde{x}(t)$ 的 FS：

$$\tilde{X}[k] = \frac{1}{T}\int_0^T \tilde{x}(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_0^{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$

比較 $X(j\omega)$ 和 $\tilde{X}[k]$ ：

$$X(j\omega) = \int_0^{T_0} x(t)\, e^{-j\omega t}\, dt$$

FT 頻譜

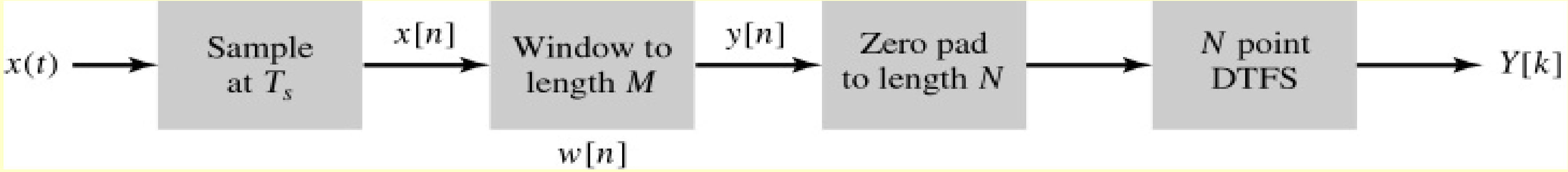
$$\tilde{X}[k] = \frac{1}{T} \int_0^{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$

FS 頻譜

可得:

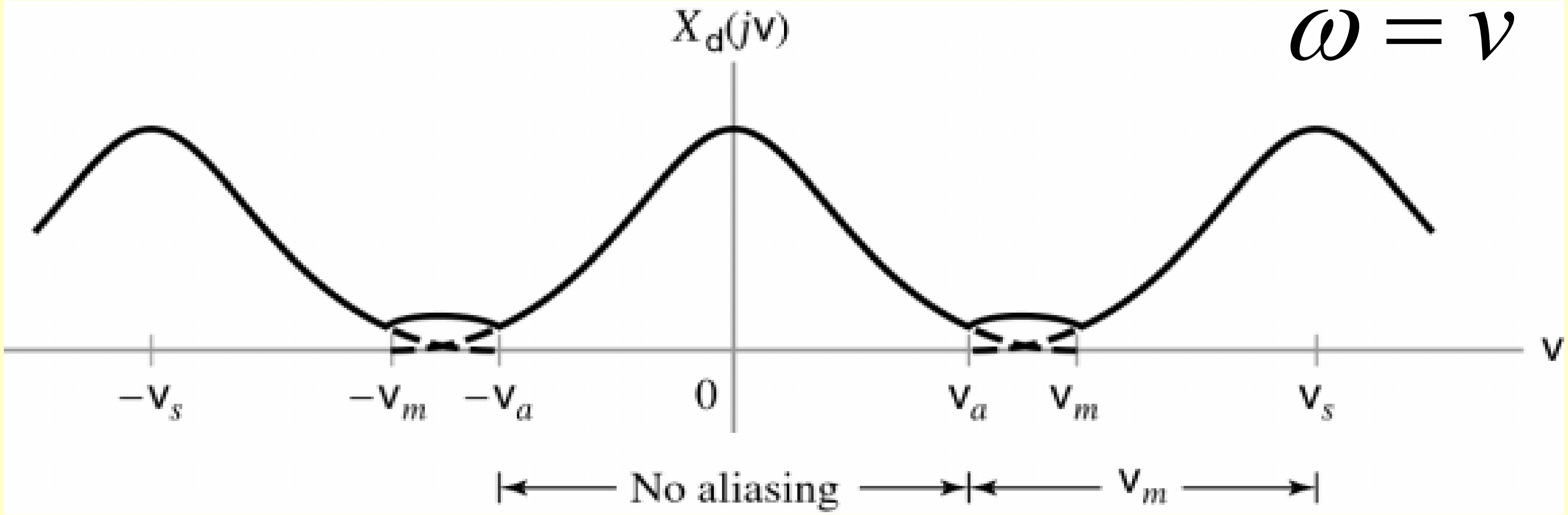
$$\tilde{X}[k] = \frac{1}{T_s} X(j\omega) \Big|_{\Omega = k\Omega_0}$$

Block Diagram Depicting the Sequence of Operations Involved in Approximating the FT with the DTFS



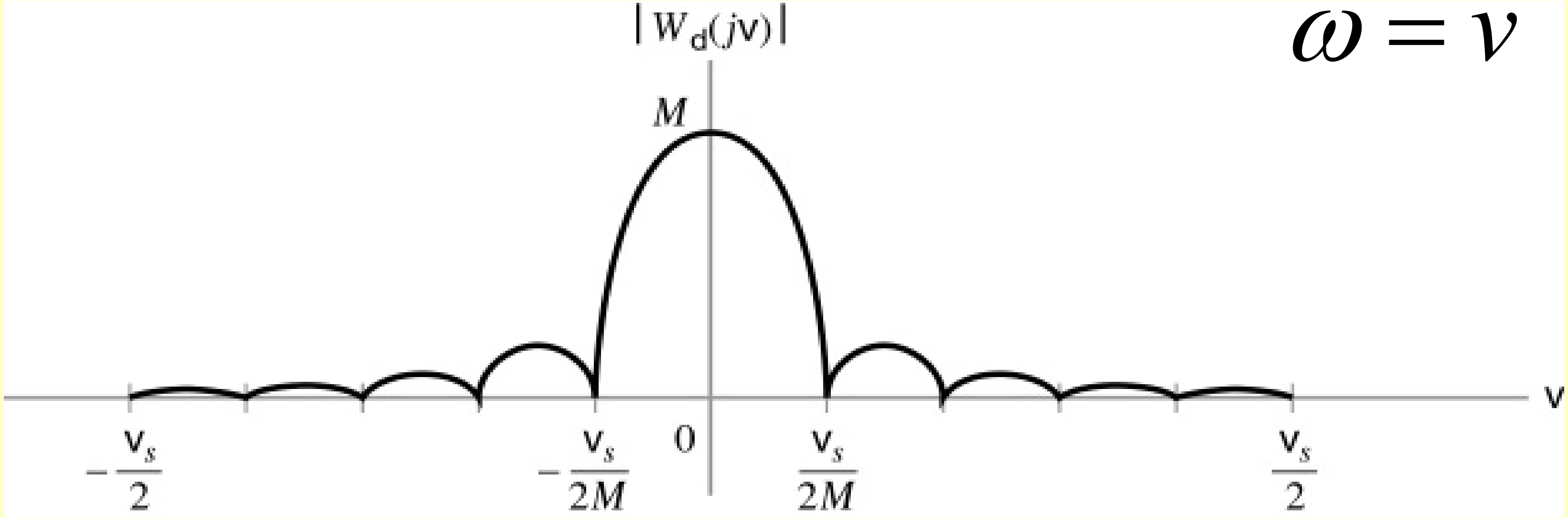
Audio & DSP Lab.

Effect of Aliasing



Audio & DSP Lab.

Magnitude response of *M*-point window

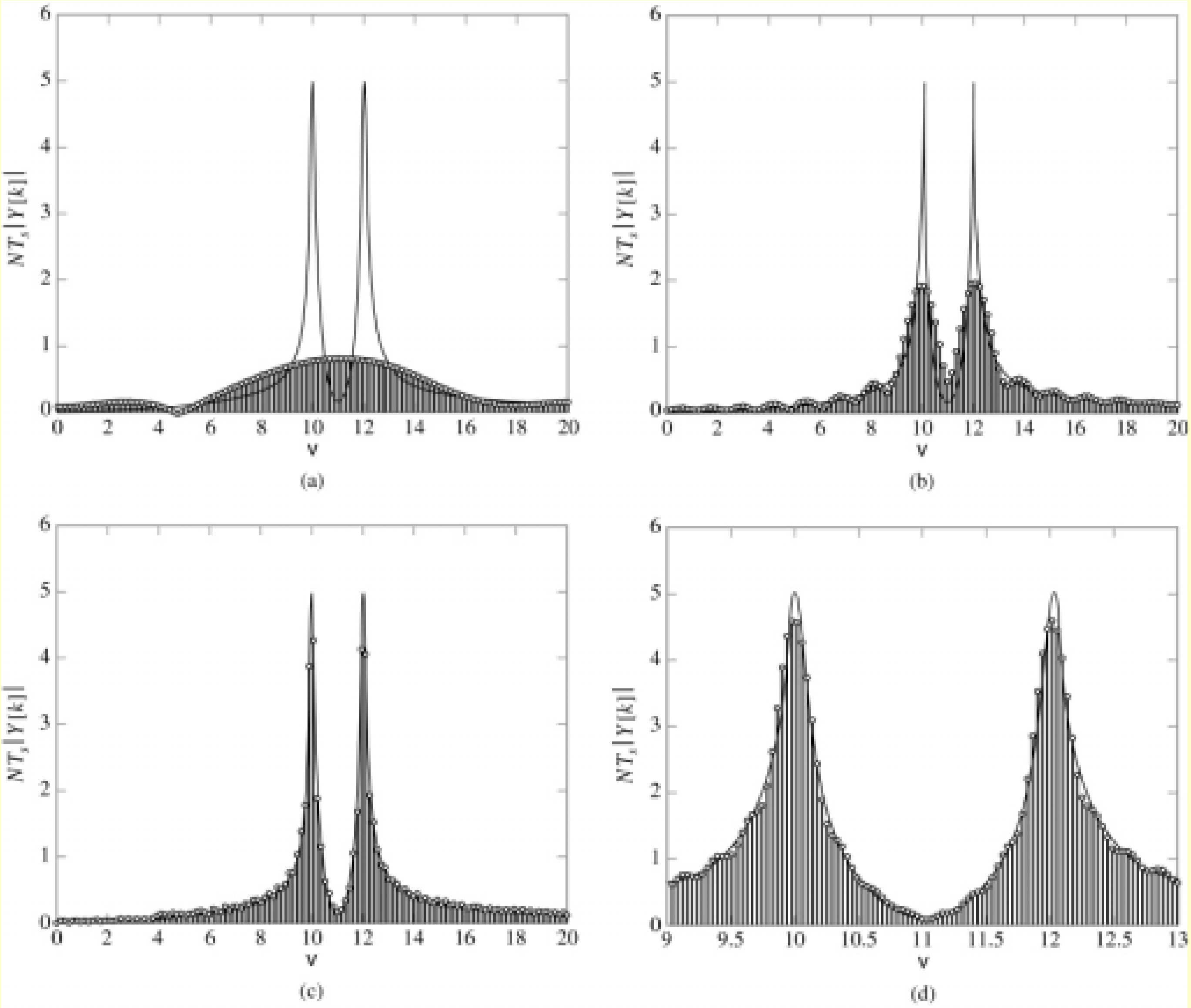


$\omega = \nu$

DTFT Approximation to the FT

The DTFS approximation to the FT of $x(t)$ = e-1/10 $u(t)(\cos(10t) + \cos(12t)$. The solid line is the FT $|X(j\omega)|$, and the stems denote the DTFS approximation $NT_s|Y[k]|$. Both $|X(j\omega)$ and $NT_s|Y[k]|$ have even symmetry, so only $0 < \omega < 20$ is displayed.

(a) $M = 100$, $N = 4000$.
(b) $M = 500$, $N = 4000$.
(c) $M = 2500$, $N = 4000$.
(d) $M = 2500$, $N = 16{,}0000$ for $9 < \omega < 13$.


6
5
4
3
2
1
0
0 2 4 6 8 10 12 14 16 18 20
v
$NT_s|Y[k]|$
9 9.5 10 10.5 11 11.5 12 12.5 13

(a)

(b)

(c)

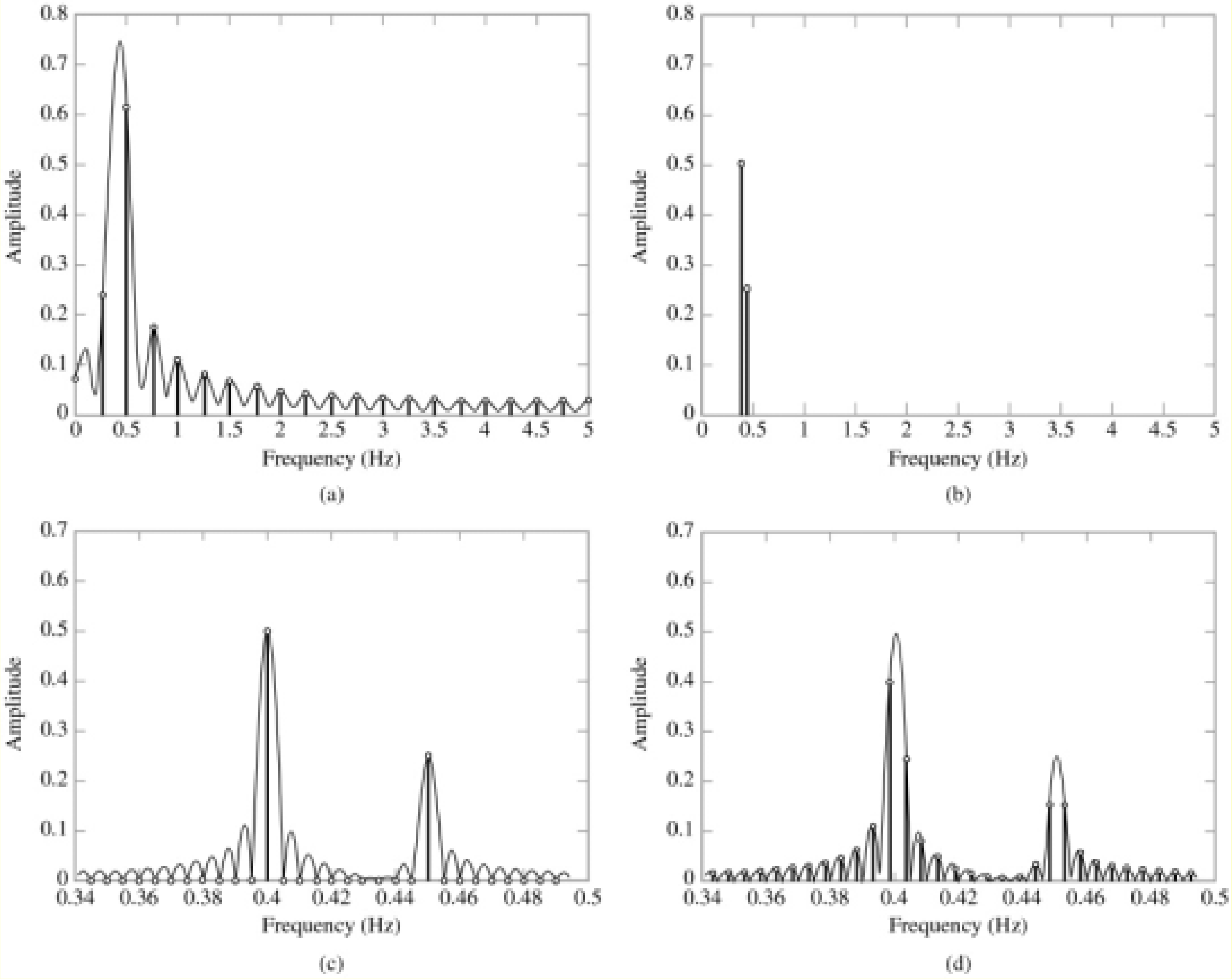
(d)

DTFS Approximation to the FT

The DTFS approximation to the FT of $x(t) = \cos(2\pi(0.4)t) + \cos(2\pi(0.45)t)$. The stems denote $|Y[k]|$, while the solid lines denote $(1/M|Y\delta(j\omega)|$. The frequency axis is displayed in units of Hz for convenience, and only positive frequencies are illustrated.

(a) $M = 40$.
(b) $M = 2000$. Only the stems with nonzero amplitude are depicted.
(c) Behavior in the vicinity of the sinusoidal frequencies for $M = 2000$.
(d) Behavior in the vicinity of the sinusoidal frequencies for $M = 2010$.

Audio & DSP Lab.



DTFS Decompositions

Block diagrams depicting the decomposition of an inverse DTFS as a combination of lower order inverse DTFS's.

(a) Eight-point inverse DTFS represented in terms of two four-point inverse DTFS's.
(b) four-point inverse DTFS represented in terms of two-point inverse DTFS's.
(c) Two-point inverse DTFS.

Efficient Algorithms for Evaluating the DTFS- Fast Fourier Transform (FFT)

- FFT 是有計算 DTFS (離散訊號、頻譜) 有效率的演算法
- 將 DTFS 分割成系列較低階的 DTFS
- 應用複數弦波 $e^{jk2\pi n}$ 的對稱性與週期性

DTFS 公式：

$$X[k]=\frac{1}{N}\sum_{n=0}^{N-1}x[n]\,e^{-jk\Omega_0 n}$$

$$x[n]=\sum_{k=0}^{N-1}X[k]\,e^{jk\Omega_0 n}$$

若忽略正規化因子 N 及複數指數符號，上述公式可用同一種演算法來計算

分解 DTFS 公式 (Even Number N)

$$X_{even}[k] = X[2k], \quad 0 \le k \le \frac{N}{2} - 1$$

$$X_{odd}[k] = X[2k+1], \quad 0 \le k \le \frac{N}{2} - 1$$

DTFS 對應公式：

$$x_{even}[n] \overset{DTFS;\Omega_0}{\longleftrightarrow} X_{even}[k]$$

$$x_{odd}[n] \overset{DTFS;\Omega_0}{\longleftrightarrow} X_{odd}[k]$$

$$x[n] = \sum_{k=0}^{N-1} X[k]\, e^{jk\Omega_0 n}$$

$$= \sum_{k=0}^{\frac{N}{2}-1} X[2k]\, e^{j2k\Omega_0 n} + \sum_{k=0}^{\frac{N}{2}-1} X[2k+1]\, e^{j(2k+1)\Omega_0 n}$$

$$= \sum_{k=0}^{\frac{N}{2}-1} X[2k]\, e^{j2k\Omega_0 n} + e^{j\Omega_0 n} \sum_{k=0}^{\frac{N}{2}-1} X[2k+1]\, e^{j2k\Omega_0 n}$$

$$= \sum_{k=0}^{\frac{N}{2}-1} X[2k]\, e^{j2k\Omega_0 n} + e^{j\Omega_0 n} \sum_{k=0}^{\frac{N}{2}-1} X[2k+1]\, e^{j2k\Omega_0 n}$$

$$let\ N' = N/2, \quad \Omega_0' = 2\Omega_0 = 2\pi / N',$$

$$X_{even}[k] = X[2k], \quad X_{odd}[k] = X[2k+1]$$

分解成偶與奇項次

$$x[n] = \sum_{k=0}^{N'-1} X_{even}[k]\, e^{jk\Omega_0' n} + e^{j\Omega_0 n} \sum_{k=0}^{N'-1} X_{odd}[k]\, e^{jk\Omega_0' n}$$

$$= x_{even}[n] + e^{j\Omega_0 n} \cdot x_{odd}[n]$$

x[n] 是週期性訊號，分析週期性關係：

$$x_{even}[n] = x_{even}[n+N'], \quad x_{odd}[n] = x_{odd}[n+N']$$

$$e^{j\Omega_0[n+N']} = e^{j\frac{2\pi}{N}[n+N']} = e^{j\Omega_0 n} \cdot e^{j\frac{2\pi}{N}\frac{N}{2}}$$

$$= e^{j\Omega_0 n} \cdot e^{j\pi} = -e^{j\Omega_0 n}$$

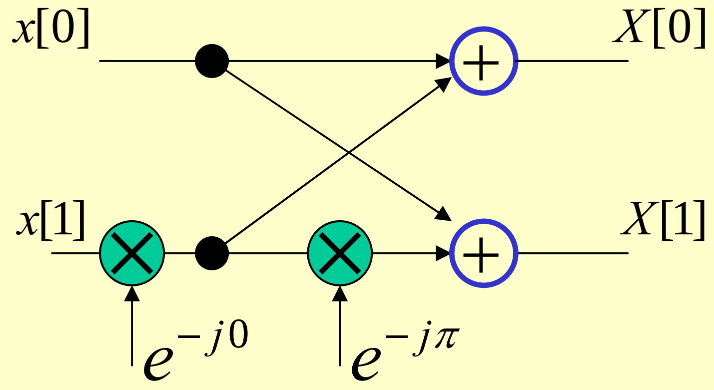
x[n] 週期性訊號前半段：x[0] ~ x[N'-1]

$$x[n] = x_{even}[n] + e^{j\Omega_0 n} \cdot x_{odd}[n], \quad 0 \le n \le N'-1$$

x[n] 週期性訊號後半段：x[N'] ~ x[N-1]

$$x[n+\mathrm{N'}] = x_{even}[n] - e^{j\Omega_0 n} \cdot x_{odd}[n], \quad 0 \le n \le N'-1$$

N=2 Case



N = 2 case: 計算 DTFS X[k] = ?

N^2 = 4 個乘法, N(N-1) = 2 個加法

$$X[k]=\sum_{n=0}^{N-1}x[n]e^{-jk\frac{2\pi}{N}n}$$

$$N=2 \quad case$$

$$X[k]=\sum_{n=0}^{1}x[n]e^{-jk\pi n}$$

$$\Rightarrow X[0]=\sum_{n=0}^{1}x[n]=x[0]\cdot e^{-j0}+x[1]\cdot e^{-j0}=x[0]+x[1]$$

$$\Rightarrow X[1]=\sum_{n=0}^{1}x[n]e^{-jk\pi n}=x[0]\cdot e^{-j0}+x[1]\cdot e^{-j\pi}=x[0]-x[1]$$

For FFT: 2 個加法

2 個乘法

N=4 Case

N = 4 case: 計算 X[k] = ?

N^2 =16 個乘法, N(N-1)=12 個加法

$$X[k]=\sum_{n=0}^{3}x[n]e^{-jk\frac{\pi}{2}n}$$

$$X[0]=\sum_{n=0}^{3}x[n]\cdot e^{-j0n}=x[0]+x[1]+x[2]+x[3]$$

$$X[2]=\sum_{n=0}^{3}x[n]\cdot e^{-j\pi n}=x[0]-x[1]+x[2]-x[3]$$

$$X[1]=\sum_{n=0}^{3}x[n]e^{-j\frac{\pi}{2}n}=x[0]-e^{j\frac{\pi}{2}}\cdot x[1]-x[2]+e^{j\frac{\pi}{2}}\cdot x[3]$$

$$X[3]=\sum_{n=0}^{3}x[n]e^{-j\frac{3\pi}{2}n}=x[0]+e^{j\frac{\pi}{2}}\cdot x[1]-x[2]-e^{j\frac{\pi}{2}}\cdot x[3]$$

FFT for N = 4 case:

計算 X[k] = ?

$4\log_2 4 = 8$ 個乘法

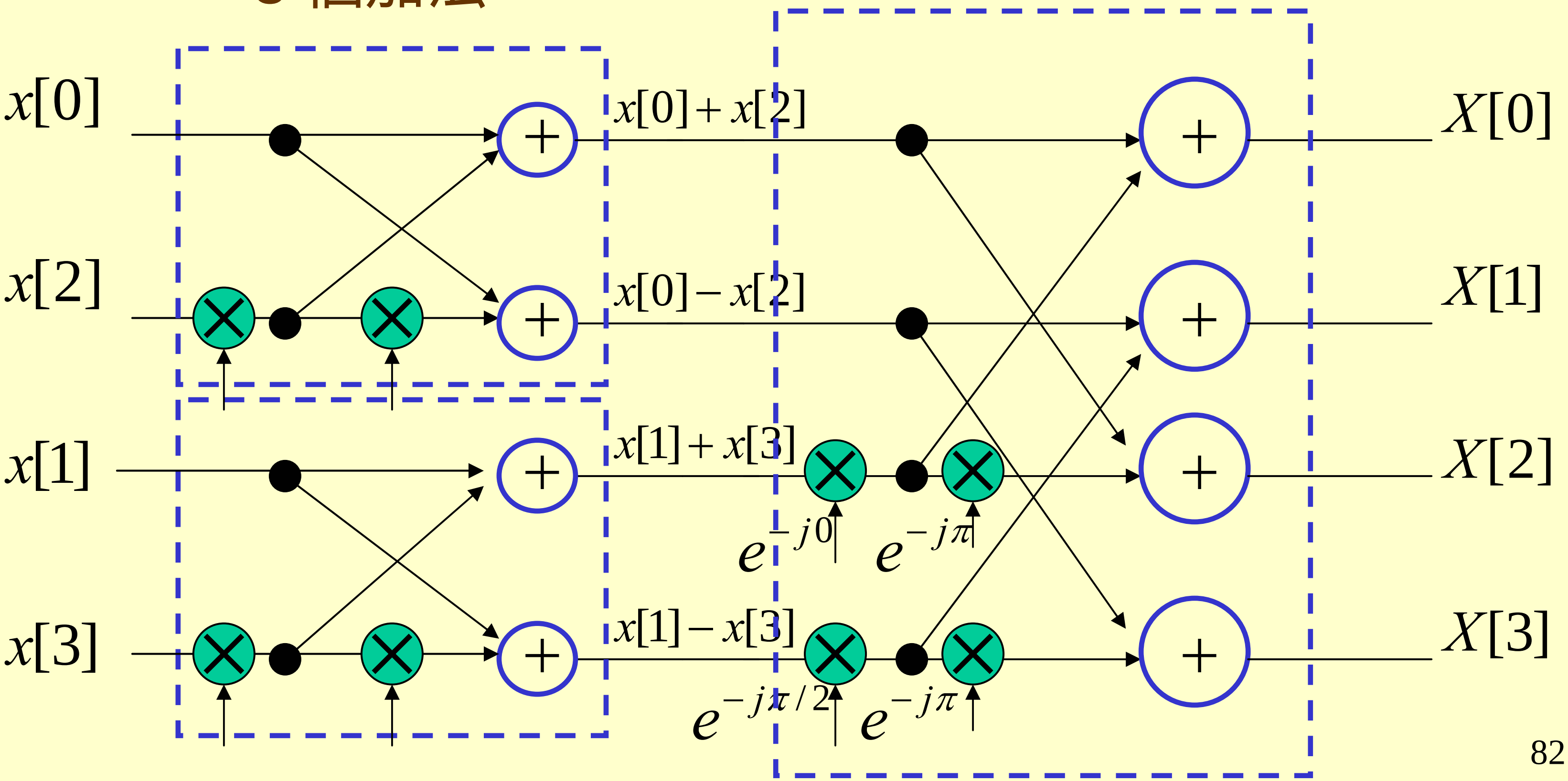
8 個加法

$$X[0] = x[0] + x[1] + x[2] + x[3]$$

$$X[2] = x[0] - x[1] + x[2] - x[3]$$

$$X[1] = x[0] - e^{j\frac{\pi}{2}} \cdot x[1] - x[2] + e^{j\frac{\pi}{2}} \cdot x[3]$$

$$X[3] = x[0] + e^{j\frac{\pi}{2}} \cdot x[1] - x[2] - e^{j\frac{\pi}{2}} \cdot x[3]$$



N = 4 case: 計算 DTFS X[k] = ? 計算 運算量

$$X[k]=\sum_{n=0}^{N-1}x[n]e^{-jk\frac{2\pi}{N}n},\ for\ k=0,\cdots N-1$$

計算 每一個 X[k] 需 N=4 個乘法 N-1=3 個加法

若計算 N 個 X[k] 需 N^2=16 個乘法 N(N-1)=12 個加法

使用 FFT, N = 4 case: 計算 X[k] = ? 計算 運算量

計算 4 個 X[k] 需 4 $\log_2$ 4 = 8 個乘法 8 個加法

Total : N $\log_2$ N

位元倒置 Bit Reversal

將輸入端的係數分割成索引值 Even Index 和 Odd Index

以二進位位元表示索引值

Example:

Old Sequence: x[0], x[1], x[2], x[3], x[4], x[5], x[6], x[7]

At k = 0, 1, 2, 3, 4, 5 , 6, 7

→ k = 000, 001, 010, 011, 100, 101, 110, 111

Bit reversal

→ k’ = 000, 100, 010, 110, 001, 101, 011, 111

At k’ = 0, 4, 2, 6, 1, 5, 3, 7

New Sequence: x[0], x[4], x[2], x[6], x[1], x[5], x[3], x[7]

位元倒置 Bit Reversal (cont.)

分割成 Even Index 和 Odd Index

Example 驗證:

Original: x[0], x[1], x[2], x[3], x[4], x[5], x[6], x[7]

<u>Even: x[0], x[2], x[4], x[6]</u>

Even: x[0], x[4]

Odd: x[2], x[6]

<u>Odd: x[1], x[3], x[5], x[7]</u>

Even: x[1], x[5]

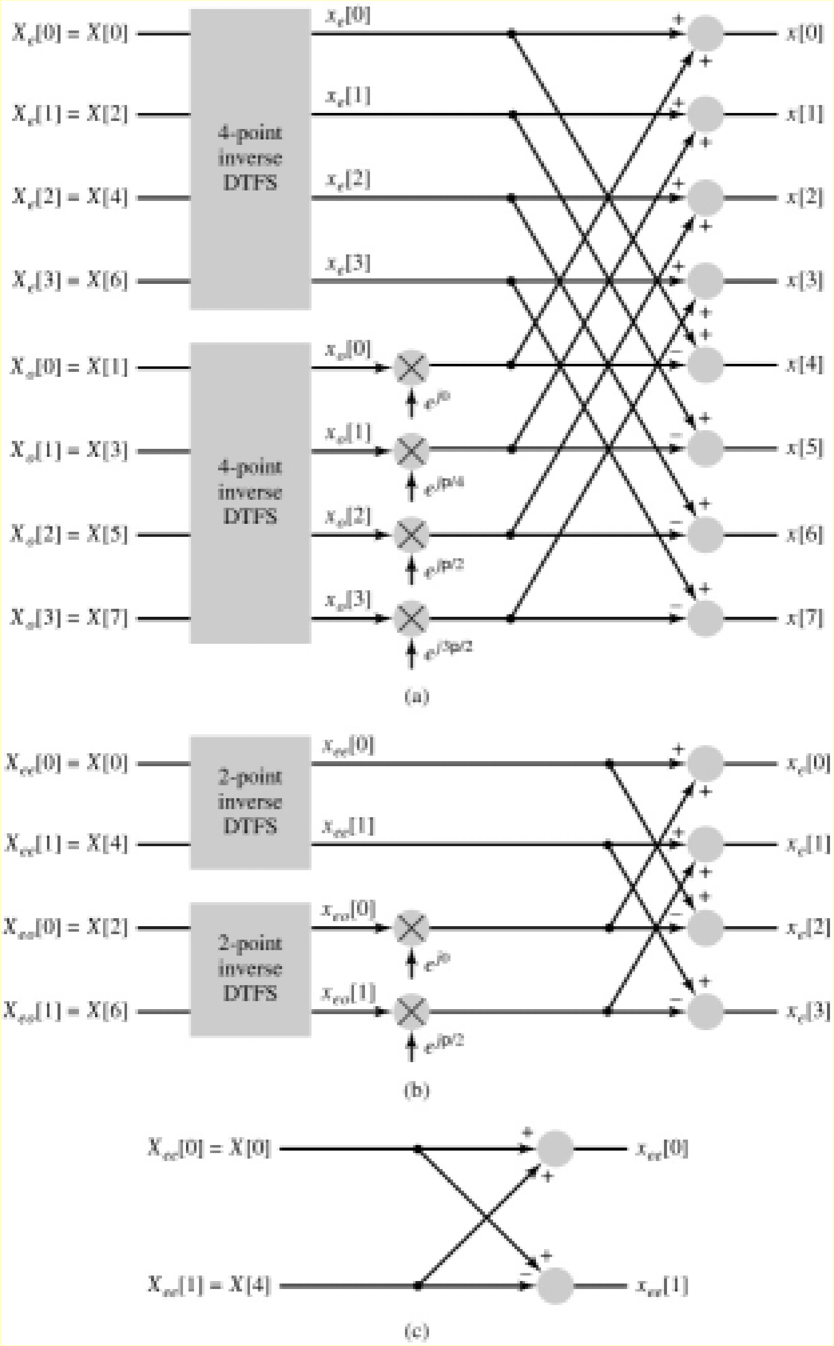
Odd: x[3], x[7]

Final :x[0], x[4], x[2], x[6], x[1], x[5], x[3], x[7]

N=8

N=4

N=2


4-point inverse DTFS
4-point inverse DTFS
(a)
2-point inverse DTFS
2-point inverse DTFS
(b)
(c)

Find x[n] From X[k] (N=2 Case)

$$x[n] = \sum_{k=0}^{N-1} X[k] e^{jk\frac{2\pi}{N}n}$$

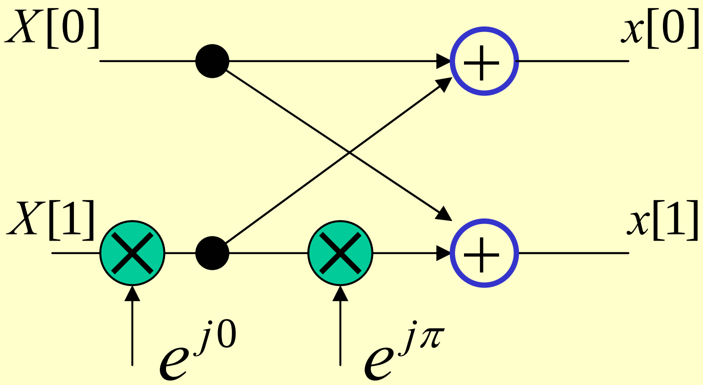
$$N = 2 \quad case$$

$$x[n] = \sum_{k=0}^{1} X[k] e^{jk\pi n}$$

$$\Rightarrow x[0] = \sum_{k=0}^{1} X[k] e^{jk\pi n} = X[0] \cdot e^{j0} + X[1] \cdot e^{j0} = X[0] + X[1]$$

$$\Rightarrow x[1] = \sum_{k=0}^{1} X[k] e^{jk\pi n} = X[0] \cdot e^{j0} + X[1] \cdot e^{j\pi} = X[0] - X[1]$$

FFT Butterfly for N = 2



Find x[n] From X[k] (N=4 Case)

$$x[n]=\sum_{k=0}^{3}X[k]e^{jk\pi n/2}$$

$$\Rightarrow x[0]=\sum_{k=0}^{3}X[k]e^{jk\pi n/2}=X[0]\cdot e^{j0}+X[1]\cdot e^{j0}+X[2]\cdot e^{j0}+X[3]\cdot e^{j0}$$

$$\Rightarrow x[1]=\sum_{k=0}^{3}X[k]e^{jk\pi n/2}=X[0]\cdot e^{j0}+X[1]\cdot e^{j\pi/2}+X[2]\cdot e^{j\pi}+X[3]\cdot e^{j3\pi/2}$$

$$\Rightarrow x[2]=\sum_{k=0}^{3}X[k]e^{jk\pi n/2}=X[0]\cdot e^{j0}+X[1]\cdot e^{j\pi}+X[2]\cdot e^{j2\pi}+X[3]\cdot e^{j3\pi}$$

$$\Rightarrow x[3]=\sum_{k=0}^{3}X[k]e^{jk\pi n/2}=X[0]\cdot e^{j0}+X[1]\cdot e^{j3\pi/2}+X[2]\cdot e^{j3\pi}+X[3]\cdot e^{j9\pi/2}$$

FFT Butterfly For N = 4

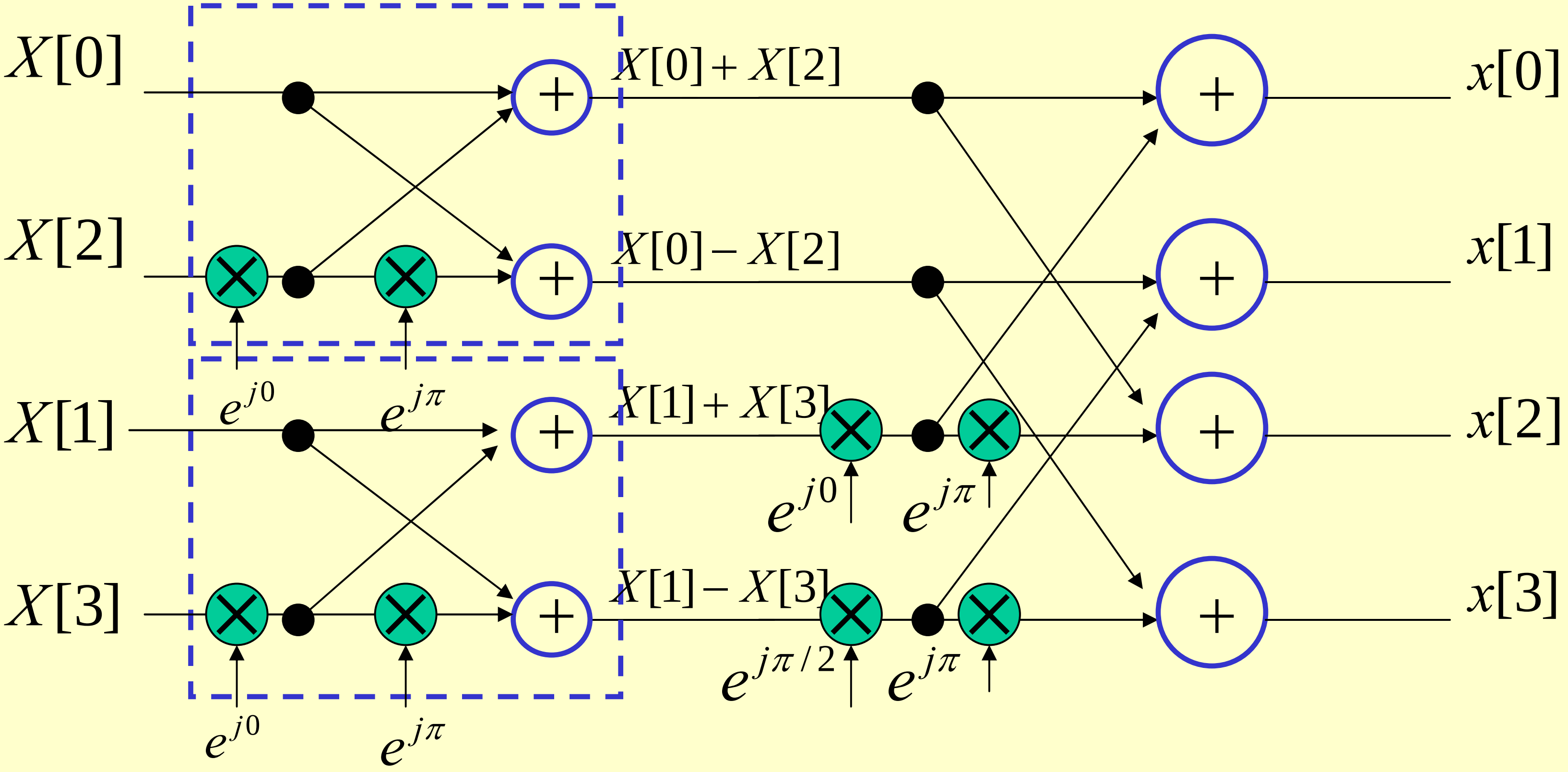


Diagram of the FFT algorithm for computing $x[n]$ from $X[k]$ for $N = 8$