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DMA Technical Report

ERROR THEORY AS APPLIED TO MAPPING, CHARTING, AND GEODESY

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Error theory provides the precision index which identifies the error distribution and specifies the probability that the true error does not exceed a certain value. These indexes are applicable to the evaluation of cartographic and geodetic information and make possible accuracy statements having a uniform interpretation. Standardized procedures and supporting theory for computing linear, circular, and spherical precision indexes are presented. The recommended procedure for computing the circular or spherical standard error from the linear standard errors in their components is to average the linear standard errors. Other precision indexes in the same error distribution are easily computed from the linear, circular, and spherical standard errors.

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DEFENSE MAPPING AGENCY TECHNICAL REPORT 8400.1

ERROR THEORY AS APPLIED TO MAPPING, CHARTING, AND GEODESY

FOREWORD

1. This publication is issued to disseminate timely and useful technical information to organizations engaged in cartography, geodesy, and related subjects.

2. DMA TR 8400.1 is based on ACIC Technical Report No. 96, Principles of Error Theory and Cartographic Applications, February 1962; ACIC Reference Publication No. 28, User's Guide to Understanding Chart and Geodetic Accuracies, September 1971; and ACIC Study No. 6, Circular Error Probability of a Quantity Affected by a Bias, June 1963. This report draws on material in these three reports.

3. This report has been approved for release to United States Government Agencies and their Contractors. Copies may be obtained by forwarding a request to the Defense Technical Information Center, Cameron Station, Alexandria, VA 22304-6145.

Charles M. Rose
CHARLES M. ROSE
Colonel, USAF
Acting Chief of Staff

PREFACE

Optimum utilization of Defense Mapping Agency (DMA) production requires that the accuracy of the source material, interim and final products be considered. This accuracy is expressed by an error statement which indicates whether the product is reliable and acceptable or should be used with discretion. Therefore, the error statement must be representative of the product and have a sound statistical basis. The purpose of this report is to present and explain the theory and procedures for providing a valid and meaningful error statement.

The normal distribution of linear errors is explained in detail because two and three-dimensional error distributions are more easily analyzed statistically by individual treatment of the linear components. The principles of the linear error distribution apply only to independent random errors, assuming that systematic errors have been eliminated or reduced sufficiently to permit treatment as random errors.

This report is based on three reports produced at the Aeronautical Chart and Information Center (now the Defense Mapping Agency Aerospace Center) . These reports are Principles of Error Theory and Cartographic Application, ACIC Technical Report No. 96, February 1962; Users Guide to Understanding Chart and Geodetic Accuracies, ACIC Reference Publication No. 28, September 1971; and Circular Error Probability of a Quantity Affected by a Bias, Studey No. 6, June 1963. This report borrows heavily from the material in these three reports.

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1. INTRODUCTION

Cartography, photogrammetry, and geodesy, as practiced at the Defense Mapping Agency (DMA), involve the measurement of physical quantities and the utilization of such measurements. Regardless of the precision of the instrument, no measurement device or method gives the true value of the quantity measured. Mechanical imperfections in instruments and the limitations introduced by human factors are such that repeated measurements of the same quantity result in different values. Variations among successive values are caused by errors in the observations. The true error of each observation is the difference between the true value of a quantity and the measured value.

While the theory of errors does not yield a true value nor improve the quality of observations, it does provide a way of estimating the most probable value for the quantity and of determining the certainty attributable to the estimate. This measure of the certainty will be called the precision index and will be the value that is attached to the DMA product to express the product's reliability. On such DMA graphics as the Air Target Chart this evaluation is expressed in the form of a reliability diagram. The reliability diagram will express the horizontal accuracy and vertical accuracy of the chart. Although some charts lack an accuracy statement, the basic concepts of horizontal and vertical accuracy apply to all series of charts and other products such as positioning data bases.

Using error theory, the user of DMA products can develop for himself accuracy estimates for products that do not furnish reliability information. This report will describe how to combine error measures to obtain the reliability estimates desired by the user. Use of error propagation techniques allow the user to relate DMA furnished accuracies to the variables of his interest.

2. ERRORS

2.1 Classes of Errors

Errors fall into three general classes which may be categorized by origin as (1) blunders, (2) systematic, and (3) random [1] [2] [3].

Blunders are mistakes or gross errors. Blunders may be caused by the wrong reading of instruments, transposing numbers or equipment failures. Blunders are usually large and easily detected. Blunders cannot be considered part of the sample from a statistical point of view. For this reason great care should be taken to avoid blunders when making observations. Observation and data collection methods should be planned to include redundancy and reasonableness checks. Since blunders are considered to be outside the population, they should be investigated and explained before being eliminated from the population. If there is no evidence of disturbance in the observations or data causing the lack of homogeneity in the population, data should not be eliminated solely on the basis of the magnitude of the error. Often data or observations are edited on the basis of a three sigma test. In this test the observations are averaged, and any that are more than three standard deviations from the mean are rejected. When an accuracy evaluation is being based on a data set there can be no automatic elimination of data as blunders.

Systematic errors affect the observations in the same way, hence they are hard to detect by repeated observation. Systematic errors may have the same sign and value, the so-called constant error or bias. In cartographic applications, systematic errors may occur due to instrumental factors or due to human limitations. Systematic errors may follow some pattern such as refraction or distortion due to the curvature of the earth. The method of compensating for systematic errors is to model them mathematically. The theorical model of the observations should attempt to mathematically compensate for all known systematic errors. If the effect of the error is not included in the theoretical model, the error should be removed from the observations.

Random errors result from accidental and unknown combination of causes beyond the control of the observer. They are characterized by: positive and negative errors occuring with equal frequency, small errors occuring more frequently than large errors, and extremely large errors occurring rarely. Random errors are the errors that remain after the blunders and systematic errors have been removed. Due to their unpredictably random errors cannot be eliminated from observations. Because of these random errors it is impossible to measure (observe) the 'true' value.

Though it is impossible to predict random errors, they have characteristics that may be expressed mathematically. The random errors of repeated observations usually display a normal frequency distribution. Because of the nature of these characteristics, the frequency distribution of random errors can be expressed mathematically by the normal distribution function. Assuming all errors are independent and random (conforming to the normal distribution function) the analysis of these errors allows us to derive accuracy information on the observation. The probability that a random error will not exceed a certain magnitude may be inferred from an analysis of the normal frequency distribution of the random errors.

2.2 Precision and Accuracy

Although the terms precision and accuracy [1] [2] [4] are often used interchangeably, there is an important difference between them. By definition, precision is the closeness with which repeated measurements made under similar conditions are grouped together; and accuracy is the closeness of the best estimated value obtained by the measurements to the "true" value of the quantity measured.

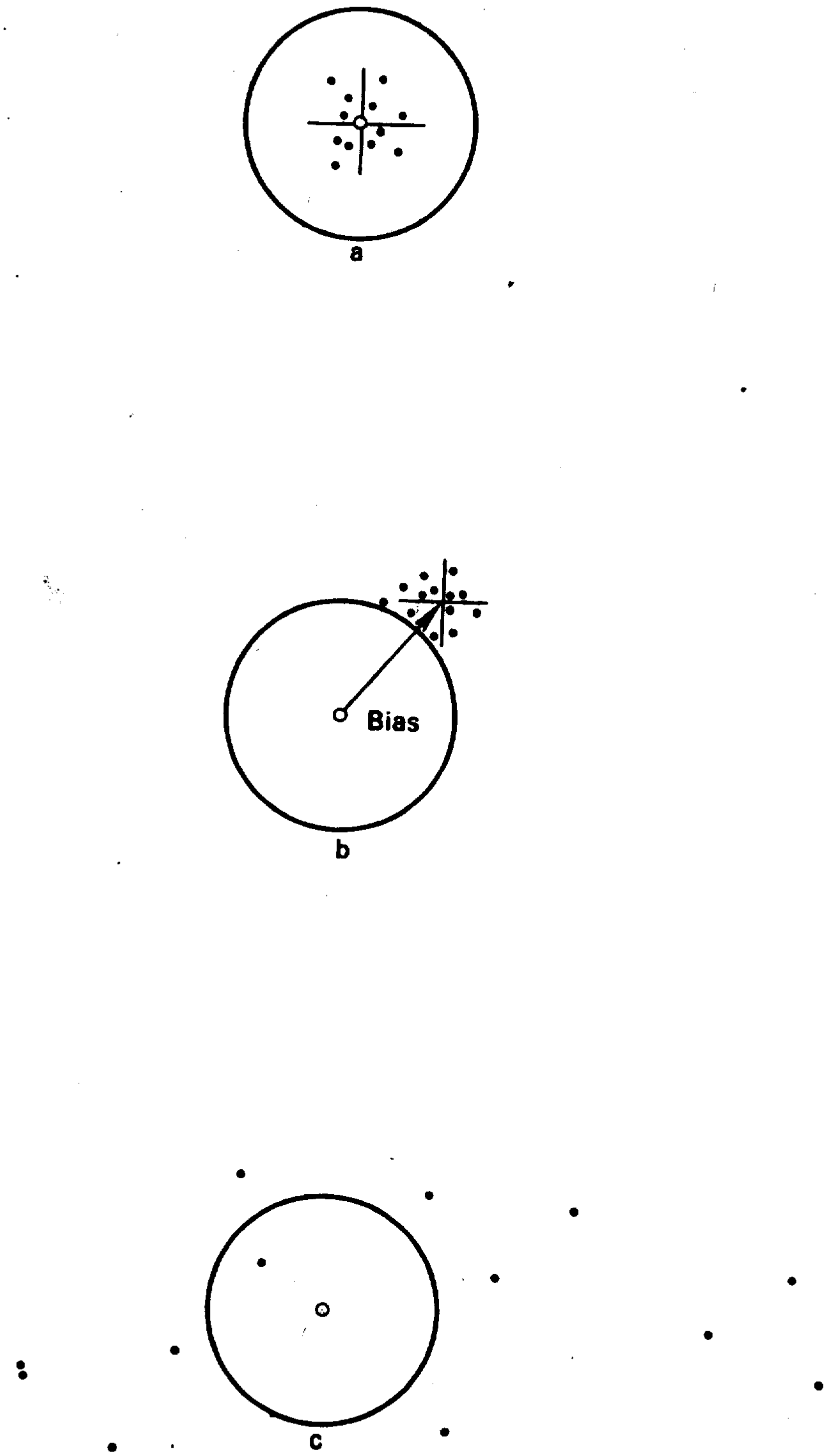


Figure 2-1. Precision and Accuracy

Precision is affected only by the random errors in the measuring process while accuracy is affected by the precision as well as the existence of unknown or systematic errors. The difference between precision and accuracy is illustrated in Figure 2-1 where the plots of errors in a circular distribution are shown. In Figure 2-1a, the points are grouped closely together and the measurement is said to be "precise".

It is also accurate because the center of the group coincides with the center of the circle. In Figure 2-1b, the grouping is still precise but inaccurate because it is not centered on the center of the circle. Instead, the mean of the points is offset by a systematic error or bias. The measurements shown in Figure 2-1b are "inaccurate" because of the bias even though they are "precise" (grouped closely). In Figure 2-1c, the points exhibit neither close grouping nor nearness to the center. They are, therefore, not precise and not accurate. Measurements can be precise and inaccurate at the same time, but they can never be accurate unless they are precise also.

The basic definition of an error distribution assumes that systematic errors and blunders have been removed and only random errors are left. However, systematic errors cannot be removed from positional information unless some means exist for their detection such as comparing this information against given control. Consequently, if systematic errors are not removed, they will have an effect (for example) on geodetic and photogrammetric measurements and the resulting positional information. It is not always possible to remove systematic errors from positioning information. To give the user some knowledge of the accuracy of the product he is using, methods have been devised to state the uncertainty of the products.

In the discussion of horizontal and vertical accuracy, the terms relative and absolute accuracy are often used. Absolute accuracy has the same meaning as accuracy, that is, how well does the measured quantity compare to the true value. DMA defines absolute horizontal accuracy as the statistical evaluation of all random and systematic errors encountered in determining the horizontal position of a single data point with respect to a specified geodetic reference datum. It is expressed as a circular error at the 90 percent probability level. Absolute vertical accuracy is defined as the statistical evaluation of all random and systematic errors encountered in determining the elevation of a single data point with respect to mean sea level. It is expressed as a linear error at the 90 percent probability level. Absolute accuracy is determined by comparing data points on a map, chart or related product to points with known horizontal positions and elevation on the specified geodetic datum and mean sea level. The absolute error will include all uncertainties and biases associated with relating the product to the specified datum. DMA defines relative (point-to-point) horizontal accuracy as the statistical evaluation of all random errors encountered in determining the horizontal position of one data point with respect to another. It is expressed as a circular error over a specified distance at the 90 percent probability level. It defines relative (point-to point) vertical accuracy as the statistical evaluation of all random errors encountered in determining the elevation of one data point with respect to another. It is expressed as a linear error over a specified distance at the 90 percent probability level. The main thing that should be pointed out about relative accuracies is that they do not translate into absolute accuracies. For instance, the relative accuracy involved in resolving a point on an image is not the absolute accuracy of positioning that point on the ground with respect to a geodetic datum.

Obviously due to their nature, relative errors will be smaller than absolute errors. If an absolute measure of the accuracy of a data set or map is required, one way to obtain it is to compare the data to an independent set of data that is known to be accurate in the geodetic frame of reference. An example of this would be comparing map points to points positioned by field survey which is on the datum for which the absolute accuracy is desired.

3. REVIEW OF THE BASIC CONCEPTS OF PROBABILITY

3.1 Probability

Probability [5] [6] is defined as the frequency of occurrence in proportion to the number of possible occurrences, or simply, the ratio of the number of successes to the number of trials. Let A and B symbolize two completely independent events. P(A) is called the probability set function of the event "A". The value of P(A) is called the probability of event "A" and the value of P(B) is called the probability of event "B". The probability of any event must be between 0 and 1. That is, zero probability means the particular event will never take place, and a probability of one means that the particular event will occur with each trial. For example, the probability of rolling the number 7 with a single die is 0.0 (an impossible event), but the probability of rolling a number from and including 1 through 6 is 1.0.

Rule 1. The probability of event A is equal to or greater than 0 but equal to or less than 1.

$$0 \leq P(A) \leq 1$$

Rule 2. The probability of a failure, or the probability of an event not occurring, is 1 minus the probability that it will occur.

$$1 - P(A) = \text{failure of event A}$$

Rule 3. The probability of two mutually exclusive events A or B occurring is equal to the sum of their individual probabilities.

$$P(A \text{ or } B) = P(A) + P(B)$$

An example is the probability of either a 3 or 4 occurring on the single roll of die:

$$P(3 \text{ or } 4) = 1/6 + 1/6 = 1/3$$

<u>Rule 4.</u> The probability of two independent events occurring simultaneously is equal to the <u>product</u> of their individual probabilities.

$$P(A \text{ and } B) = P(A) \cdot P(B).$$

If A is the value of the first die and B is the value of the second die, then the probability that A = 3 and B = 4 in a single roll of both dice,

$$P(A = 3 \text{ and } B = 4) = 1/6 \cdot 1/6 = 1/36.$$

The probabilities of occurrence of the numbers summed from each of 36 possible combinations resulting from the single roll of two dice are presented in Figure 3-1. The probability of rolling the number 7, for example, is 6/36 or 1/6 since there are six combinations which have a sum of seven. A histogram shown in Figure 3-1 is a discrete representation of the normal probability curve for the roll of two dice.

3.2 The Normal Distribution of a Continuous Random Variable [6] [7]

The area under the normal probability density curve (Figure 3-2a, page 14) represents the total probability of the occurrence of the continuous random variable x and is equal to one, or 100%. The mathematical expression of the curve is the normal probability density function, p(x):

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad (3\text{-}1)$$

where: x = the random variable

Number	Probability	Combinations
1	0	
2	1/36	(1,1)
3	2/36	(1,2) (2,1)
4	3/36	(1,3) (3,1) (2,2)
5	4/36	(1,4) (4,1) (2,3) (3,2)
6	5/36	(1,5) (5,1) (2,4) (4,2) (3,3)
7	6/36	(1,6) (6,1) (2,5) (5,2) (4,3) (3,4)
8	5/36	(2,6) (6,2) (3,5) (5,3) (4,4)
9	4/36	(3,6) (6,3) (4,5) (5,4)
10	3/36	(4,6) (6,4) (5,5)
11	2/36	(5,6) (6,5)
12	1/36	(6,6)
13	0	

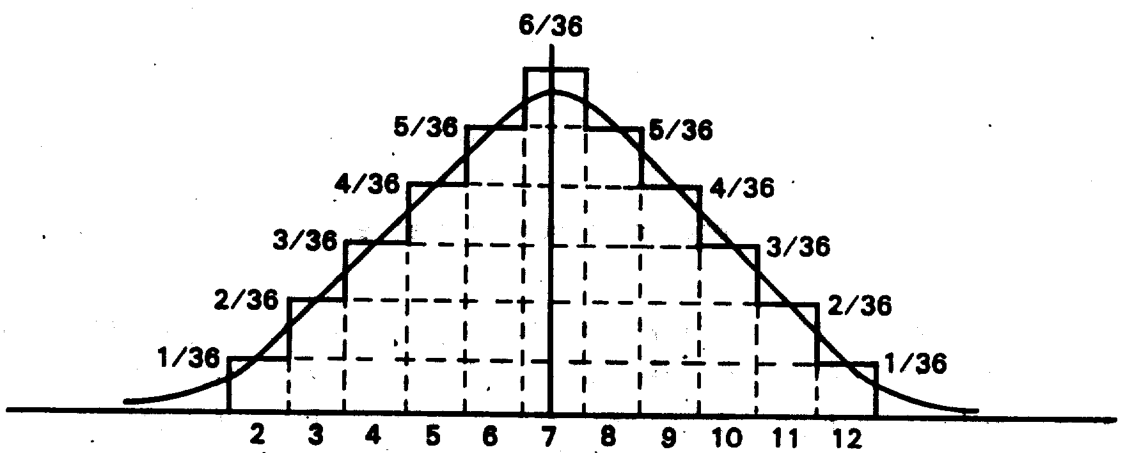


Figure 3-1. Probabilities of Numbers from the Roll of Two Dice

μ = a parameter representing the mean value of x

σ = a parameter representing the standard deviation, a measure of the dispersion of the random variable from the mean, μ. (The square of the standard deviation is called the variance.)

$\sqrt{2\pi}$ = 2.5066...

e = the base of natural logarithms, 2.71828...

The parameters are computed from an infinite number of random variables:

$$\mu = \frac{\sum_{i=1}^{n} x_i}{n} \tag{3-2}$$

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \mu)^2}{n}} \tag{3-3}$$

where:

n = the number of random variables

The normal probability <u>distribution</u> function, P(<u>x</u>), determines the probability that the random variable will assume a value within a certain interval and is derived from the normal probability <u>density</u> function by integrating between limits of the desired interval. Letting the limits range from $-\infty$ to <u>x</u>:

$$P(\underline{x}) = \int_{-\infty}^{\underline{x}} p(x)\,dx$$

$$P(\underline{x}) = \int_{-\infty}^{\underline{x}} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx \qquad (3\text{-}4)$$

The value of P(x) ranges between 0 and 1, illustrated in Figure 3-2b. As x approaches its upper limit, P(x) approaches 1; as x approaches its lower limit, P(x) approaches zero. This is true since x cannot exceed nor be less than its defined limits.

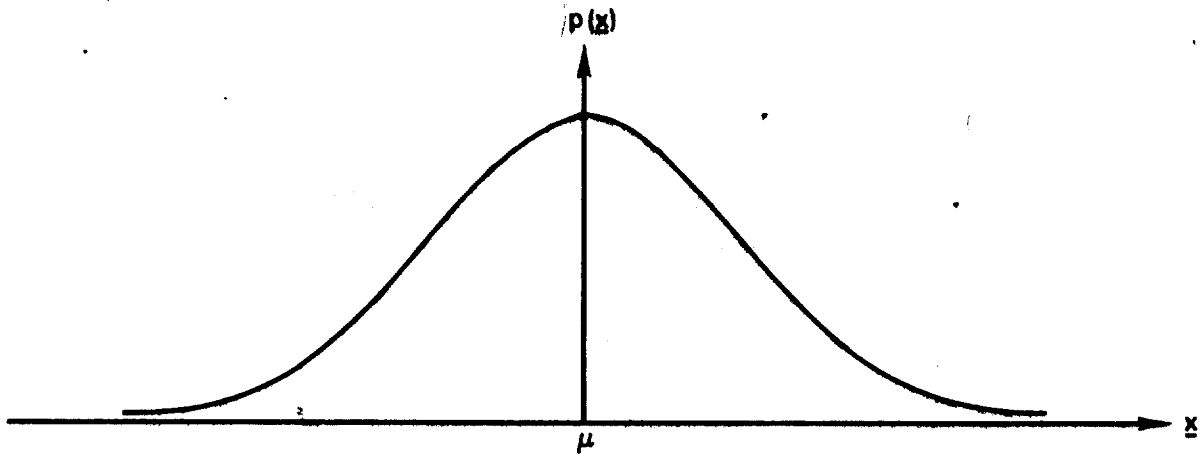


Figure 3-2a. Normal Probability Density Curve

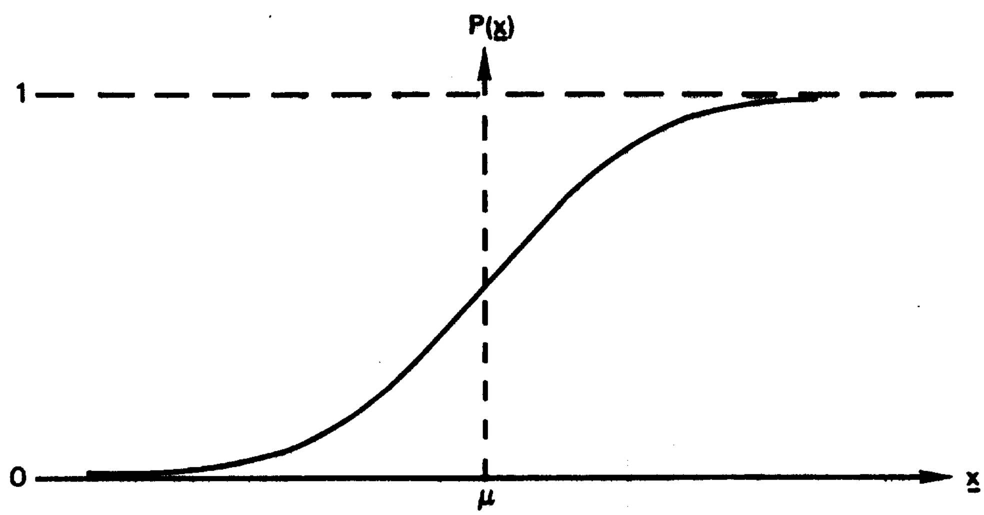


Figure 3-2b. Normal Probability Distribution Curve

4. ONE-DIMENSIONAL (LINEAR) ERRORS

4.1 Linear Errors [4] [8]

An error in a measurement is the difference between the "true" value of a quantity and the measured or derived value. The "true" value can never be determined because of instrument limitations and human fallibility, but can be estimated by taking a sufficient number of measurements. In determining the value of a quantity, only one measurement may be necessary when an approximate value is sufficient. If, on the other hand, the quantity is important enough to require a more precise value, repeated measurements are made. Variations will exist between the values obtained from several measurements. Applying the theory of the normal distribution to these measurements, the "best" value for the quantity is the mean or average of all the observed values. The differences between the mean, usually denoted by the Greek letter "mu" (μ), and the observed values are the apparent errors or residuals which are used to estimate a statement of precision for the measuring process. When the residual errors (the measurement minus the mean) here denoted by (x), are randomly distributed about the mean, the precision of the measurements is expressed by a single term, the standard deviation. The standard deviation is designated by the Greek letter "sigma" (σ)and sometimes referred to as the one sigma (1σ) error. The square of the standard deviation (σ^2) is called the variance. For a linear distribution, the standard deviation is computed by squaring all the residual errors, adding the squared values, dividing by the number of errors (less one), and taking the square root:

$$\sigma = \sqrt{\frac{\sum x^2}{n-1}}$$

4.2 Application of the Probability Density Function to Random Errors [7]

The normal probability density curve of an infinite number of measurements of the unknown quantity X is expressed by parameters analogous to those of equation (3-1). The true value, μ_x, is the mean of the distribution of the observed values X_1, X_2, X_3, $\ldots X_n$. The curve, illustrated in Figure 4-1 (page 17), has the mathematical form:

$$p(X) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(X_i - \mu_X)^2}{2\sigma^2}} \qquad (4\text{-}1)$$

where: $$\sigma = \sqrt{\frac{\sum_{i=1}^{n} (X_i - \mu_X)^2}{n}}$$

The normal probability density curve of errors has a mean of zero and is identical in form to that of the observed values. Illustrated in Figure 4-2 (page 18), the curve is described by the function:

$$p(\varepsilon) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{\varepsilon^2}{2\sigma^2}} \qquad (4\text{-}2)$$

where: ε = the true error;

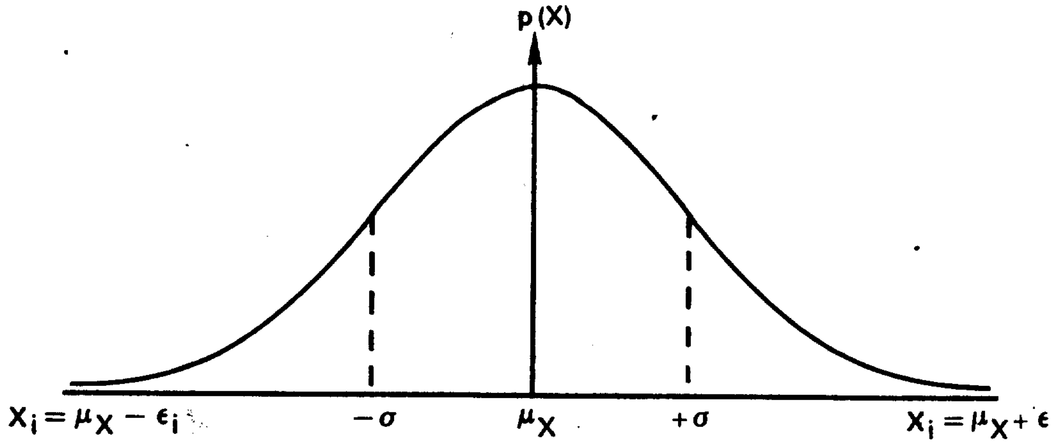
$\varepsilon = X_i - \mu_X$

σ = the standard deviation of the errors

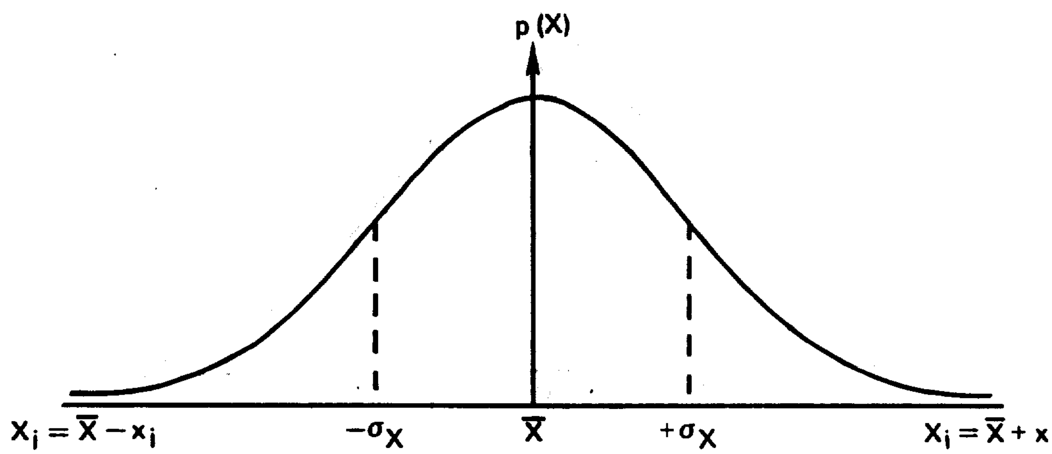
$$\sigma^2 = \frac{\sum_{i=1}^{n} \varepsilon_i^2}{n}$$

Since the true value of a quantity cannot be measured and an infinite number of measurements is impractical, estimated values obtained from a finite number or sample of measurements must be substituted for the true value and the parameters of the density function. As the number of measurements in the sample becomes larger, the reliability of the estimate increases. Often, 30 values provide an adequate estimate. The most probable value ($\overline{X}$) approximates the true value and is determined from the arithmetic mean of observed values:

$$\overline{X} = \frac{\sum_{i=1}^{n} X_i}{n} \qquad (4\text{-}3)$$



Population



Sample

Figure 4-1. Normal Probability Density Curve of Observed Values

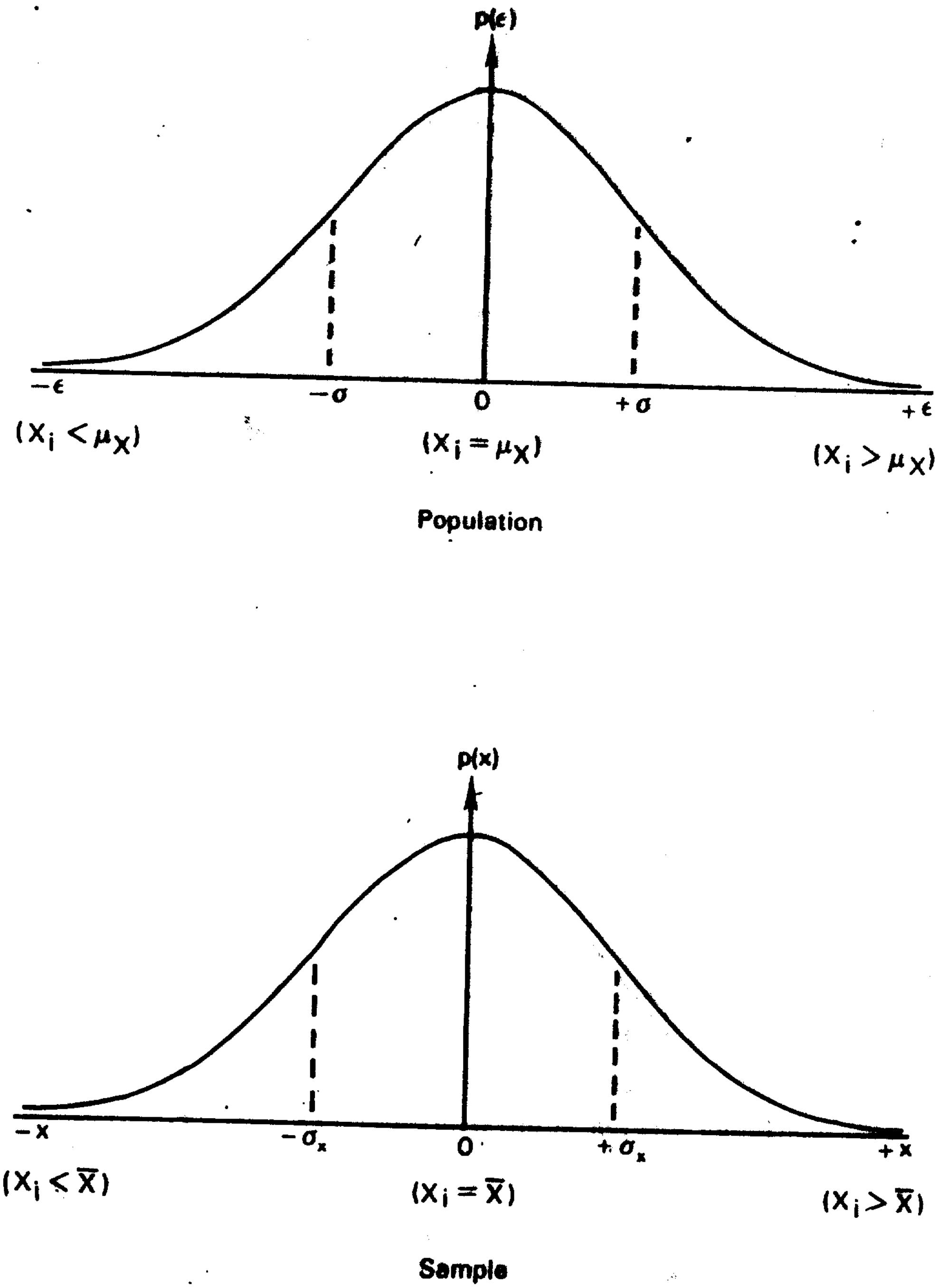


Figure 4-2. Normal Probability Density Curve of Errors

The true error is approximated by the residual "x", hereafter designated the error and defined as the difference between the observed value and the most probable value:

$$x = X_i - \overline{X} \tag{4-4}$$

The standard deviation computed from a sample (σ_x) is identified by a subscript and computed from:

$$\sigma_x = \sqrt{\frac{\sum_{i=1}^{n} (X_i - \overline{X})^2}{n-1}} = \sqrt{\frac{\sum x^2}{n - 1}} \tag{4-5}$$

This term is sometimes referred to as the standard error. The normal probability density function of errors now becomes:

$$p(x) = \frac{1}{\sigma_x \sqrt{2\pi}} \, e^{-\frac{x^2}{2\sigma_x^2}} \tag{4-6}$$

The parameters $\overline{X}$ and σ_x may assume different values as various samples are selected from the same population and are, therefore, random variables with dispersion expressed by similar parameters. The standard deviation of mean, $\sigma_{\overline{X}}$, and the standard deviation of the standard deviation, σ_σ, indicate the reliability of the estimate and help "round off" the computed values:

$$\sigma_{\overline{X}} = \sqrt{\frac{\sum_{i=1}^{n} (X_i - \overline{X})^2}{n (n - 1)}} = \frac{\sigma_x}{\sqrt{n}} \tag{4-7}$$

$$\sigma_\sigma = \sqrt{\frac{\sum_{i=1}^{n} (X_i - \overline{X})^2}{2 (n - 1)^2}} = \frac{\sigma_x}{\sqrt{2 (n-1)}} \tag{4-8}$$

4.3 Precision Indexes

A precision index [7] reveals how errors are dispersed or scattered about zero and reflects the limiting magnitude of error for various probabilities. For example, 50% of all errors in a series of measurements do not exceed ±20 feet; 90% do not exceed ±49 feet. Although different errors are given, each expresses the same precision of the measuring process (Figure 4-3, page 22). The standard deviation (σ_X) and average error (η) are two indexes with theoretical derivations. Common usage has included three additional probability levels which are, in effect, precision indexes: (1) probable error (PE), (2) map accuracy standard (MAS), and (3) and three sigma error (3σ) .

The standard deviation is the most important of the indexes and has the probability of:

$$P(x) = \int_{-\sigma_X}^{+\sigma_X} p(x)\, dx = 0.6827 \qquad (4\text{-}9)$$

Or, 68.27% of all errors will occur within the limits of $\pm\,\sigma_X$.

The average error is defined as the mean of the sum of the absolute values of all errors:

$$\eta = \frac{\sum_{i=1}^{n} |(X_i - \overline{X})|}{n} = \frac{\sum |x|}{n} \qquad (4\text{-}10)$$

The probability represented by the average error is 0.5751, or 57.51%. The average error is easily computed from the standard deviation:

$$\eta = 0.7979\, \sigma_X \qquad (4\text{-}11)$$

The probable error is that error which 50% of all errors in a linear distribution will not exceed. Specifically, the true error is equally likely to be larger or smaller than the probable error. Expressed mathematically:

$$PE = \int_a^b p(x) \; dx = 0.50 \tag{4-12}$$

The probable error is computed from the standard error:

$$PE = 0.6745 \sqrt{\frac{\sum x^2}{n - 1}} = 0.6745 \; \sigma_x \tag{4-13}$$

The U.S. National Map Accuracy Standards specify that no more than 10% of map elevations (a one-dimensional error) shall be in error by more than a given limit. The standards are commonly interpreted as limiting the size of error of which 90% of the elevations will not exceed. Therefore, the map accuracy standard is represented by:

$$MAS = \int_{a'}^{b'} p(x) \; dx = 0.90 \tag{4-14}$$

or, computed from the standard deviation:

$$MAS = 1.6449 \; \sigma_x \tag{4-15}$$

The three sigma error, as the name implies, is an error three times the magnitude of the standard deviation. The 3σ error is used because it approaches near certainty -- 0.9973 or 99.73% probability. Since the probability of a linear random error falling outside these limits is sufficient cause to consider such errors as "blunders".

The meaning of the standard deviation with respect to the normal distribution function is illustrated in Figure 4-3. The vertical axis, p(x), represents the mean value for the measured quantity. The normal error distribution function about the mean is expressed in one sigma units centered on the mean. The property of the distribution curve is twofold:

- The total area under the distribution curve is equal to unity.

- The area under the curve between any two values of x_1 and x_2 is equal to the probability of an error occurring between these limits.

- The area under the curve between the limits $x_1 = \sigma$ and $x_2 = -\sigma$ is 68.27% of the total area under the curve. Under the assumption that the errors are normally distributed, this means there is a 68.27% probability that errors in any further measurements under the same conditions will not exceed the standard deviation. The standard deviation does not indicate the probability that an error of a certain size will occur, it only indicates that approximately 68% of the errors will fall within the specified limits of plus or minus one sigma.

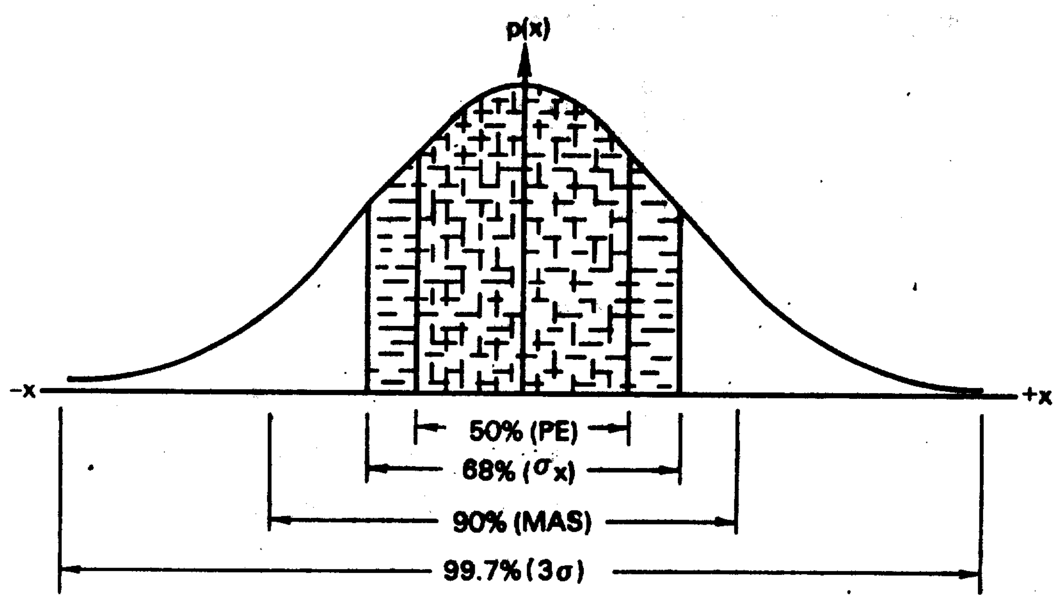


Figure 4-3. Normal Linear Distribution

Factors for converting from one probability level to another are shown in Table 4-1. For example, an error of ±20 feet in elevation at 90% probability can be converted to ±12 feet at 68.27% by multiplying 20 feet by the factor 0.6080.

Table 4-1

Linear Error Conversion Factors

From \ To	50.00%	68.27%	90.00%	99.73%
50.00%	1.0000	1.4826	2.4387	4.4475
68.27%	0.6745	1.0000	1.6449	3.0000
90.00%	0.4101	0.6080	1.0000	1.8239
99.73%	0.2248	0.3333	0.5483	1.0000

4.4 Examples of Linear Errors

The foregoing discussion demonstrates the use of the normal distribution in the analysis of random errors. There are numerous opportunities for the occurrence of random variables in cartographic and geodetic work. For example, the base lines and measured angles, observed lengths of lines, elevations, etc., resulting from geodetic triangulation, traverse, and leveling all contain error. Celestial and gravimetric observations as well as distances measured by trilateration are also examples of measures where linear errors occur. The principles of error theory can be used advantageously to analyze the results in terms of the specifications established for the survey.

At DMA, the normal linear error distribution has important applications with respect to evaluating the accuracy of positional information. In addition to the one-dimensional errors which exist in such positional data as elevations above mean sea level, the linear error components of two-dimensional positions can be analyzed by applying principles of the normal linear error distribution. The following sections contain discussions of the utility of the linear standard error for analyzing two and three-dimensional distributions.

5. TWO-DIMENSIONAL (ELLIPTICAL, CIRCULAR) ERRORS

5.1 Introduction

A two-dimensional error is the error in a quantity defined by two random variables. For example, consider the true geographic position of a point referred to the X and Y axes. Each observation of the X and Y coordinates will contain the errors "x" and "y". When assumed random and independent, each error has a probability density distribution of:

$$p(x) = \frac{1}{\sigma_x \sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_x^2}}$$

and:

$$p(y) = \frac{1}{\sigma_y \sqrt{2\pi}} e^{-\frac{y^2}{2\sigma_y^2}}$$

Applying Rule 4 of Section 3.1., the two-dimensional probability density function becomes:

$$p(x,y) = \frac{1}{2\pi\ \sigma_x\ \sigma_y} e^{-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2}\right)} \qquad (5\text{-}1)$$

Rearranging terms:

$$p(x,y)\ \sigma_x \sigma_y\ 2\pi = e^{-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2}\right)}$$

Therefore:

$$-2 \ln \left[p(x,y)\ \ \sigma_x\ \sigma_y\ 2\pi \right] = \frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} \qquad (5\text{-}2)$$

For given values of p(x,y), the left side of equation (5-2) is a constant K^2

Then:

$$K^2 = \frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} \qquad (5\text{-}3)$$

For values of p(x,y) from 0 to ∞ , a family of equal probability density ellipses are formed with axes K σ_x and K σ_y .

When $\sigma_x = \sigma_y$, equation (5-2) becomes: (5-4)

$$-2\sigma_x^2 \ln \left[p(x,y)\, \sigma_x^2\, 2\pi \right] = x^2 + y^2$$

For a given value of p(x,y), the left side of equation (5-4) is a constant which is the square of the radius of an equal probability density circle.

The probability density function integrated over a certain region becomes the probability distribution function which yields the probability that x and y will occur simultaneously within that region, or:

$$P(x,y) = \int \int p(x,y)\, dx\, dy \qquad (5\text{-}5)$$

However, since both positive and negative values of either "x" or "y" will occur with equal frequency, the errors may be considered as radial errors, designated by "r", where $r = \sqrt{x^2 + y^2}$.

5.2 Elliptical Errors

The probability of an ellipse [10] is given by the distribution function:

$$P(x,y) = 1 - e^{-\frac{K^2}{2}} \qquad (5\text{-}6)$$

The solution of equation (5-6) with values of K for different probabilities yields the results shown in Table 5-1. For a 39%

probability, the axes of the ellipse are 1.000 σ_x and 1.0000 σ_y ; for a 50% probability, the axes are 1.1774 σ_x and 1.1774 σ_y .

Table 5 - 1

Values of the Constant K

Probability	K
39.35%	1.0000
50.00%	1.1774
63.21%	1.4142
90.00%	2.1460
99.00%	3.0349
99.78%	3.5000

The use of the error ellipse is complicated by the problem of axes orientation and propagation of elliptical errors. Therefore, the ellipse is commonly replaced by a circular approximation which is easier to use and understand.

5.3 Circular Errors

The probability distribution function [9] of the radial error expressing the probability that "r" will be equal to or less than radius R, or the probability that the point (x,y) will be contained within a circle of radius R, is derived in Appendix D and stated as:

$$P(R) = \frac{1}{\sigma_x \sigma_y} \int_0^R r e^{-\frac{r^2}{4\sigma_y^2}\left[1 + \frac{\sigma_y^2}{\sigma_x^2}\right]} I_0 \left[\frac{r^2}{4\sigma_y^2}\left(\frac{\sigma_y^2}{\sigma_x^2} - 1\right)\right] dr \qquad (5\text{-}7)$$

A special case of the P(R) function (5-7) is formed when r=R, and $\sigma_x = \sigma_y = \sigma_r = \sigma_c$. From Appendix D, part 2:

$$P(R) = P_c = 1 - e^{-\frac{R^2}{2\sigma_c^2}} \qquad (5\text{-}8)$$

where:

P_c = the circular probability distribution function, a special case of P(R)

R = the radius of the probability circle

σ_c = the circular standard error, a special case of σ_r when $\sigma_r = \sigma_x = \sigma_y$.

When σ_x and σ_y are not equal, the P(R) function, (5-7), is modified by letting "a" equal the ratio σ_x/σ_y where σ_x is the smaller standard error of the two. Then from Appendix D, part 3:

$$P(R) = \frac{2a}{1 + a^2} \int_0^x e^{-v} I_0 (vk) \; dv \qquad (5\text{-}9)$$

where:

$$x = \frac{R^2}{4\sigma_y^2} \left[\frac{1 + a^2}{a^2}\right]$$

$$v = \frac{r^2}{4\sigma_y^2} \left[\frac{1 + a^2}{a^2}\right]$$

$$k = \frac{1 - a^2}{1 + a^2}$$

Equation (5-9) can be solved for different probabilities or values of P(R) representing precision indexes of the error distribution.

5.4 Circular Precision Indexes

The precision indexes [10] illustrated in Figure 5.1 (page 35) are measures of the dispersion of errors in a distribution and represent the error which is unlikely to be exceeded for a given probability. The preferred circular precision indexes, consistent with indexes used in the linear distribution, are: (1) the circular standard error (σ_c), (2) the circular error probable (CEP), (3) the circular map accuracy standard (CMAS), and (4) the circular near-certainty error, three point five sigma (3.5 σ_c). The mean square positional error (MSPE), an additional index which has been used at DMA, is not recommended because the probability represented varies when σ_x and σ_y are not equal.

The probability of the <u>circular standard error</u> is found by solving equation (5-8) for P_c when $\sigma_c = R$, thus:

$$P_c = 1 - e^{-\frac{\sigma_c^2}{2\sigma_c^2}}$$

$$P_c = 1 - e^{-\frac{1}{2}}$$

$$P_c = 1 - 0.60653$$

$$\therefore P_c = 0.3935 \qquad (5\text{-}10)$$

That is, 39.35% of all errors in a circular distribution are not expected to exceed the circular error.

For a truly circular distribution, the linear standard errors are equal and identical to the circular standard error $(\sigma_x = \sigma_y = \sigma_c)$. When σ_x and σ_y are not equal, a normal circular error distribution may be substituted for the elliptical distribution. The substitution is satisfactory for error analysis within specified $\sigma_{min}/\sigma_{max}$ ratios. Because of distortion in the error distribution for low ratios, however, the circular concept should be used with discretion.

Table 5-2

Solution of P(R) Function for P(R) = 39.35%

$\frac{\sigma_{min}}{\sigma_{max}}$	$\frac{\sigma_c}{\sigma_{max}}$
1.0000	1.0000
0.8165	0.9063
0.6547	0.8197
0.5000	0.7323
0.3333	0.6327
0.2294	0.5727
0.1005	0.5274
0.0	0.5151

Note: When P(R) = 39.35%, $R \sim \sigma_c$.

Table 5-3

Solution of P(R) Function for P(R) = 50.00%

$\frac{\sigma_{min}}{\sigma_{max}}$	$\frac{CEP}{\sigma_{max}}$
1.000	1.1774
0.8165	1.0683
0.6547	0.9690
0.5000	0.8707
0.3333	0.7696
0.2294	0.7174
0.1005	0.6835
0.0	0.6745

Note: When P(R) = 50.00%, R ~ CEP

An approximate circular standard error is determined from equation (5-9) by letting P(R) = 39.35% and R = σ_c . Values of σ_c/σ_{max} for ratios of $\sigma_{min}/\sigma_{max}$ from 0.0 to 1.0 are contained in Table 5-2 and plotted in Figure 5-2 (page 39). For the $\sigma_{min}/\sigma_{max}$ ratio between 1.0 and 0.6, the curve is a straight line with the equation:

$$\sigma_c \sim (0.5222\ \sigma_{min} + 0.4778\ \sigma_{max}) . \qquad (5\text{-}11)$$

A rapid approximation gives a slightly larger σ_c value for the same $\sigma_{min}/\sigma_{max}$ ratio:

$$\sigma_c \sim 0.5000\ (\sigma_x + \sigma_y) . \qquad (5\text{-}12)$$

As $\sigma_{min}/\sigma_{max}$ approaches zero, the 39.35% probability curve follows a transition from circular, through elliptical, to the linear distribution form. The curve does not effectively represent a circular standard error for $\sigma_{min}/\sigma_{max}$ ratios less than 0.6 because it is not compatible with other circular precision indexes. For example, the factor 1.774 converts a circular error at 39% probability to a circular error at 50% probability when $\sigma_{min}/\sigma_{max}$ = 1.0, but when σ_{min} = 0, the factor converting a linear error at 39% probabiltiy to a linear error at 50% probability is 1.3094. The circular standard error computed from equation (5-12), however, can be converted to other circular precision indexes by constant circular conversion factors for $\sigma_{min}/\sigma_{max}$ ratios between <u>1.0 and 0.2</u> and is, therefore, the preferred method for approximating the circular standard error.

The <u>circular error probable</u> is the circular error which 50% of all errors in a circular distribution will not exceed, or the value of R in equation (5-5) which makes P_c = 0.5. The CEP in a truly circular distribution (i.e. $\sigma_x = \sigma_y = \sigma_c$) is computed by:

$$0.5 = 1 - e^{-\frac{R^2}{2\sigma_c^2}}$$

$$1 - 0.5 = e^{-\frac{R^2}{2\sigma_c^2}}$$

$$\ln 0.5 = -\frac{R^2}{2\sigma_c^2}$$

$$R^2 = 0.69315\ (2\sigma_c^2)$$

$$R = 1.1774\ \sigma_c$$

$$CEP = 1.1774\ \sigma_c \qquad (5\text{-}13)$$

When σ_x and σ_y are not equal, an approximate CEP is determined from equation (5-9) by letting P(R) = 50.00% and R = CEP. Values of CEP/σ_{max} for ratios of $\sigma_{min}/\sigma_{max}$ from 1.0 to 0.0 are tabulated in Table 5-3. The 50% probability curve plotted in Figure 5-3 is approximated by a series of straight lines for different ratios of $\sigma_{min}/\sigma_{max}$ with the equations:

$$CEP \sim (0.6142\ \sigma_{min} + 0.5632\ \sigma_{max})$$

when $\sigma_{min}/\sigma_{max}$ is between 1.0 and 0.3

$$CEP \sim (0.4263\ \sigma_{min} + 0.6196\ \sigma_{max})$$

when $\sigma_{min}/\sigma_{max}$ is between 0.3 and 0.2

A rapid approximation of the CEP plots as a straight line which intersects the 50% probability curve at the point where $\sigma_{min}/\sigma_{max} = 0.2$ and has the equation:

$$CEP \sim 0.5887\ (\sigma_x + \sigma_y)$$

when $\sigma_{min}/\sigma_{max}$ is between 1.0 and 0.2

The CEP computed by this equation is compatible with the circular standard error computed by equation (5-12) and is, therefore, the preferred method for approximating the circular probable error within the specified limits.

Although a circular error concept is not recommended for $\sigma_{min}/\sigma_{max}$ ratios less than 0.2, a near-linear 50% probabilty error may be computed to represent a CEP for lower ratios when a comparison of circular errors derived from different sources is required:

$$CEP \sim (0.2141\ \sigma_{min} + 0.6621\ \sigma_{max})$$
when $\sigma_{min}/\sigma_{max}$ is between 0.2 and 0.1

$$CEP \sim (0.0900\ \sigma_{min} + 0.6745\ \sigma_{max})$$
when $\sigma_{min}/\sigma_{max}$ is between 0.1 and 0.0

$$CEP \sim 0.6745\ \sigma_{max}$$
when $\sigma_{min} = 0$

The following alternate methods of computing an approximate CEP are not recommended because of limited applicability:

$$CEP \sim 1.1774 \sqrt{\frac{\sigma_x^2 + \sigma_y^2}{2}}$$

and $$CEP \sim 0.8325 \sqrt{\sigma_x^2 + \sigma_y^2}$$
when $\sigma_{min}/\sigma_{max}$ is between 1.0 and 0.8

The <u>mean square positional error</u> is defined as the radius of the error circle equal to $1.4142\sigma_c$ and has little significance in a truly circular error distribution. However, when σ_x and σ_y are approximately equal, the MSPE defines the error in a geographic position and is computed:

$$MSPE = \sqrt{\sigma_x^2 + \sigma_y^2} \qquad (5\text{-}14)$$
when $\sigma_{min}/\sigma_{max}$ is between 1.0 and 0.8

The probability represented by the MSPE can be found by solving equation (5-8) for P_c , when R = MSPE and σ_c is approximated by equation (5-12), thus:

$$P_c = 1 - e^{-\frac{R^2}{2\sigma_c^2}}$$

$$P_c = 1 - e^{-\frac{(\sigma_x^2 + \sigma_y^2)}{2\sigma_c^2}} \qquad (5\text{-}15)$$

When $\sigma_x = \sigma_y$:

$$P_c = 1 - e^{-1.0}$$

$$P_c \doteq 1 - 0.3679$$

$$P_c = 63.21\% \qquad (5\text{-}16)$$

When $\sigma_x \neq \sigma_y$, the solution of (5-15) yields values of P_c ranging from 64% when $\sigma_{min}/\sigma_{max} = 0.8$ to 77% when $\sigma_{min}/\sigma_{max} = 0.3$. Because of the variation in probability, the MSPE is not recommended for use as a precision index.

The <u>2drms</u> accuracy standard has been suggested [11] for use in the navigation community. The 2drms is defined by:

$$2\text{drms} = 2\sqrt{\sigma_x^2 + \sigma_y^2} \;.$$

Hence, it will have a radius twice the MSPE. The percentage level for the 2drms is 95.4%. Like the MSPE, it will vary in probability and may be skewed by large errors.

$$2\text{drm} = 2.99\ \sigma_c$$

for $\sigma_x = \sigma_y$.

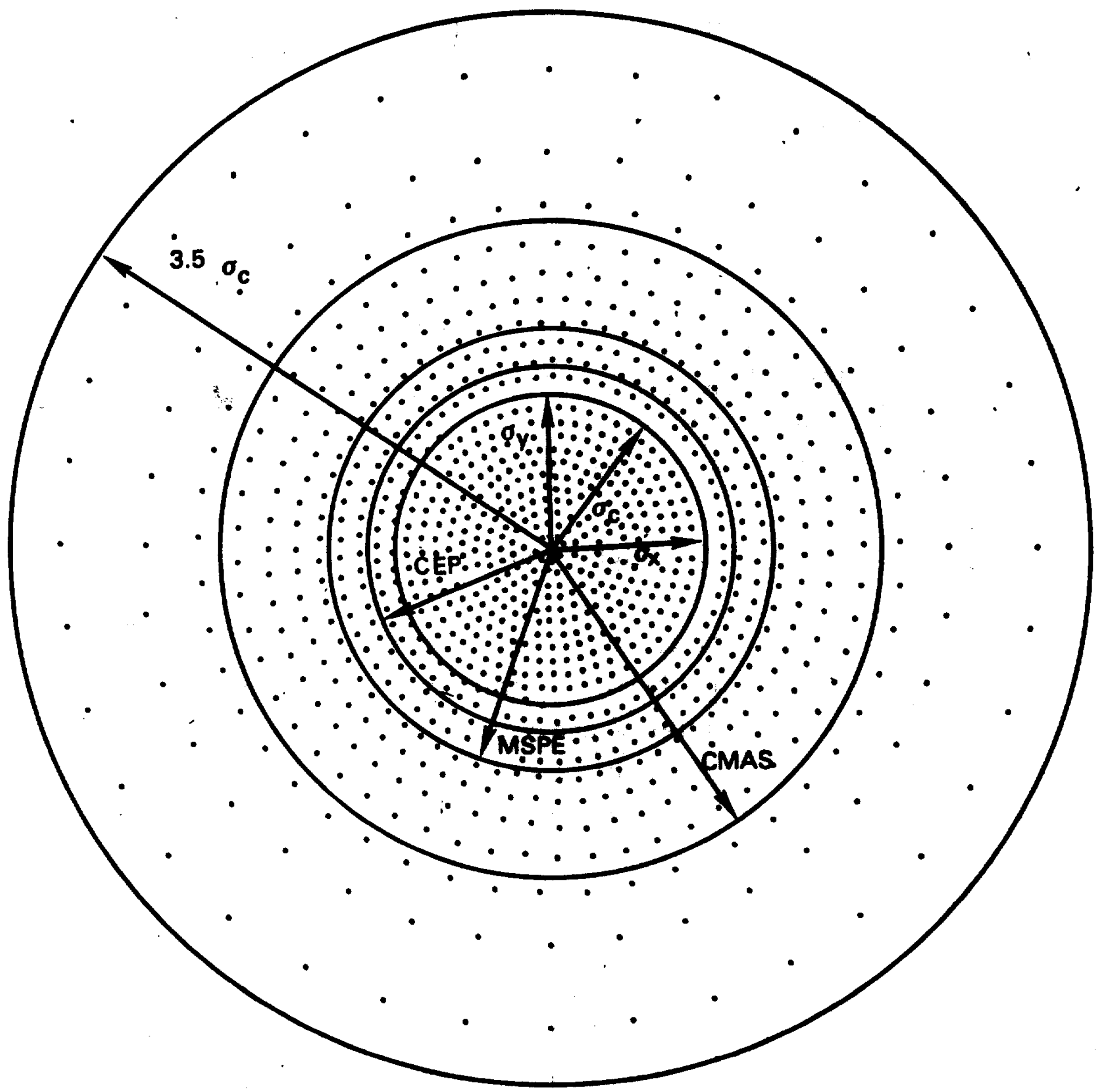


Figure 5-1. **Normal Circular Distribution**

The circular map accuracy standard is based on the percentage level in use by the U.S. National Map Accuracy Standards which specify that no more than 10% of the well-defined points in a map will exceed a given error. The standards are commonly interpreted as limiting the size of error which 90% of the well-defined points will not exceed. Therefore, the circular map accuracy standard is represented by the value of R in equation (5-8) when $P_c = 0.90$, and is computed:

$$\text{CMAS} = 2.1460\ \sigma_c \tag{5-17}$$

or

$$\text{CMAS} = 1.8227\ \text{CEP} \tag{5-18}$$

The three-five sigma error, representing a circular probability of 99.78% approaches near-certainty in a circular distribution and has a magnitude 3.5 times that of the circular standard error.

5.5 Discussion of Circular Errors

The normal circular error distribution is derived from the bivariate or two-dimensional distribution of errors. For typical applications, the variables x and y are random errors defined as Eastings and Northings, downrange and crossrange components, or latitude and longitude converted to some metric unit. It is rare that these errors are uncorrelated and their standard errors equal. As such, they are referred to as elliptical distributions. To simplify probability calculations, the elliptical distribution is converted to an equivalent circular distribution. For the following discussion, it will be assumed such a conversion has taken place according to the following formula which is adequate for cases where the larger standard error does not exceed five times the smaller:

$$\sigma_c = 0.5\ (\sigma_x + \sigma_y) \tag{5-19}$$

The quantity (σ_C) is defined as the circular standard error and is the basic statistical parameter used in probability estimates based upon the normal circular distribution.

The probability associated with circular errors is used in the same way as for linear errors. However, instead of the area under the normal distribution curve, the probability in a circular distribution is a function of the radius of a circle centered on the mean of the error distribution (Figure 5-1, page 35). A circle with a radius equal to the circular standard error represents 39.35% probability in a normal circular distribution. Other probability levels can be defined by circles of larger radii. For example, a circle with a radius of 1.1774 times the circular standard error represents 50% probability and is known as the circular error probable (CEP) which is common to missile, bombing, and artillery error terminology.

The radius can be increased further to describe a circle representing 90% probability. The latter, which is approximately twice the circular standard error (or $2.1460\sigma_C$) ,is known as the circular map accuracy standard (CMAS). Circular errors (CE) at the 90% level are sometimes expressed as CE 90%. The expression "90% assurance" is another term used to express confidence in an estimate. However, probability rather than assurance is the preferred term because it has a statistical meaning.

Circular distribution probabilities can be converted from one level to another by the factors in Table 5-4. For example, a geographic position is assigned a CMAS of 500 feet. The CEP (50% probability) of the position is estimated by multiplying 500 feet by the factor 0.5486 from Table 5-5 to obtain 274 feet. This does not imply that there is a 50% probability that an error of 274 feet will occur, rather it means that there is a 50% probability that the error will not be larger than 274 feet.

5.6 Circular Conversion Factors

The relationships of the circular standard error to other circular precision indexes [10] are summarized in Table 5-4. Conversion factors (Table 5-5) computed from these relationships convert a circular error at a given probability to a circular error at another probability. When a circular error distribution is substituted for an elliptical distribution, the circular conversion factors are retained.

Table 5-4

Summary of Circular Precision Indexes

Symbol	Probability	Derivation
σ_c	.3935	1.0000 σ_c
CEP	.5000	1.1774 σ_c
MSPE	.6321	1.4142 σ_c
CMAS	.9000	2.1460 σ_c
3.5 σ_c	.9978	3.5000 σ_c

Table 5-5

Circular Error Conversion Factors

From \ To	39.35%	50.00%	63.21%	90.00%	99.78%
39.35%	1.0000	1.1774	1.4142	2.1460	3.5000
50.00%	0.8493	1.0000	1.2011	1.8227	2.9726
63.21%	0.7071	0.8325	1.0000	1.5174	2.4749
90.00%	0.4660	0.5486	0.6590	1.0000	1.6309
99.78%	0.2857	0.3364	0.4040	0.6131	1.0000

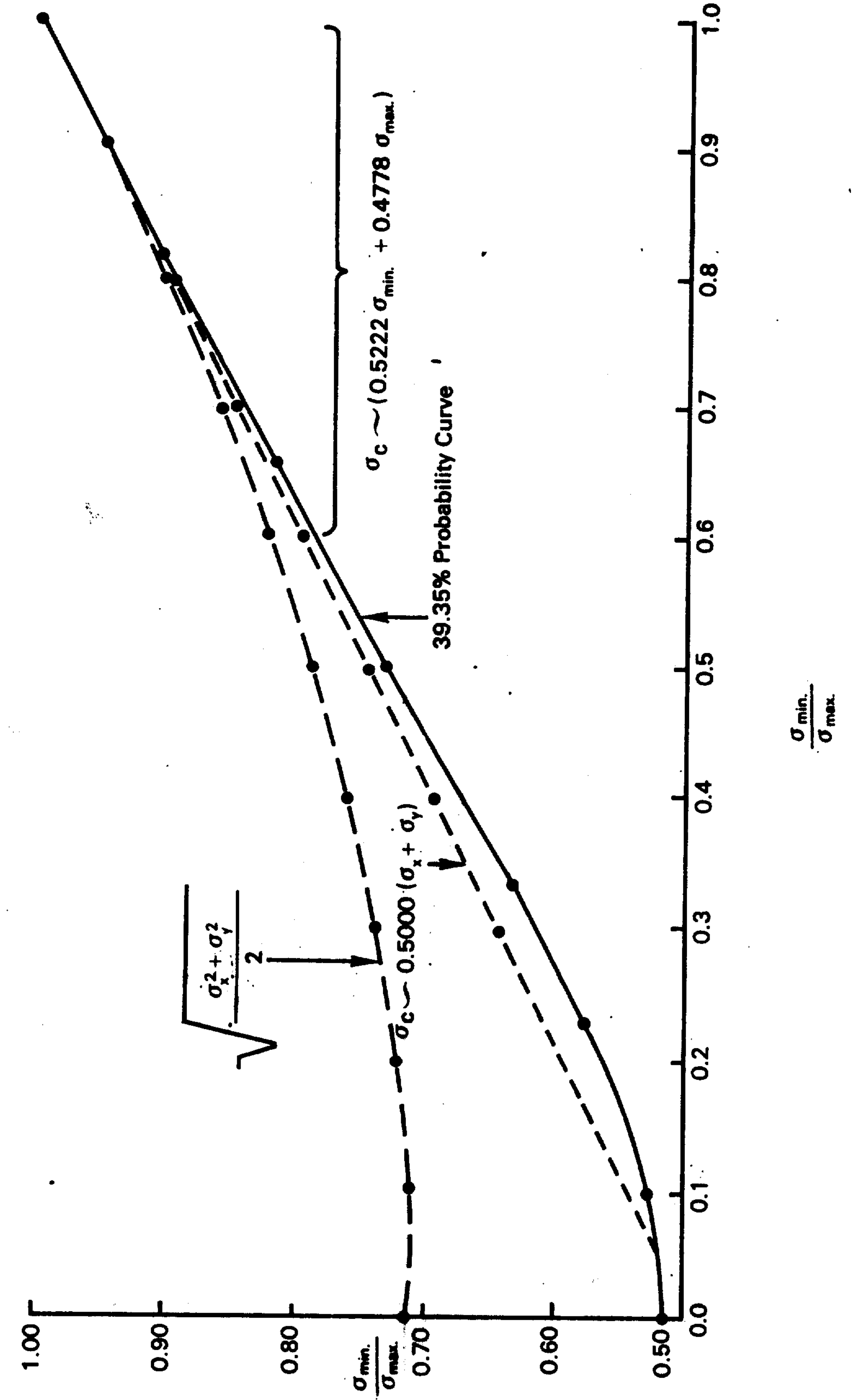


Figure 5-2. Curve of the P(R) Function When P(R) = 39.35%

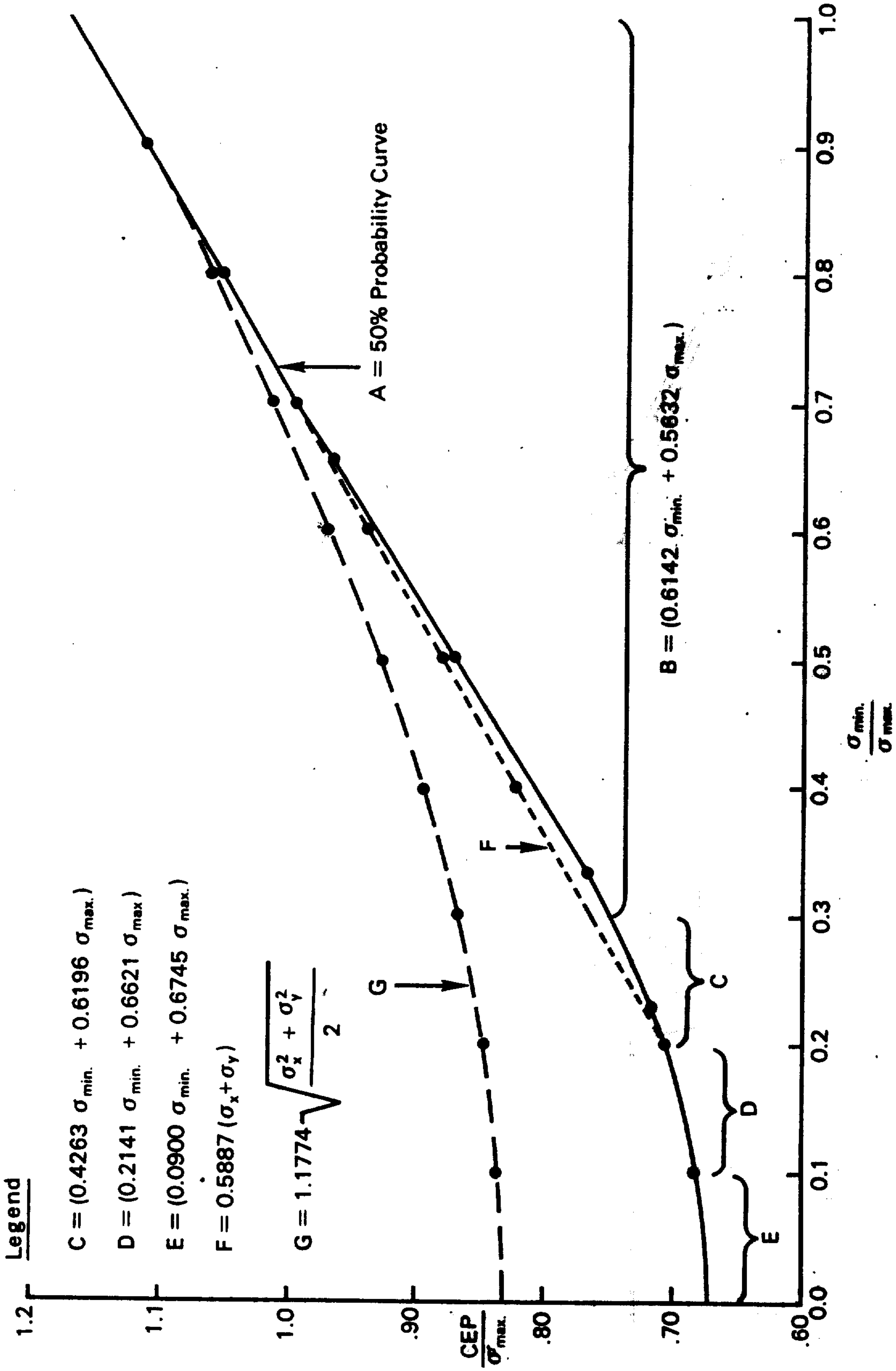


Figure 5-3. Curve of the P(R) Function When P(R) = 50.00%

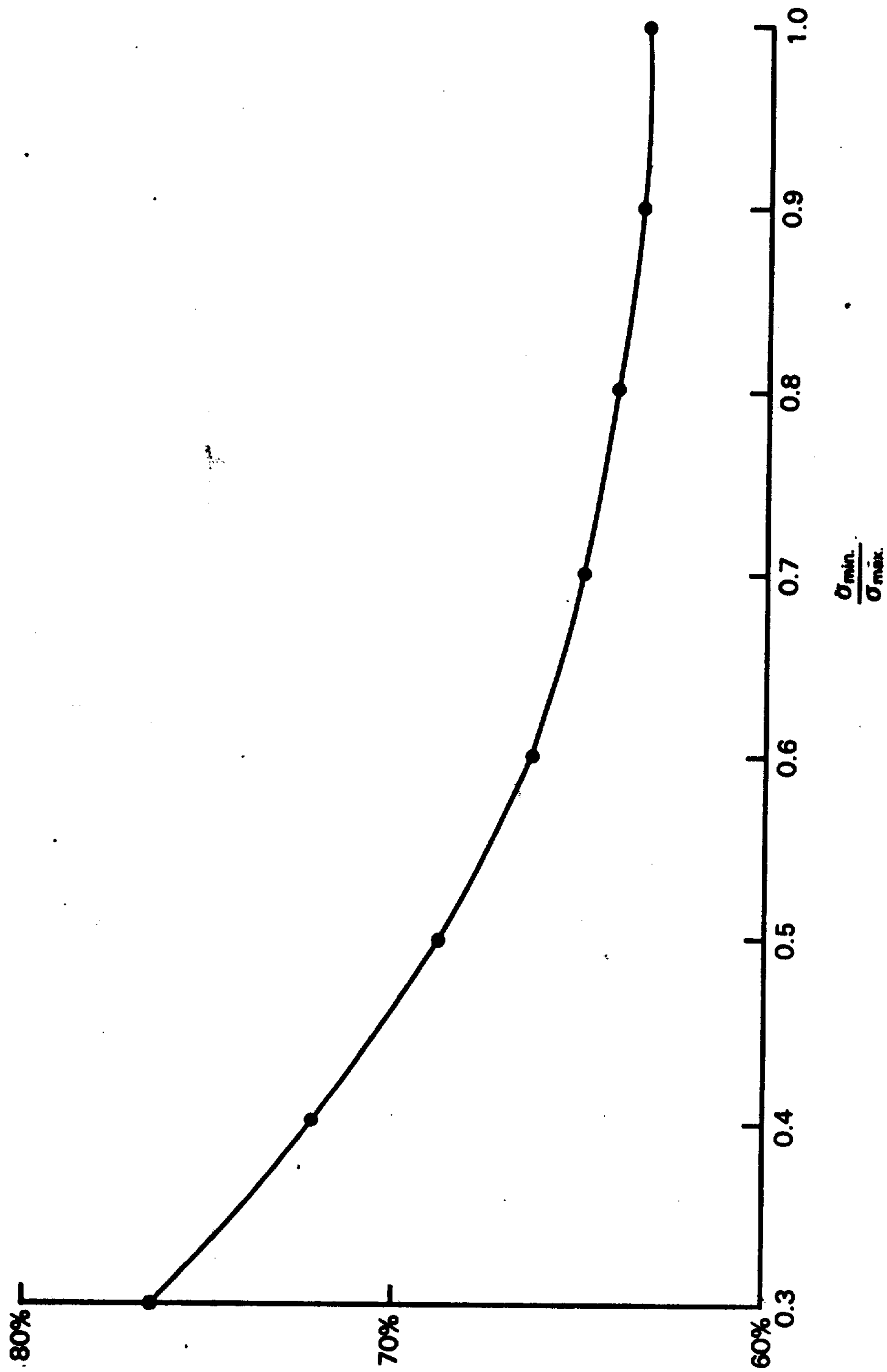


Figure 5–4. MSPE Probability Curve

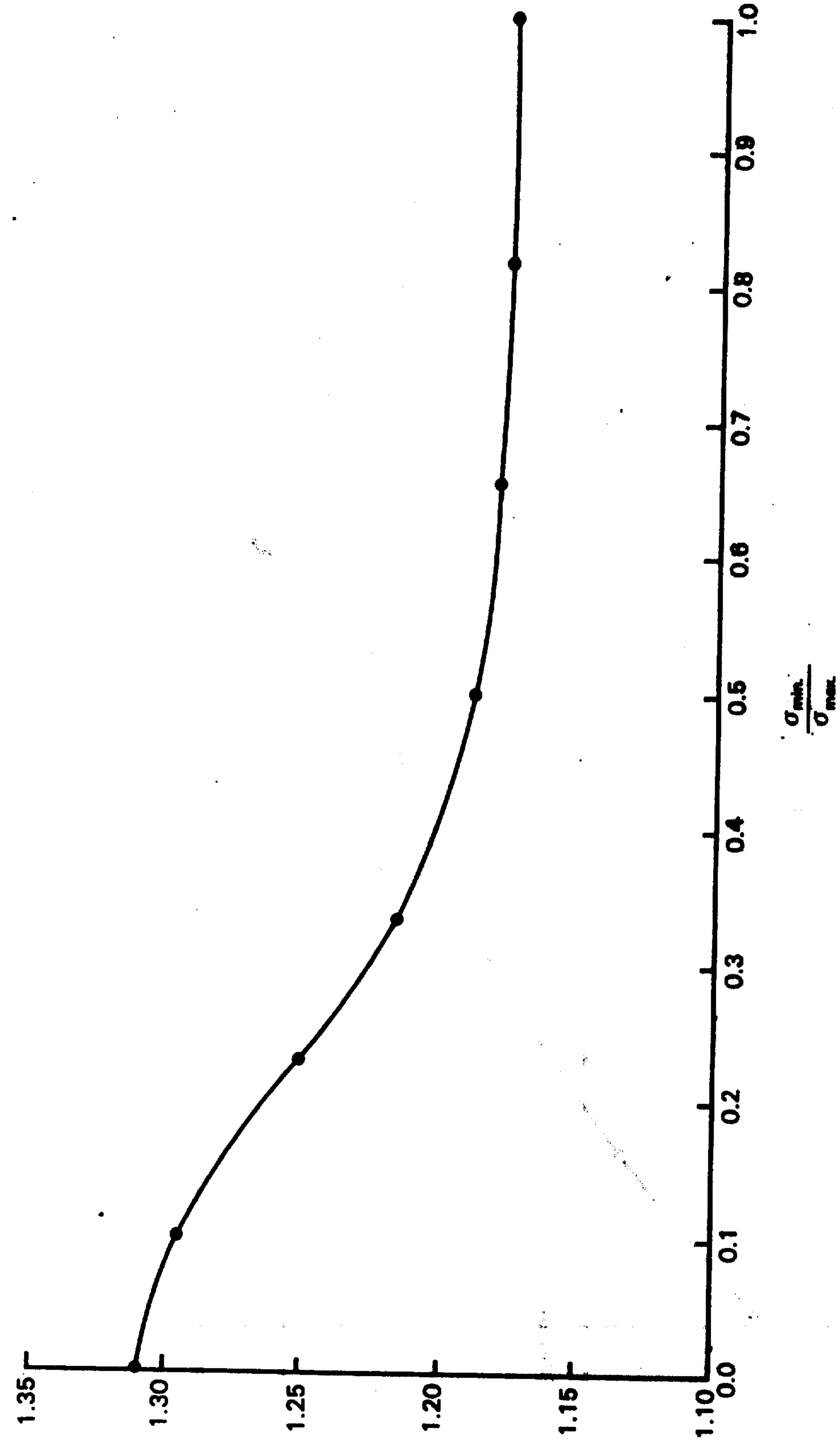


Figure 5-5. Graph of Conversion Factors for 39.35% to 50% Probability

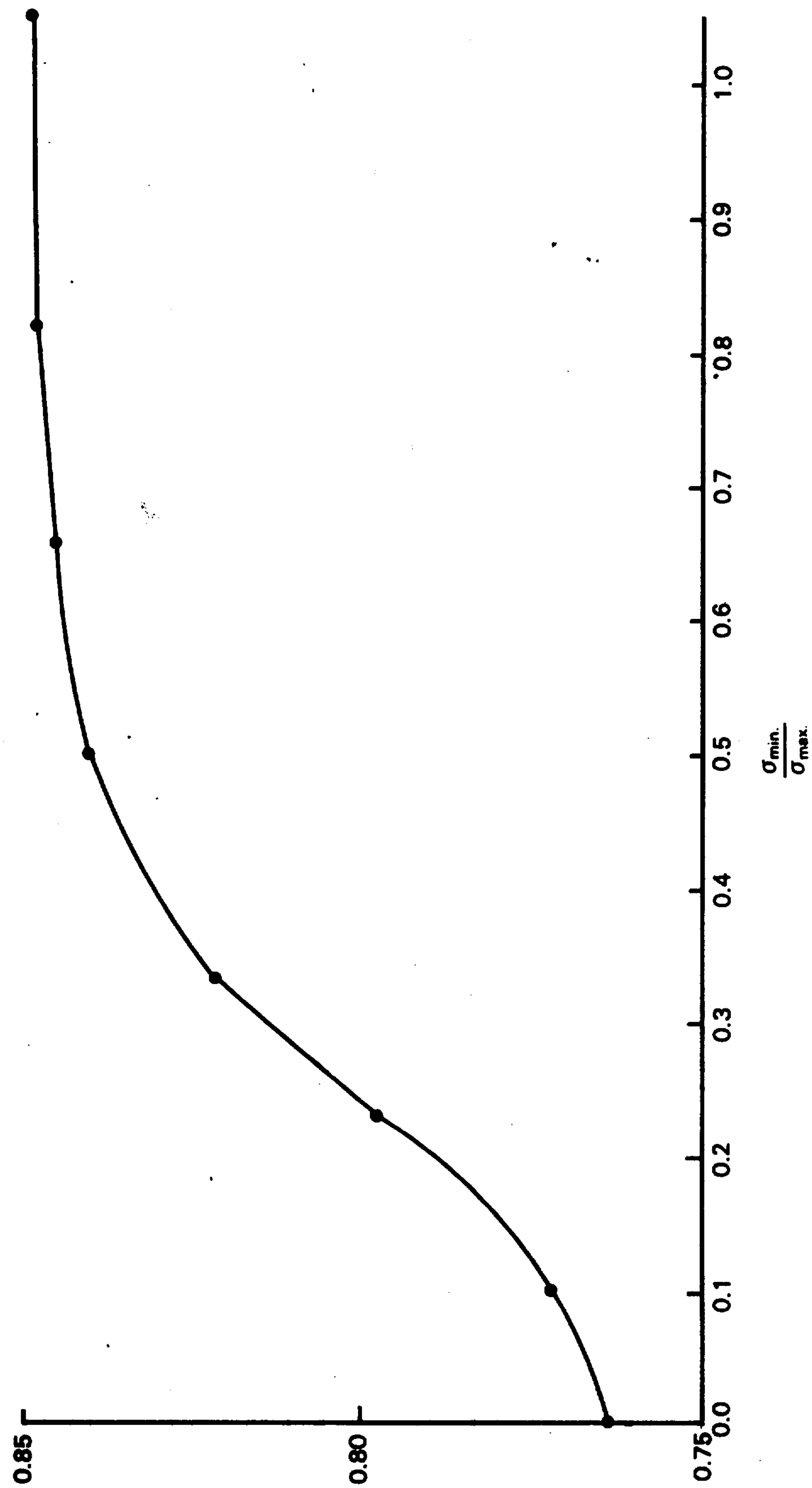


Figure 5-6. Graph of Conversion Factors for 50.00% to 39.35% Probability

6. THREE DIMENSIONAL (ELLIPSOIDAL, SPHERICAL) ERRORS

6.1 Introduction

A three-dimensional error is the error in a quantity defined by three random variables. Expanding on the example in Section 5.1., a point is referred to X, Y, and Z axes which establish the spatial position of the point. When random and independent, the errors x, y, and z each have a linear probability distribution. The three-dimensional probability density function is expressed by:

$$p(x,y,z) = \frac{1}{(2\pi)^{\frac{3}{2}} \sigma_x \sigma_y \sigma_z} e^{-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} + \frac{z^2}{\sigma_z^2}\right)} \tag{6-1}$$

Similar to Section 5.1, the probability density function can be written:

$$W^2 = \frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} + \frac{z^2}{\sigma_z^2} \tag{6-2}$$

where:

$$W^2 = -2 \ln \left(p(x,y,z) \sigma_x \sigma_y \sigma_z (2\pi)^{\frac{3}{2}} \right)$$

For values of the constant W^2 from $0 \rightarrow \infty$, the density function defines a family of ellipsoids of equal probability density.

6.2 Ellipsoidal Errors

The probability of an error ellipsoid is given [10] by the probability distribution function:

$$P(s) = \sqrt{\frac{2}{\pi}} \int_0^W t^2 e^{-\frac{1}{2} t^2} dt \tag{6-3}$$

where: s = the radial error;

$$s = \sqrt{x^2 + y^2 + z^2}$$

$$t = \frac{s}{\sigma_{rs}^{\frac{3}{2}}}$$

σ_{rs} = standard error of the radial error "s"

The solution of equation (6-3) for W yields the values given in Table 6-1.

Table 6-1
Values for the Constant W

Probability	W
19.9%	1.000
50	1.538
60.8	1.732
90	2.500
99	3.368
99.89	4.000

6.3 Spherical Probability Distribution Function

When $\sigma_x = \sigma_y = \sigma_z = \sigma_{rs} = \sigma_s$, equation (6-1) becomes the spherical probability density function [10]:

$$p(s) = \frac{1}{(2\pi)^{\frac{3}{2}} \sigma_s^3} e^{-\frac{s^2}{2\sigma_s^2}} \qquad (6\text{-}4)$$

where: σ_s = spherical standard error

Integrating p(s) from s = 0 to s = S, equation (6-4) becomes the spherical probability distribution function:

$$P(S) = \sqrt{\frac{2}{\pi}} \left[\left(\frac{S}{\sigma_s}\right) - e^{-\frac{s^2}{2\sigma_s^2}} + \int_0^S \frac{e}{s}^{-\frac{s^2}{2\sigma_s^2}} ds \right] \qquad (6\text{-}5)$$

where: S = radius of the probability sphere

Equation (6-5) can be solved by an approximation formula:

$$P(S) \sim \sqrt{\frac{2}{\pi}} \left(1.253 - Ce^{-\frac{C^2}{2}} - \frac{e^{-\frac{C^2}{2}}}{C + 0.8\, e^{-0.4C}} \right) \qquad (6\text{-}6)$$

where: $C = \frac{S}{\sigma_s}$

6.4 Spherical Precision Indexes

A spherical error distribution is represented by indexes [10] similar to those in Sections 4.4. and 5.4. Preferred spherical precision indexes include: (1) the spherical standard error (σ_s), (2) the spherical error probable (SEP), (3) the spherical accuracy standard (SAS), and (4) the spherical near-certainty error, four sigma ($4\sigma_s$). The mean radial spherical error (MRSE), an index which has been used at DMA, is not recommended because the probability represented varies when σ_x, σ_y, and σ_z are not equal.

The probability of an error sphere of radius equal to the spherical standard error is computed by equation (6-6) for the condition

$C = \frac{S}{\sigma_s} = 1$ as follows:

$$\sqrt{\frac{2}{\pi}} = 0.7978846$$

$$e^{-\frac{1}{2}} = 0.60653$$

$$e^{-0.4} = 0.67032$$

$$0.8e^{-0.4} = 0.53626$$

$$P(S) \sim 0.79788\ (1.253 - 0.6065 - 0.3948)$$

$$\therefore\ P(S) \sim 0.20 \text{ or } 20\% \qquad (6\text{-}7)$$

For a truly spherical distribution, the linear standard error are equal and identical to the spherical standard error ($\sigma_x = \sigma_y = \sigma_z \equiv \sigma_s$) . When σ_x, σ_y, and σ_z are not equal, the spherical standard error is approximated by:

$$\sigma_s \sim 0.333\ (\sigma_x + \sigma_y + \sigma_z)$$

when $\sigma_{min}/\sigma_{max}$ is between 1.0 and 0.35

The substitution of a spherical form for an ellipsoidal distribution is not recommended when the $\sigma_{min}/\sigma_{max}$ ratio is less than 0.35.

The following alternate method of approximating σ_s is not recommended because of limited applicability:

$$\sigma_s \sim \sqrt{\frac{\sigma_x^2 + \sigma_y^2 + \sigma_z^2}{3}} \qquad (6\text{-}8)$$

when $\sigma_{min}/\sigma_{max}$ is between 1.0 and 0.9

The <u>spherical error probable</u> is defined as the magnitude of the spherical radius S when the function P(S) = 0.5 or 50%. Expressed in the form $S = C\sigma_s$, the spherical probable error is computed by:

$$SEP = 1.5382\ \sigma_s \qquad (6\text{-}9)$$

The P(R) function for two-dimensional errors is solved by the use of Grad and Solomon's tables. Expanding this method into the spherical distribution, the radius S for a 50% probability sphere ($S_{50\%}$) was computed in terms of σ_{max} for ratios of $\sigma_{min}/\sigma_{max}$ and $\sigma_{mid}/\sigma_{max}$ and tabulated in Table 6-2. Utilizing these values, an approximation of the spherical probable error can be computed:

$$SEP \sim 0.5127\ (\sigma_x + \sigma_y + \sigma_z) \qquad (6\text{-}11)$$

when $\sigma_{min}/\sigma_{max}$ is between 1.0 and 0.35

The <u>mean radial spherical error</u> is the radius of the error sphere equal to $1.732\sigma_s$, or $\sqrt{3}\ \sigma_s$, in a truly spherical distribution. When $\sigma_x \neq \sigma_y \neq \sigma_z$, the MRSE is computed by:

$$MRSE \sim \sqrt{\sigma_x^2 + \sigma_y^2 + \sigma_z^2} \qquad (6\text{-}12)$$

when $\sigma_{min}/\sigma_{max}$ is between 1.0 and 0.9

The probabilities represented by the MRSE are computed by equation (6-6). Because of the variation in probability, the MRSE is not recommended for use as a precision index.

The spherical accuracy standard is defined as the magnitude of the spherical radius S when the function P(S) = 0.9 or 90%. Expressed in the form $S = C\ \sigma_S$, the spherical accuracy standard is computed by:

$$SAS = 2.500\sigma_S \qquad (6\text{-}13)$$

The four sigma error, representing a spherical probability of 99.89%, approaches near-certainty in a spherical distribution and has a magnitude four times that of the spherical standard error.

Table 6-2
Solution of P(S) Function for P(S) = 50.00%

$\sigma_{mid}/\sigma_{max}$	$\sigma_{min}/\sigma_{max}$	SEP ~ $S_{50\%}$	SEP ~ $0.512/(\sigma_x + \sigma_y + \sigma_z)$ Letting $\sigma_{max} = 1$
0.866	0.866	1.4016 σ_{max}	1.4007
1.0	0.707	1.3892 σ_{max}	1.3879
0.775	0.632	1.2341 σ_{max}	1.2341
0.577	0.577	1.1016 σ_{max}	1.1044
0.894	0.447	1.2104 σ_{max}	1.2002
0.707	0.408	1.0894 σ_{max}	1.0844
0.535	0.378	0.9791 σ_{max}	0.9808
0.354	0.354	0.8689 σ_{max}	0.8757

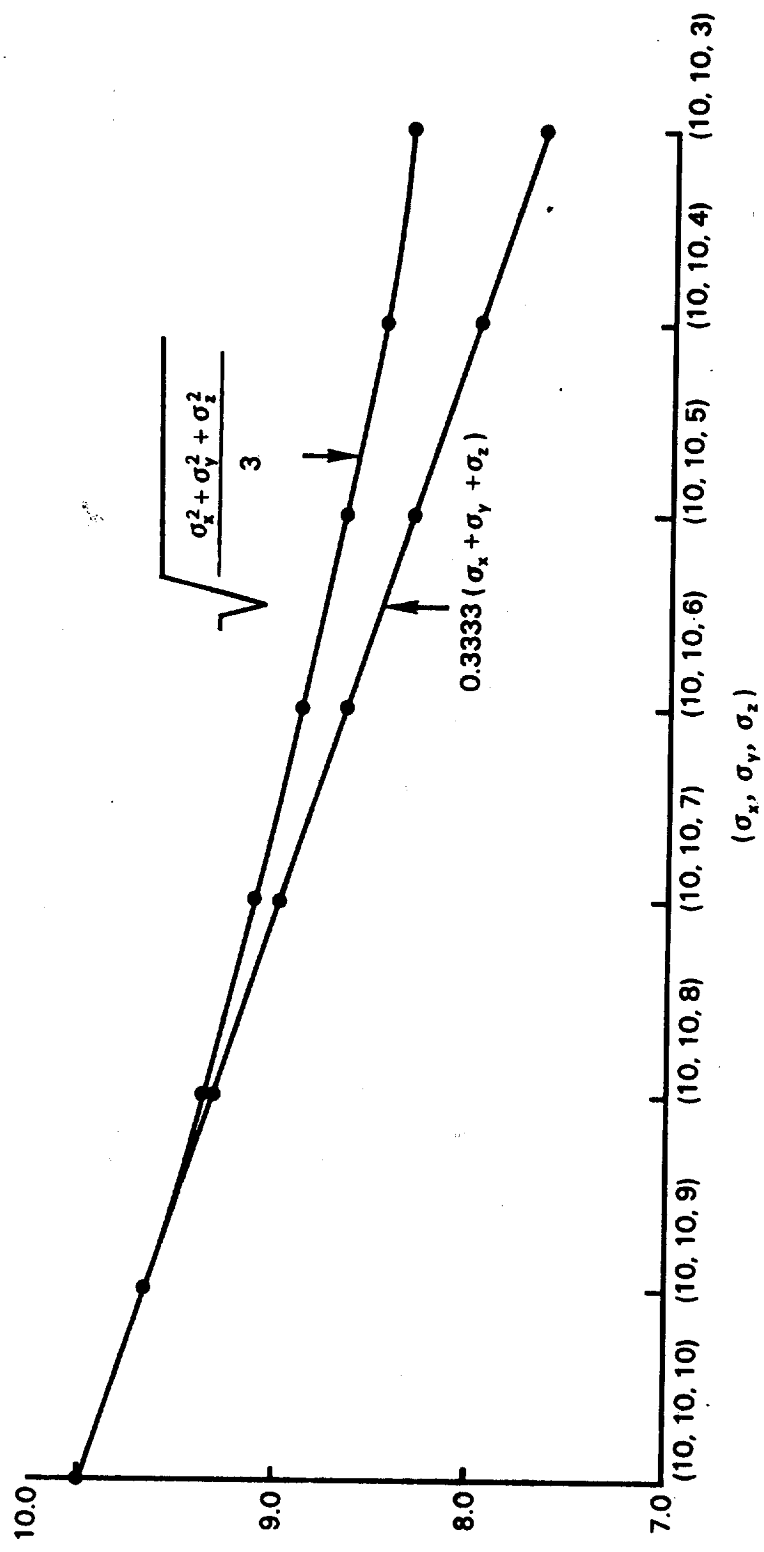


Figure 6-1. Comparison of Spherical Standard Error Approximation Methods

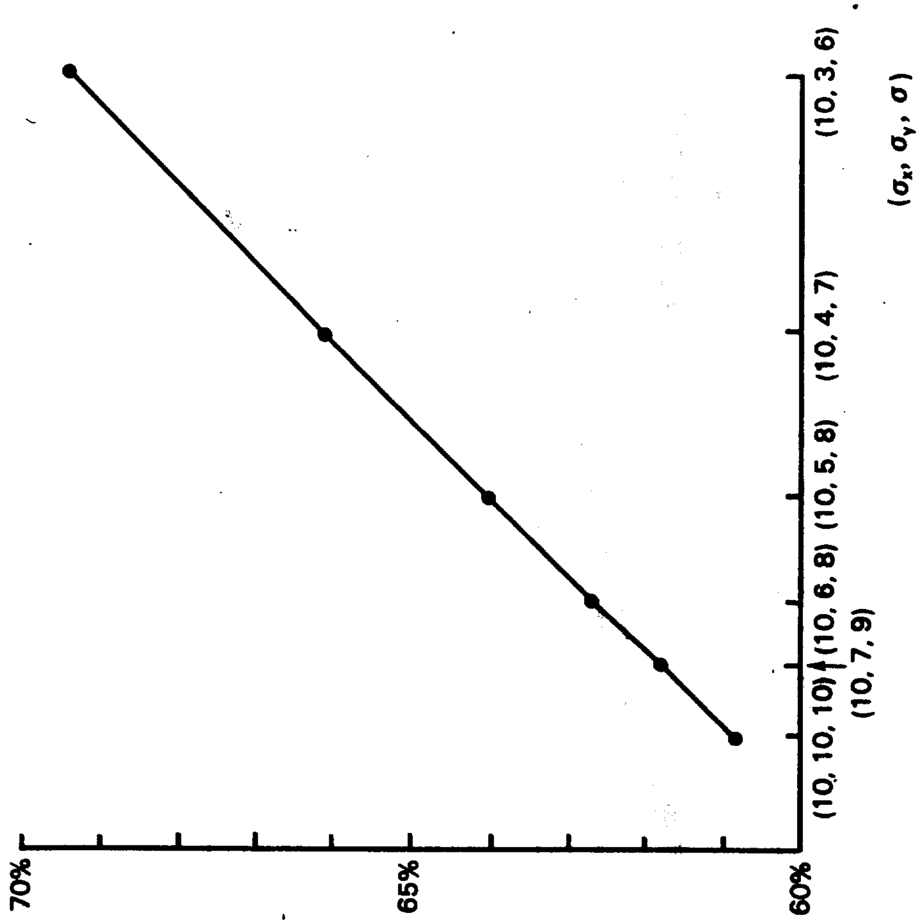


Figure 6–2 MRSE Probability Curve

6.5 Spherical Conversion Factors

The relationships of the spherical standard error to other spherical precision indexes [10] are summarized in Table 6-3. Conversion factors (Table 6-4) computed from these relationships convert a spherical error at a given probability to a spherical error at another probability.

Table 6-3
Summary of Spherical Precision Indexes

Symbol	Probability	Derivation
σ_s	.199	1.000 σ_s
SEP	.50	1.538 σ_s
MRSE	.608	1.732 σ_s
SAS	.90	2.500 σ_s
4 σ_s	.9989	4.000 σ_s

Table 6-4

Spherical Error Conversion Factors

From \ To	19.9%	50%	60.8%	90%	99.89%
19.9%	1.000	1.538	1.732	2.500	4.000
50%	0.650	1.000	1.126	1.625	2.600
60.8%	0.577	0.888	1.000	1.443	2.309
90%	0.400	0.615	0.693	1.000	1.600
99.89%	0.250	0.385	0.433	0.625	1.000

7. PROPAGATION OF ERRORS

7.1 Variance and Covariance Propagation

The propagation of errors [1] [2] is sometimes called the propagation of variances and covariances. Variances, the measure of dispersion of a random variable, has already been defined. The covariance is a measure of the mutual variation of two random variables. Covariance describes the correlation between two variables. Covariance is computed from the equation:

$$\sigma_{xy} = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \mu_x)(y_i - \mu_y) \tag{7-1}$$

where x_i, y_i are the observations of the variables x, y

μ_x, μ_y mean of the set of observations

n the number of observations

The covariance matrix is a square symmetric matrix of the variance and covariances of the random variables we are using. The matrix may be any dimension depending on the number of random variables.

$$\Sigma_x = \begin{bmatrix} \sigma^2_{x_1} & \sigma_{x_1 x_2} & \sigma_{x_1 x_3} \\ \sigma_{x_1 x_2} & \sigma^2_{x_2} & \sigma_{x_2 x_3} \\ \sigma_{x_1 x_3} & \sigma_{x_2 x_3} & \sigma^2_{x_3} \end{bmatrix} \tag{7-2}$$

If the random variables y_1, and y_z are linear functions of the random variables, x_1, and x_2, they may be written:

$$y_1 = a_0 + a_1 x_1 + a_2 x_2$$

$$y_2 = b_0 + b_1 x_1 + b_2 x_2$$

or written in matrix form:

$$\bar{y} = c + C\bar{x} \qquad (7\text{-}3)$$

where $\bar{y} = [y_1 \ y_2]^t$

$\bar{x} = [x_1 \ x_2]^t$

$c = [a_o \ b_o]^t$

$$C = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$$

It can be shown [2] that the covariance matrices are related by:

$$\Sigma_{yy} = C\Sigma_{xx} C^t \qquad (7\text{-}4)$$

or $$\begin{bmatrix} \sigma^2_{y_1} & \sigma_{y_1y_2} \\ \sigma_{y_1y_2} & \sigma^2_{y_2} \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \begin{bmatrix} \sigma^2_{x_1} & \sigma_{x_1x_2} \\ \sigma_{x_1x_2} & \sigma^2_{x_2} \end{bmatrix} \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

The matrix C is called the Jacobian of $\bar{y}$ with respect to $\bar{x}$, denoted by J_{yx}.

$$J_{yx} = \frac{\partial \bar{y}}{\partial \bar{x}} \qquad (7\text{-}5)$$

It should be noted that even if there is no covariance in $\bar{x}$, that is

$$\sigma_{x_1 x_2} = 0 ,$$

there may still be a covariance in $\bar{y}$.

7.2 Propagation Through Known Functions

Another method for the propagation of errors [1] is the propagation of known errors through known functions. This method assumes the errors are independent and known. The equations for this propagation are derived and shown in Appendix C. A quantity, x, is computed from two measured quantities u and v, where x = f(u,v) denotes a function of u and v. The error σ_x of x is affected by the errors in both u and v. When σ_u and σ_v are randomly distributed, the propagated error, σ_x , can be computed by the general equation:

$$\sigma_x = \sqrt{\left(\frac{\partial x}{\partial u}\right)^2 \sigma_u^{\,2} + \left(\frac{\partial x}{\partial v}\right)^2 \sigma_v^{\,2}} \qquad (7\text{-}6)$$

where: σ_x = the standard error of x

σ_u, σ_v = the standard errors of u and v

$\frac{\partial x}{\partial u}$, $\frac{\partial x}{\partial u}$ = partial derivatives of x, with respect to u and v.

Application of the general equation to specific conditions produces the following rules for the function f(u,v):

Rule 1. Addition and Subtraction

$$x = au \pm bv$$

$$\sigma_x = \sqrt{a^2 \sigma_u^{\,2} + b^2 \sigma_v^{\,2}} \qquad (7\text{-}7)$$

Rule 2. Multiplication and Division

$$x = \pm auv$$

$$\frac{\sigma_x}{x} = \sqrt{\frac{\sigma_u^{\,2}}{u^2} + \frac{\sigma_u^{\,2}}{v^2}} \qquad (7\text{-}8)$$

Rule 3. Powers

$$x = u^a v^b$$

$$\frac{\sigma_x}{x} = \sqrt{a^2 \left(\frac{\sigma_u}{u}\right)^2 + b^2 \left(\frac{\sigma_v}{v}\right)^2} \qquad (7\text{-}9)$$

Indexes other than the standard error can be used to propagate errors. For example, using Rule 1:

$$(PE)_x = \sqrt{(PE)_u^2 + (PE)_v^2}$$

$$\eta_x = \sqrt{\eta_u^2 + \eta_v^2}$$

$$\sigma_x = \sqrt{\sigma_u^2 + \sigma_v^2}$$

$$(MAS)_x = \sqrt{(MAS)_u^2 + (MAS)_v^2}$$

$$(3\sigma)_x = \sqrt{(3\sigma)_u^2 + (3\sigma)_v^2}$$

However, note that the index must be consistent throughout the formula. Similarly precision indexes in two and three dimensions may be propagated. For example:

$$\sigma_{c_x} = \sqrt{\sigma_{c_u}^2 + \sigma_{c_v}^2 + \sigma_{c_w}^2} \; .$$

7.3 Application of Error Propagation Methods

The principle source of covariance information to be used in the propagation of errors is least squares adjustment. In the case of positional information for DMA products these adjustments may be analytical photogrammetric triangulation. In strict terms, the inverse matrix (N^{-1}) is not a covariance matrix, but a cofactor matrix usually denoted by Q.

$$Q = N^{-1}$$

A cofactor matrix is related to the covariance matrix by a scalar multiplier.

$$Q = k\Sigma$$

where $k = \frac{1}{\sigma_0^2}$,

and σ_0^2 is called the reference variance.

It should be pointed out that only random errors will be propagated by:

$$\Sigma_y = J_{yx} Q \, J^T_{yx} \quad .$$

Hence this will be a measure of the precision of the adjustment.

Another method is called error propagation from sample statistics. This method evaluates the product by comparison to diagnostic control points. This method will take into account biases in the product, random errors in the product, random errors in the diagnostic control and random error in measuring the control on the product. In this method a 6 x 6 cross-covariance matrix is computed. Let this matrix be denoted by Q.

$$Q = \begin{bmatrix} \sigma^2_{\phi_1} & \sigma_{\phi_1\lambda_1} & \sigma_{\phi_1 h_1} & \sigma_{\phi_1\phi_2} & \sigma_{\phi_1\lambda_2} & \sigma_{\phi_1 h_2} \\ \sigma_{\phi_1\lambda_1} & \sigma^2_{\lambda_1} & \sigma_{\lambda_1 h_1} & \sigma_{\lambda_1\phi_2} & \sigma_{\lambda_1\lambda_2} & \sigma_{\lambda_1 h_2} \\ \sigma_{\phi_1 h_1} & \sigma_{\lambda_1 h_1} & \sigma^2_{h_1} & \sigma_{h_1\phi_2} & \sigma_{h_1\lambda_2} & \sigma_{h_1 h_2} \\ \sigma_{\phi_1\phi_2} & \sigma_{\lambda_1\phi_2} & \sigma_{h_1\phi_2} & \sigma^2_{\phi_2} & \sigma_{\phi_2\lambda_2} & \sigma_{\phi_2 h_2} \\ \sigma_{\phi_1\lambda_2} & \sigma_{\lambda_1\lambda_2} & \sigma_{h_1\lambda_2} & \sigma_{\phi_2\lambda_2} & \sigma^2_{\lambda_2} & \sigma_{\lambda_2 h_2} \\ \sigma_{\phi_1 h_2} & \sigma_{\lambda_1 h_2} & \sigma_{h_1 h_2} & \sigma_{\phi_2 h_2} & \sigma_{\lambda_2 h_2} & \sigma^2_{h_2} \end{bmatrix} \qquad (7\text{-}10)$$

Q should include the effects of residuals from the adjustment, the covariance of the control used as diagnostics, a covariance of the measurement errors, and the covariance of the comparisons. The points (ϕ_1, λ_1, h_1) (ϕ_2, λ_2, h_2) should be chosen in pairs separated by the distance over which the point-to-point relative accuracy is desired.

To compute the absolute accuracy, consider the 6 x 6 Q matrix to be partitioned such that:

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q^T_{12} & Q_{22} \end{bmatrix} . \qquad (7\text{-}11)$$

The absolute accuracy may be determined from Q_{11} or Q_{22}. The units of the matrix should all be meters. Assume that the horizontal and vertical components of Q_{11} are independent but the latitude and longitude may be correlated. That is:

$$Q_{11} = \begin{bmatrix} \sigma^2_\phi & \sigma_{\phi\lambda} & 0 \\ \sigma_{\phi\lambda} & \sigma^2_\lambda & 0 \\ 0 & 0 & \sigma^2_h \end{bmatrix} \qquad (7\text{-}12)$$

The 90% LE or MAS value for the vertical accuracy is computed from:

$$90\%\ LE = 1.6449 \left| \sqrt{\sigma^2_h} \right| . \qquad (7\text{-}13)$$

The factor 1.6449 is from Table 4.1 relating the one sigma error to the 90% error.

To compute the circular accuracy in this case, we will consider the latitude and longitude as correlated. The equations to relate these variances and covariances to two independent variables, u,v, in the horizontal plane are:

by eigenvalues

$$\sigma^2_u = \frac{1}{2} (\sigma_\phi^2 + \sigma_\lambda^2) + \sqrt{\frac{1}{4} (\sigma^2_\phi + \sigma^2_\lambda)^2 + (\sigma_{\phi\lambda})^2}$$

and

$$\sigma_v^2 = \frac{1}{2} (\sigma^2_\phi + \sigma_\lambda^2) - \sqrt{\frac{1}{4} (\sigma_\phi^2 + \sigma_\lambda^2)^2 + (\sigma_{\phi\lambda})^2} \qquad (7\text{-}14)$$

Now

$$\sigma_c = \frac{1}{2} (\sigma_u + \sigma_v)$$

and

$$90\%\ CE = 2.146\ \sigma_c$$

for $.2 \leq \sigma_v/\sigma_u \leq 1$.

To compute the relative accuracy using the 6 x 6 covariance matrix Q, change the variables such that:

$$\Delta\phi = \phi_2 - \phi_1$$

$$\Delta\lambda = \lambda_2 - \lambda_1$$

$$\Delta h = h_2 - h_2 .$$

The point-to-point covariance matrix will be:

$$Q_{pp} = JQJ^T \qquad (7\text{-}15)$$

where

$$J = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}$$

Assuming the horizontal and vertical components are independent,

$$Q_{pp} = \begin{bmatrix} \sigma^2_{\Delta\phi} & \sigma_{\Delta\phi\Delta\lambda} & 0 \\ \sigma_{\Delta\phi\Delta\lambda} & \sigma^2_{\Delta\lambda} & 0 \\ 0 & 0 & \sigma^2_{\Delta h} \end{bmatrix}. \qquad (7\text{-}16)$$

From here the horizontal circular relative error and vertical linear relative error are determined with the same formulas used for the absolute accuracies.

8. APPLICATION OF ERROR THEORY TO POSITIONAL INFORMATION

8.1 Positional Errors

By the use of error theory in the evaluation of positional information, it is possible to establish a meaningful accuracy subject to uniform interpretation. To provide a logical and acceptable basis for computation and comparison, positional errors are assumed to follow a normal distribution. This assumption is valid because positional error components generally follow a normal distribution pattern when sufficient data is available.

The statistical treatment of errors is applied to measureable quantities found in the sources of positional information. The differences between diagnostic surveyed values of ground control (not used in the photogrammetric adjustment) and the coordinates of that control from a photogrammetric adjustment are considered to be the errors in the photogrammetric adjustment. Analysis of the linear components, latitude and longitude expressed in meters, provide a two-dimensional expression for the accuracy of the adjustment. When all the linear standard errors occurring in the adjustment are combined and converted to a circular distribution, as described in the Chapter 7 on error propagation, it will be part of the accuracy statement for the product. In the case of maps, the final product accuracy will be based on this adjustment error and the errors associated with map construction. In the case of digital data, the accuracy statement contained in the header record is for any single data point in the cell. Although the accuracy is based on comparing diagnostic points to data interpolated from the digital data, the accuracy does not include any interpolation caused errors of the user.

Because of the different sources contributing to the accuracy of various DMA products, it is important for the user to carefully choose the right product to fulfill his positioning requirements. If there is a requirement for high absolute accuracy for point positioning, the recommended product is the Point Positioning Data Base. This product allows positioning directly from a data base that avoids the errors caused by map construction and the representation of the positioning data on a paper copy.

The statement [12] for a map of descriptor 1 is that 90 percent of all well-defined planimetric features are located within 0.5mm (0.02 inches) of their geographic position with reference to a prescribed datum. Table 8-1 gives the ground distance in feet of 0.02 inches on a map for various scales of DMA maps. This number forms a limit for the 90 percent accuracy of a map at the given scale. These limiting case accuracies for descriptor 1 products should not be confused with general product accuracy which will usually be less accurate.

Among the positioning errors in photogrammetric adjustments or on maps, there are often those which are not measureable or able to be modeled and which must be estimated by empirical methods. When this is necessary, an additional assumption must be made to the effect that such data is compatible with the computed data and that the empirically derived error data will also follow the theoretical normal error distribution.

Table 8-1
Ground Distance Equivalent to
0.02 of an Inch at Chart Scale

Chart	Scale	Distance (feet)
ONC	1:1,000,000	1667
TPC/PC	1:500,000	833
JOG	1:250,000	417
ATC	1:200,000	333
Topo	1:100,000	167
Topo	1: 50,000	83

Various types of points require different parameters to establish precise positions. These have been discussed as one, two, or three dimensional coordinates. For example, a vertical position (elevation) requires only a one-dimensional coordinate - the height of the point above a reference datum; a geodetic position is expressed by two-dimensional coordinates - latitude and longitude referenced to a specific datum; and spatial positions require three-dimensional coordinates such as the x, y, z coordinates in a rectangular system. The errors accumulated in the process of determining the various positions must be evaluated in the same dimensions required to express the position. Errors for vertical positioning can be assumed to follow a normal linear distribution; those for a geodetic position - a circular distribution; and the errors for a spatial point can be assumed to follow a normal spherical distribution.

8.2 The Accuracy Statement

Three major groups of data fall within Department of Defense positioning requirements: (1) maps, charts and other graphics; (2) data bases of digital data such as Digital Terrain Elevation Data (DTED); and (3) specific points. By the use of error theory, a horizontal accuracy evaluation of the DMA positioning product as a whole can be obtained. In the case of maps or graphics, a specified, probability (90 percent) that the true errors in well-defined planimetry will not exceed, is used as an accuracy statement for the map. Map accuracy can also be interpreted as a percentage - the percentage of well-defined points which contain errors not exceeding the given magnitude. Similarly, vertical accuracy is stated as a 90 percent probability that the linear errors in vertical position will not exceed a specific value. DMA also expresses the accuracy of its digital data bases in terms of 90 percent probability. For some data bases, for instance Digital Terrain Elevation Data (DTED), the header record contains both absolute and point-to-point (relative) error values. These accuracies are given as 90 percent horizontal (circular) and 90 percent vertical (linear) errors. The accuracy statement does not mean that the error in position is exactly the value shown, rather it expresses the probability that the true error in a given position will not be larger than the error given.

Positional error should be expressed by precision indexes which immediately identify the form and probability represented by a given error. The precision index chosen should be that which has been conventionally used for the type of data represented. For instance, the 90 percent probability used at DMA for map errors. Another example is the error in position associated with artillery or missile testing, or weapons delivery. In these cases the errors have conventionally been expressed as circular error probable (CEP). Using error theory and the factors given in this report, the different accuracies and precision indexes can be related. For example, if an accuracy is given by a weapons analyst as 100 feet CEP, we know the form of the error is circular and the probability is 50 percent. This implies a 50-50 chance that the geodetic position in question does not vary more than 100 feet from the true geodetic position. If we are interested in the equivalant error at the 90 percent probability level, multiply 100 ft. by 1.8227 (from Table 5-5) to yield a 90 percent probability that the positional error will not exceed 182 ft.

Errors in different forms are more easily understood when precision indexes common to linear, circular, and spherical error distributions are used. Precision indexes suitable for expressing positional error include (1) the linear, circular, and spherical standard errors representing 68.27%, 39.35%, and 19.9% probabilities, respectively, (2) the linear probable error, circular error probable, and spherical error probable representing 50% probability in each distribution, (3) the map accuracy standard, circular map accuracy standard, and spherical accuracy standard representing a 90% probability level, and (4) a probability level approaching near-certainty for each distribution which the positional error is theoretically unlikely to exceed; (a) three sigma (linear, 99.73%), (b) three-five sigma (circular, 99.78%), and (c) four sigma (spherical, 99.89%). Since error values are easily converted from one precision index to another in the same distribution, the use of any index is largely a matter of choice. However, in presenting positional information, the position error is best expressed by the precision index that is conventional for the type of positional error being described.

Summary of Formulas and Conversion Factors

Linear Error Formulas			
Precision Index	Symbol	Percentage Probability	Formula
Standard Error	σ	68.27%	$\sigma_x = \sqrt{\frac{\sum (X_i - \overline{X})^2}{n-1}} = \sqrt{\frac{\sum x^2}{n-1}}$ where: X_i = measured value of the quantity X; X_1, X_2 ... X_n $\overline{X}$ = the most probable value (arithmetic mean) of X $\overline{X} = \frac{\sum X_i}{n}$ x = the error; $x = X_i - \overline{X}$ n = number of measurements
Probable Error	PE	50%	$PE = 0.6745 \ \sigma_x$
Map Accuracy Standard	MAS	90%	$MAS = 1.6449 \ \sigma_X$
Near-Certainty Error (Three sigma)	3 σ	99.73%	$3.0000 \ \sigma_x$

Linear Error Conversion Factors				
From \ To	50%	68.27%	90%	99.73%
50%	1.0000	1.4826	2.4387	4.4475
68.27	0.6745	1.0000	1.6449	3.0000
90	0.4101	0.6080	1.0000	1.8239
99.73	0.2248	0.3333	0.5483	1.0000

Circular Error Formulas			
Precision Index	Symbol	Percentage Probability	Formula
Circular Standard Error	σ_c	39.35%	$\sigma_c = 0.5000\ (\sigma_x + \sigma_y)$, when $\sigma_{min}/\sigma_{max} \geq 0.2$
Circular Error Probable	CEP	50%	$CEP = 1.1774\ \sigma_c$ $CEP = 0.5887\ (\sigma_x + \sigma_y)$ when $\sigma_{min}/\sigma_{max} \geq 0.2$ $CEP \sim (0.2141\ \sigma_{min} + 0.6621\ \sigma_{max})$ when $0.1 \leq \sigma_{min}/\sigma_{max} \leq 0.2$ $CEP \sim (0.0900\ \sigma_{min} + 0.6745\ \sigma_{max})$ when $0.0 \leq \sigma_{min}/\sigma_{max} \leq 0.1$
Circular Map Accuracy Standard	CMAS	90%	$CMAS = 2.1460\ \sigma_c$ $CMAS = 1.0730\ (\sigma_x + \sigma_y)$ when $\sigma_{min}/\sigma_{max} \geq 0.2$
Circular Near-Certainty Error (Three-five sigma)	$3.5\ \sigma_c$	99.78%	$3.5000\ \sigma_c$

Circular Error Conversion Factors					
From \ To	39.35%	50%	63%	90%	99.78%
39.35%	1.0000	1.1774	1.4142	2.1460	3.5000
50	0.8493	1.0000	1.2011	1.8227	2.9726
63	0.7071	0.8325	1.0000	1.5174	2.4749
90	0.4660	0.5486	0.6590	1.0000	1.6309
99.78	0.2857	0.3364	0.4040	0.6131	1.0000

Spherical Error Formulas			
Precision Index	Symbol	Percentage Probability	Formula
Spherical Standard Error	σ_s	19.9%	$\sigma_s' = 0.3333(\sigma_x + \sigma_y + \sigma_z)$ when $\sigma_{min}/\sigma_{max} \geq 0.35$
Spherical Error Probable	SEP	50%	SEP = $1.5382\ \sigma_s$
			SEP = $0.5127\ (\sigma_x + \sigma_y + \sigma_z)$ when $\sigma_{min}/\sigma_{max} \geq 0.35$
Spherical Accuracy Standard	SAS	90%	SAS = $2.5003\ \sigma_s$
Spherical Near-Certainty Error (Four sigma)	$4\sigma_s$	99.89%	$4.0000\ \sigma_s$

Spherical Error Conversion Factors					
From \ To	19.9%	50%	61%	90%	99.89%
19.9%	1.000	1.538	1.732	2.500	4.000
50	0.650	1.000	1.126	1.625	2.600
61	0.577	0.888	1.000	1.443	2.309
90	0.400	0.615	0.693	1.000	1.600
99.89	0.250	0.385	0.433	0.625	1.000

Appendix A

PERCENTAGE PROBABILITY FOR STANDARD ERROR INCREMENTS

The following table presents the increments of linear (σ_X), circular (σ_C), and spherical (σ_S) standard errors for intervals of one percent probability. Percentage levels corresponding to precision indexes are underlined.

Factors for converting the error at one percentage probability to another within the same distribution are derived by dividing the standard error increment of the new percentage probability by the standard error increment of the given percentage probability. An example is the conversion from the circular map accuracy standard (90%) to the circular probable error (50%):

$$CEP = 1.1774\ \sigma_C$$

$$CMAS = 2.1460\ \sigma_C$$

$$CEP = \frac{1.1774}{2.1460}\ CMAS$$

$$\therefore\ CEP = 0.5486\ CMAS$$

TABLE A-1

Conversion Factors for Converting Standard Errors to Various Different Percentage Probabilities

%	σ_X	σ_C	σ_S
00	0.0000	0.0000	0.0000
01	0.0125	0.1418	0.3389
02	0.0251	0.2010	0.4299

TABLE A-1 (cont'd.)

%	σ_X	σ_C	σ_S
03	0.0376	0.2468	0.4951
04	0.0502	0.2857	0.5479
05	0.0627	0.3203	0.5932
06	0.0753	0.3518	0.6334
07	0.0878	0.3810	0.6699
08	0.1004	0.4084	0.7035
09	0.1130	0.4343	0.7349
10	0.1257	0.4590	0.7644
11	0.1383	0.4828	0.7924
12	0.1510	0.5056	0.8192
13	0.1637	0.5278	0.8447
14	0.1764	0.5492	0.8694
15	0.1891	0.5701	0.8932
16	0.2019	0.5905	0.9162
17	0.2147	0.6105	0.9386
18	0.2275	0.6300	0.9605
19	0.2404	0.6492	0.9818
19.9			1.0000
20	0.2533	0.6680	1.0026
21	0.2663	0.6866	1.0230
22	0.2793	0.7049	1.0430
23	0.2924	0.7230	1.0627
24	0.3055	0.7409	1.0821
25	0.3186	0.7585	1.1012
26	0.3319	0.7760	1.1200
27	0.3451	0.7934	1.1386
28	0.3585	0.8106	1.1570
29	0.3719	0.8276	1.1751
30	0.3853	0.8446	1.1932
31	0.3989	0.8615	1.2110
32	0.4125	0.8783	1.2288
33	0.4261	0.8950	1.2464
34	0.4399	0.9116	1.2638
35	0.4538	0.9282	1.2812
36	0.4677	0.9448	1.2985
37	0.4817	0.9613	1.3158
38	0.4959	0.9778	1.3330
39	0.5101	0.9943	1.3501
39.35		1.0000	
40	0.5244	1.0108	1.3672
41	0.5388	1.0273	1.3842
42	0.5534	1.0438	1.4013
43	0.5681	1.0603	1.4183
44	0.5828	1.0769	1.4354
45	0.5978	1.0935	1.4524

TABLE A-1 (cont'd)

%	σ_x	σ_c	σ_s
46	0.6128	1.1101	1.4695
47	0.6280	1.1268	1.4866
48	0.6433	1.1436	1.5037
49	0.6588	1.1605	1.5209
50	0.6745	1.1774	1.5382
51	0.6903	1.1944	1.5555
52	0.7063	1.2116	1.5729
53	0.7225	1.2288	1.5904
54	0.7388	1.2462	1.6080
55	0.7554	1.2637	1.6257
56	0.7722	1.2814	1.6436
57	0.7892	1.2992	1.6616
57.51	0.7979		
58	0.8064	1.3172	1.6797
59	0.8239	1.3354	1.6980
60	0.8416	1.3537	1.7164
60.82			1.7321
61	0.8596	1.3723	1.7351
62	0.8779	1.3911	1.7540
63	0.8965	1.4101	1.7730
63.21		1.4142	
64	0.9154	1.4294	1.7924
65	0.9346	1.4490	1.8119
66	0.9542	1.4689	1.8318
67	0.9741	1.4891	1.8519
68	0.9945	1.5096	1.8724
68.27	1.0000		
69	1.0152	1.5305	1.8932
70	1.0364	1.5518	1.9144
71	1.0581	1.5735	1.9360
72	1.0803	1.5956	1.9580
73	1.1031	1.6182	1.9804
74	1.1264	1.6414	2.0034
75	1.1503	1.6651	2.0269
76	1.1750	1.6894	2.0510
77	1.2004	1.7145	2.0757
78	1.2265	1.7402	2.1012
79	1.2536	1.7667	2.1274
80	1.2816	1.7941	2.1544
81	1.3106	1.8225	2.1825
82	1.3408	1.8519	2.2114
83	1.3722	1.8825	2.2416
84	1.4051	1.9145	2.2730
85	1.4395	1.9479	2.3059

TABLE A-1 (cont'd.)

%	σ_X	σ_C	σ_S
86	1.4758	1.9830	2.3404
87	1.5141	2.0200	2.3767
88	1.5548	2.0593	2.4153
89	1.5982	2.1011	2.4563
90	1.6449	2.1460	2.5003
91	1.6954	2.1945	2.5478
92	1.7507	2.2475	2.5998
93	1.8119	2.3062	2.6571
94	1.8808	2.3721	2.7216
95	1.9600	2.4477	2.7955
96	2.0537	2.5373	2.8829
97	2.1701	2.6482	2.9912
98	2.3263	2.7971	3.1365
99	2.5758	3.0349	3.3683
99.73	3.0000		
99.78		3.5000	
99.89			4.0000
99.9	3.2905	3.7169	4.0345
99.99	3.8905	4.2919	4.6094

Appendix B
THE MOST PROBABLE VALUE

Since the true value of a measured quantity is never known, the most probable value of the quantity must be determined from the observed values. The following proof [4] will show that the arithmetic mean of the observed values is the most probable value of the quantity:

Symbols:

X = an unknown quantity

X_i = the observed values of the unknown quantity;

$$X_i = X_1, X_2, X_3 \ldots X_n \qquad (1)$$

$\overline{X}$ = the arithmetic mean of the observed values;

$$\overline{X} = \sum_{i=1}^{n} \frac{X_i}{n}, \text{ or } n\,\overline{X} = \sum_{i=1}^{n} X_i \qquad (2)$$

x_i = the error in an observation; (3)

$$x_i = X_i - \overline{X}$$

Proof:

$$x_1 = X_1 - \overline{X}$$

$$x_2 = X_2 - \overline{X}$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$x_n = X_n - \overline{X}$$

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} X_i - n\overline{X}$$

From equation (2);

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} X_i - \sum_{i=1}^{n} X_i = 0 \qquad (4)$$

This shows that the sum of the differences about the mean is zero, which was expected, but if equation (3) is squared and then summed:

$$x_1^2 = X_1^2 - 2X_1 \bar{X} + \bar{X}^2 \qquad (5)$$

$$x_2^2 = X_2^2 - 2X_2 \bar{X} + \bar{X}^2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$x_n^2 = X_n^2 - 2X_n \bar{X} + \bar{X}^2$$

$$\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} X_i^2 - 2\bar{X} \sum_{i=1}^{n} X_i + n\bar{X}^2 \qquad (6)$$

The most probable value will be found when $\sum_{i=1}^{n} x_i^2 = 0$, or the most probable value of $\bar{X}$ will be that which makes $\sum_{i=1}^{n} x_i^2$ = a minimum. In order to find this minimum, differentiate equation (6) with respect to $\bar{X}$ and equate to 0:

$$\frac{d}{d\bar{X}} \sum_{i=1}^{n} x_i^2 = -2 \sum_{i=1}^{n} X_i + 2n\bar{X} = 0$$

$$\therefore \bar{X} = \sum_{i=1}^{n} \frac{X_i}{n} \qquad (7)$$

Equation (7) proves that the mean value $\bar{X}$ is the most probable value of a set of independent observations. Therefore, in the determination of the residual value it is correct to use the mean value as an approximation of the true value.

Appendix C

PROPAGATION OF ERRORS THROUGH KNOWN FUNCTIONS

Suppose that we wish to determine a quantity, x, which is a function of at least two other variables, u and v, which are actually observed (measured). We will determine the errors in x from those for u and v and from the functional dependence [1] [2] [4].

$$x = f(u,v, \ldots) \qquad \text{(C-1)}$$

The errors in x can be determined by considering the dispersion in the values of x resulting from combining the individual measurements u_i, v_i . . . into individual results x_i.

$$x_1 = f(u_1, v_1, \ldots)$$

$$x_2 = f(u_2, v_2, \ldots)$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$x_n = f(u_n, v_n \ldots) \qquad \text{(C-2)}$$

It will be assumed that the most probable value of x, $\bar{x}$ is:

$$\bar{x} = f(\bar{u}, \bar{v} \ldots) \qquad \text{(C-3)}$$

where $\bar{u}, \bar{v}$ are averages of the measured variables.

It is shown [1] that the variance of x, σ_x^2 will be given by:

$$\sigma_x^2 = \sigma_u^2 \left(\frac{\partial x}{\partial u}\right)^2 + \sigma_v^2 \left(\frac{\partial x}{\partial v}\right)^2 + 2\sigma_{uv} \left(\frac{\partial x}{\partial u}\right) \left(\frac{\partial x}{\partial v}\right) \qquad \text{(C-4)}$$

If u and v are independent ($\sigma_{uv} = 0$) then the standard deviation of x is:

$$\sigma_x = \sqrt{\sigma^2_u \left(\frac{\partial x}{\partial u}\right)^2 + \sigma_v^2 \left(\frac{\partial x}{\partial v}\right)^2} \, . \qquad \text{(C-5)}$$

Special Formulas for Error Propagation

Rule 1. Addition and Subtraction

The parameter's a and b are defined as positive constants. If x is the weighted sum of u and v,

$$x = au \pm bv$$

Then (C-5) becomes:

$$\sigma_x = \sqrt{a^2\sigma_u^2 + b^2\sigma_v^2} \qquad \text{(C-6)}$$

or in the special case where a = b = 1

$$\sigma_x = \sqrt{\sigma^2_u + \sigma_v^2} \qquad \text{(C-7)}$$

Rule 2. Multiplication and Division

If x is the weighted product of u and v

$$x = \pm auv$$

Then (C-5) becomes:

$$\sigma_x = \sqrt{a^2v^2\sigma_u^2 + a^2u^2\sigma_v^2}$$

or more symmetrically,

$$\frac{\sigma_x}{x} = \sqrt{\frac{\sigma_u^2}{u^2} + \frac{\sigma_v^2}{v^2}} \qquad \text{(C-8)}$$

Rule 3. <u>Powers</u>

If x is obtained by factors raised to various powers

$$x = u^a v^b$$

Then (C-5) becomes:

$$\sigma_x = \sqrt{a^2 u^{2a-2} v^{2b} \sigma_u^2 + b^2 u^{2a} v^{2b-2} \sigma_v^2}$$

or symmetrically (obtained by dividing by $x = u^a v^b$)

$$\frac{\sigma_x}{x} = \sqrt{a^2 \left(\frac{\sigma_u}{u}\right)^2 + b^2 \left(\frac{\sigma_v}{v}\right)^2} \qquad \text{(C-9)}$$

Appendix D

DERIVATION AND SOLUTION OF THE TWO-DIMENSIONAL PROBABILITY DISTRIBUTION FUNCTION

1. <u>Derivation.</u> The probability density functions [9] of the independent errors "x" and "y" are :

$$p(x) = \frac{1}{\sigma_x \sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_x^2}}, \text{ and } p(y) = \frac{1}{\sigma_y \sqrt{2\pi}} e^{-\frac{y^2}{2\sigma_y^2}}$$

Using Rule 4, Section 2.1.:

$$p(x,y) = \frac{1}{2\pi \sigma_x \sigma_y} e^{-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2}\right)}$$

$$P(x,y) = \frac{1}{2\pi \sigma_x \sigma_y} \int_x \int_y e^{-\frac{1}{2}\left[\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2}\right]} dx\, dy \qquad (1)$$

Using polar coordinates:

$$x^2 = r^2 \cos^2 \theta$$

$$y^2 = r^2 \sin^2 \theta$$

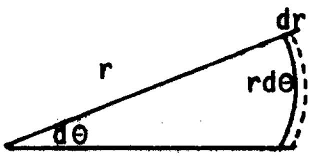
where r is the radial error and $r = \sqrt{x^2 + y^2}$

$$P(r) = P\left(r = \sqrt{x^2 + y^2} \leq R\right) = P(xy < R) \qquad (2)$$

where R is the radius of the probability circle.

The two-dimensional probability distribution functions is:

$$P(R) = \frac{1}{2\pi\ \sigma_x\ \sigma_y} \int_{r=0}^{R} \int_{\theta=0}^{2\pi} e^{-\frac{r^2}{2}\left[\frac{\sin^2\theta}{\sigma_y^2} + \frac{\cos^2\theta}{\sigma_x^2}\right]} r\ dr\ d\theta$$



r dθ dr (small increment resulting from dx and dy)

Using identities:

$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$$

$$P(R) = \frac{1}{2\pi\ \sigma_x\ \sigma_y} \int_{r=0}^{R} \int_{\theta=0}^{2\pi} e^{-\frac{r^2}{4}\left[\frac{1 - \cos 2\theta}{\sigma_y^2} + \frac{1 + \cos 2\theta}{\sigma_x^2}\right]} r\ dr\ d\theta$$

Rearranging terms:

$$P(R) = \frac{1}{2\pi\ \sigma_x\ \sigma_y} \int_{r=0}^{R} r e^{-\frac{r^2}{4}\left[\frac{1}{\sigma_y^2} + \frac{1}{\sigma_x^2}\right]} 4 \int_{\theta=0}^{\frac{\pi}{2}} e^{-\frac{r^2}{4}\left[\frac{1}{\sigma_x^2} - \frac{1}{\sigma_y^2}\right] \cos 2\theta} d\theta dr$$

let $\phi = 2\theta$,

$d\phi = 2d\theta$

Then:

$$P(R) = \frac{1}{2\pi\ \sigma_x\ \sigma_y} \int_{r=0}^{R} re^{-\frac{r^2}{4}\left[\frac{1}{\sigma_y^2} + \frac{1}{\sigma_x^2}\right]}\ 4 \int_{\phi=0}^{\pi} e^{-\frac{r^2}{4}\left[\frac{1}{\sigma_x^2} - \frac{1}{\sigma_y^2}\right] \cos\phi}\ \frac{d\phi}{2}\ dr$$

Rearranging terms:

$$P(R) = \frac{1}{\sigma_x\ \sigma_y} \int_{r=0}^{R} re^{-\frac{r^2}{4\sigma_y^2}\left[1 + \frac{\sigma_y^2}{\sigma_x^2}\right]} \left[\frac{1}{\pi} \int_{\phi=0}^{\pi} e^{-\frac{r^2}{4\sigma_y^2}\left[\frac{\sigma_y^2}{\sigma_x^2} - 1\right] \cos\phi}\ d\phi\right] dr$$

Let:

$$\left[\frac{1}{\pi} \int_{0}^{\pi} e^{-\frac{r^2}{4\sigma_y^2}\left[\frac{\sigma_y^2}{\sigma_x^2} - 1\right] \cos\phi}\ d\phi\ \right] = I_0\ \left[\frac{r^2}{4\sigma_y^2}\left(\frac{\sigma_y^2}{\sigma_x^2} - 1\right)\right]$$

where I_0 is a Bessel Function, zero order, modified first kind.

Therefore:

$$P(R) = \frac{1}{\sigma_x\ \sigma_y} \int_{r=0}^{R} re^{-\frac{r^2}{4\sigma_y^2}\left[1 + \frac{\sigma_y^2}{\sigma_x^2}\right]}\ I_0\left[\frac{r^2}{4\sigma_y^2}\left(\frac{\sigma_y^2}{\sigma_x^2} - 1\right)\right] dr \qquad (3)$$

2. <u>Special Case of Two-Dimensional Probability Distribution Function.</u>

When $\sigma_x = \sigma_y = \sigma_r$ from equation (3):

$$P(R) = \frac{1}{\sigma_r^2}\int_0^R re^{-\frac{r^2}{2\sigma_r^2}} I_o\left[\frac{r^2}{4\sigma_r^2}\left(\frac{\sigma_r^2}{\sigma_r^2} - 1\right)\right] dr$$

$$P(R) = \frac{1}{\sigma_r^2}\int_0^R re^{-\frac{r^2}{2\sigma_r^2}} I_o(0)\ dr$$

$$I_o(0) = 1$$

$$P(R) = \int_0^R \frac{r}{\sigma_r^2} e^{-\frac{r^2}{2\sigma_r^2}} dr$$

Since:

$$\frac{d}{dr}\ e^{-\frac{r^2}{2\sigma_r^2}} = -\frac{r}{\sigma_r^2}\ e^{-\frac{r^2}{2\sigma_r^2}}$$

Then:

$$\int \frac{r}{\sigma_r^2} e^{-\frac{r^2}{2\sigma_r^2}} dr = -e^{-\frac{r^2}{2\sigma_r^2}}$$

$$P(R) = -e^{-\left[\frac{r^2}{2\sigma_r^2}\right]_0^R} = 1 - e^{-\frac{R^2}{2\sigma_r^2}}$$

$$\therefore P(R) = 1 - e^{-\frac{R^2}{2\sigma_r^2}}$$

3. Modified Form of the Two-Dimensional Probability Distribution Function. To solve equation (3) by the use of tables, the equation must be modified. From S.O. Rice's "Properties of Sine Wave Plus Noise" [13]:

$$I_e (kx) = \int_0^x e^{-v} \; I_o (vk) \, dv \qquad (5)$$

Modifying equation (3):

$$\frac{1}{\sigma_x \; \sigma_y} \int_{r=0}^{R} re^{-\frac{r^2}{4\sigma_y^2}\left[1 + \frac{\sigma_y^2}{\sigma_x^2}\right]} I_o \left[\frac{r^2}{4\sigma_y^2} \left(\frac{\sigma_y^2}{\sigma_x^2} - 1\right)\right] dr$$

Step A

Letting:

$$v = \frac{r^2}{4\sigma_y^2} \left(1 + \frac{\sigma_y^2}{\sigma x^2}\right)$$

$$dv = \frac{2r}{4\sigma_y^2} \left(1 + \frac{\sigma_y^2}{\sigma_x^2}\right) dr$$

$$4\sigma_y^2 \, dv = 2r \left(1 + \frac{\sigma_y^2}{\sigma_x^2}\right) dr$$

$$rdr = \frac{2\sigma_y^2}{1 + \frac{\sigma_y^2}{\sigma_x^2}} dv$$

Step B

To get the quantity $\left[\frac{r^2}{4\sigma_y^2}\left(\frac{\sigma_y^2}{\sigma_x^2} - 1\right)\right]$ in the form of (vk):

$$v = \frac{r^2}{4\sigma_y^2}\left(1 + \frac{\sigma_y^2}{\sigma_x^2}\right)$$

$$\left(\frac{\sigma_y^2}{\sigma_x^2} - 1\right) = \left(1 + \frac{\sigma_y^2}{\sigma_x^2}\right) k$$

$$\therefore \quad k = \frac{\left(\frac{\sigma_y^2}{\sigma_x^2} - 1\right)}{\left(1 + \frac{\sigma_y^2}{\sigma_x^2}\right)}$$

Let $a = \frac{\sigma_x}{\sigma_y}$ where σ_x is the smaller of the two:

$$k = \frac{\left(\frac{\sigma_y^2}{\sigma_x^2} - \frac{\sigma_x^2}{\sigma_x^2}\right)}{\left(\frac{\sigma_x^2}{\sigma_x^2} + \frac{\sigma_y^2}{\sigma_x^2}\right)} = \frac{\left(\frac{1}{a^2} - 1\right)}{\left(1 + \frac{1}{a^2}\right)} = \frac{\left(\frac{1 - a^2}{a^2}\right)}{\left(\frac{1 + a^2}{a^2}\right)} = \frac{1 - a^2}{1 + a^2} \tag{6}$$

Step C

Getting σ_x and σ_y in terms of a:

$$\frac{1}{\sigma_x \sigma_y} \cdot \frac{2\sigma_y^2}{\left(+ \frac{\sigma_y^2}{\sigma_x^2}\right)} = 2\left[\frac{1}{\sigma_x \sigma_y} \cdot \frac{\sigma_y^2}{\frac{\sigma_x^2 + \sigma_y^2}{\sigma_x^2}}\right] = 2\left[\frac{1}{\sigma_x \sigma_y} \cdot \frac{\sigma_x^2 \sigma_y^2}{\sigma_x^2 + \sigma_y^2}\right]$$

$$2\left[\frac{\frac{1}{\sigma_y^2}}{\frac{1}{\sigma_y^2}} \cdot \frac{\frac{\sigma_x\ \sigma_y}{1}}{\frac{\sigma_x^2 + \sigma_y^2}{1}}\right] = 2\left[\frac{\frac{\sigma_x\ \sigma_y}{\sigma_y^2}}{\frac{\sigma_x^2 + \sigma_y^2}{\sigma_y^2}}\right] = 2\left[\frac{\frac{\sigma_x}{\sigma_y}}{\frac{\sigma_x^2}{\sigma_y^2} + 1}\right] = \frac{2a}{a^2 + 1} \tag{7}$$

Step D

$$1 + \frac{\sigma_y^2}{\sigma_x^2} = \frac{\sigma_x^2 + \sigma_y^2}{\sigma_x^2} = \left[\frac{\frac{1}{\sigma_y^2}}{\frac{1}{\sigma_y^2}} \cdot \frac{\frac{\sigma_x^2 + \sigma_y^2}{1}}{\frac{\sigma_x^2}{1}}\right] = \left[\frac{\frac{\sigma_x^2 + \sigma_y^2}{\sigma_y^2}}{\frac{\sigma_x^2}{\sigma_y^2}}\right] = \frac{a^2 + 1}{a^2} \tag{8}$$

Combining Steps A, B, C, D and equation (3):

$$P(R) = \frac{2a}{1+a^2} \int_0^{\frac{R^2}{4\sigma_y^2}\left[\frac{1+a^2}{a^2}\right]} e^{-\frac{r^2}{4\sigma_y^2}\left[\frac{1+a^2}{a^2}\right]}\ I_o\left[\frac{r^2}{4\sigma_y^2}\left(\frac{1+a^2}{a^2}\right)\left(\frac{1-a^2}{1+a^2}\right)\right] dv. \tag{9}$$

Rewriting equation (9):

$$P(R) = \frac{2a}{1 + a^2} \int_0^x e^{-v}\ I_o\ (vk)\ dv \tag{10}$$

where:

$$x = \frac{R^2}{4\sigma_y^2}\left[\frac{1 + a^2}{a^2}\right] \quad ; \qquad v = \frac{r^2}{4\sigma_y^2}\left[\frac{1 + a^2}{a^2}\right] \quad ; \qquad k = \left(\frac{1 - a^2}{1 + a^2}\right)$$

4. Solution of Modified Function. To compute the CEP [13] [14] (CEP = R when P(R) = 0.5) for values of $\sigma_y = \sqrt{.6}$ and $\sigma_x = \sqrt{.4}$, two methods are available:

Method 1:

To determine the value for x by Rice's table of I_e (vk) dv, enter the table with values of k and the required probability.

P(R) = 50% probability; $a = \frac{\sigma_x}{\sigma_y} = 0.8165$; $a^2 = 0.6667$; $k = \frac{1-a^2}{1+a^2} = 0.2$

$$P(R) = \frac{2a}{1 + a^2} \int_0^x e^{-v} \; I_o \, (vk) \; dv$$

$$\frac{.50 \; (1 + a^2)}{2a} = \int_0^x = e^{-v} \; I_o \, (vk) dv$$

$$0.5103 = \int_0^x e^{-v} \; I_o \, (vk) \; dv$$

Enter the table with k = 0.2 and interpolate for 0.5103 to get the value of x.

.6 = 4517
5103
.8 = 5516
$\therefore$ x = 0.71732

$$x = \frac{R^2}{4\sigma_y 2} \left[\frac{1 + a^2}{a^2}\right] = 0.71732$$

$$\frac{R}{\sigma_y} = 1.0713$$

The radius of the 50% probability circle (CEP) resulting from σ_x, σ_y is $R = 1.0713\ \sigma_y$.

<u>Method 2</u>:

Using tables computed by Arthur Grad and Herbert Solomon [14]:
From equation (2):

$$P(R) = P\left(\sqrt{x^2 + y^2} \leq R\right) = P\left(x^2 + y^2 \leq R^2\right)$$

Since x and y have unit standard errors, they can be written as:

$x = \sigma_x\ x$ and $y = \sigma_y\ y$.

Therefore:

$$P(R) = P\left(\sigma_y^2\ y^2 + \sigma_x^2\ x^2 \leq R^2\right)$$

$$= P\left(y^2 + \frac{\sigma_x^2}{\sigma_y^2} x^2 \leq \frac{R^2}{\sigma_y^2}\right) \qquad (11)$$

From Grad and Solomon Tables:

$$P\left(a_1\ y_1^2 + a_2\ y_2^2 \leq t\right) \qquad a_1 + a_2 = 1$$

$$P\left(y_2^2 + \frac{a_1}{a_2}\ y_1^2 \leq \frac{t}{a_2}\right) \qquad (12)$$

Correlation between equations (11) and (12) will permit use of the tabled values.

$$\frac{\sigma_x}{\sigma_y} = \sqrt{\frac{a_1}{a_2}}; \quad \frac{R}{\sigma_y} = \sqrt{\frac{t}{a_2}}$$

Enter the tables with values of a_1, a_2 and the required probability. Then interpolate for values of

$$\frac{R}{\sigma_y} = \sqrt{\frac{t}{a_2}}.$$

Since $\frac{\sigma_x}{\sigma_y} = \sqrt{\frac{.4}{.6}}$, then $a_1 = .4$; $a_2 = .6$

t =	.6 =	4559
		5000
	.7	5080

$$\frac{t}{a_2} = \frac{0.68464}{.6} = \frac{R^2}{\sigma_y^2}$$

$$\frac{R}{\sigma_y} = \sqrt{\frac{0.68464}{.6}} = 1.068$$

$$R = 1.068\ \sigma_y$$

Appendix E
DERIVATION OF THE SPHERICAL PROBABILITY DISTRIBUTION FUNCTION

The combined probability density distribution function of the independent errors x, y and z are:

$$p(x,y,z) = \frac{1}{\sigma_x \sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_x^2}} \cdot \frac{1}{\sigma_y \sqrt{2\pi}} e^{-\frac{y^2}{2\sigma_y^2}} \cdot \frac{1}{\sigma_z \sqrt{2\pi}} e^{-\frac{z^2}{2\sigma_z^2}} \qquad (1)$$

In the spherical case where $\sigma_x = \sigma_y = \sigma_z = \sigma_s$:

$$p(x,y,z)dx\ dy\ dz = \frac{1}{\sigma_s^3 \ (2\pi)^{\frac{3}{2}}} e^{-\frac{x^2 + y^2 + z^2}{2\sigma_s^2}} dx\ dy\ dz \qquad (2)$$

Converting to 3-dimensional coordinates:

$$x^2 = s^2 \cos^2 \Psi \cos^2 \lambda$$

$$y^2 = s^2 \cos^2 \Psi \sin^2 \lambda$$

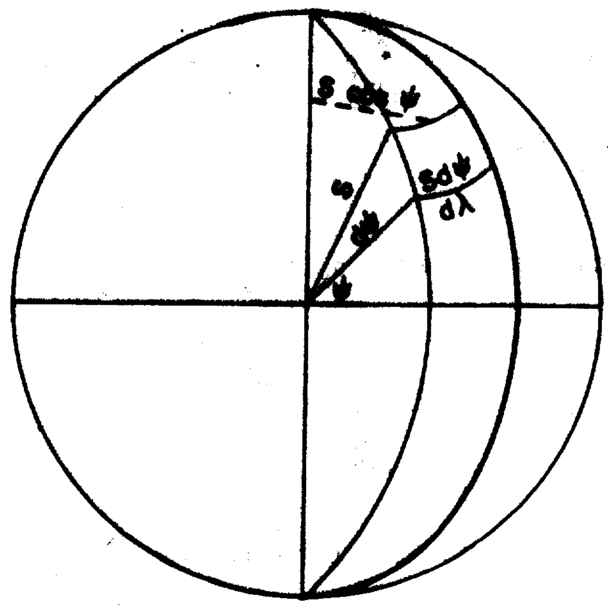
$$z^2 = s^2 \sin^2 \Psi$$

$$x^2 + y^2 + z^2 = s^2 \cos^2 \Psi \cos^2 \lambda + s^2 \cos^2 \Psi \sin^2 \lambda + s^2 \sin^2 \Psi$$

$$= s^2 \cos^2 \Psi (\cos^2 \lambda + \sin^2 \lambda) + s^2 \sin^2 \Psi$$

$$= s^2$$

Let S = radius of sphere, replacing radial error s.



Then: $dS \; Sd\Psi \; S \cos \Psi \; d\lambda = S^2 \cos \Psi \; d\Psi \; d\lambda \; dS$

$$P(S) = \int_{S=0}^{S} \int_{\lambda=0}^{2\pi} \int_{\Psi=-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{(2\pi)^{\frac{3}{2}} \sigma_s^3} e^{-\frac{S^2}{2\sigma_s^2}} S^2 \cos \Psi \; d\Psi \; d\lambda \; dS \quad (3)$$

$$P(S) = \frac{1}{(2\pi)^{\frac{3}{2}}} (2) \; (2\pi) \int_{S=0}^{S} \frac{S^2}{\sigma_s^3} e^{-\frac{s^2}{2\sigma_s^2}} \; dS$$

$$\therefore\ P(S) = \sqrt{\frac{2}{\pi}} \int_0^S \frac{S^2}{\sigma_S^3} e^{-\frac{S^2}{2\sigma_S^2}} dS$$

Integrating by parts:

Let $u = \frac{S}{\sigma_S}$, $dv = \frac{S}{\sigma_S^2} e^{-\frac{S^2}{2\sigma_S^2}} dS$

$du = \frac{dS}{\sigma_S}$, $v = -e^{-\frac{S^2}{2\sigma_S^2}}$

$$P(S) = \sqrt{\frac{2}{\pi}} [(\frac{S}{\sigma_S}) (-e^{-\frac{S^2}{2\sigma_S^2}}) + \int_0^S \frac{e}{\sigma_S}^{-\frac{S^2}{2\sigma_S^2}} dS] \qquad (5)$$

In order to use approximation formula [14], P(S) must be transformed to the integral of $e^{-t^2/2}dt$.

Letting $C = \frac{S}{\sigma_S}$, $dS = \sigma_S\, dC$, where σ_S = constant :

$$P(S) = \sqrt{\frac{2}{\pi}} [- Ce^{-\frac{C^2}{2}} + \int_{C=0}^{C=\frac{S}{\sigma_S}} e^{-\frac{C^2}{2}} dC] \qquad (6)$$

From above reference when $x \geq 0$:

$$\int_x^{\infty} e^{-\frac{t^2}{2}}\,dt \sim \frac{e^{-\frac{x^2}{2}}}{x + 0.8\,e^{-.4x}} \qquad (7)$$

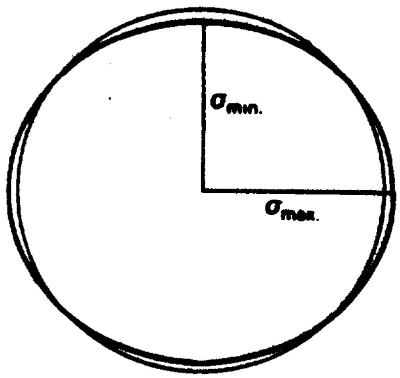
$$\int_0^{\infty} e^{-\frac{C^2}{2}}\,dC = 1.253$$

$$\int_0^{x} = \int_0^{\infty} - \int_x^{\infty}$$

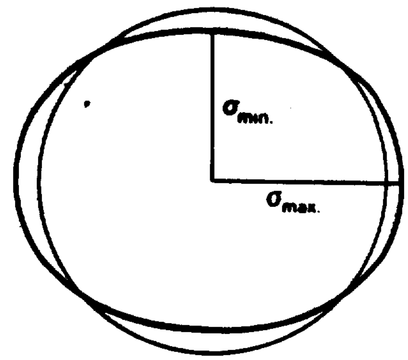
$$\therefore P(S) = \sqrt{\frac{\pi}{2}}\left[- C\,e^{-\frac{C^2}{2}} + 1.253 - \frac{e^{-\frac{C^2}{2}}}{C + 0.8\,e^{-.4C}} \right] \qquad (8)$$

Appendix F

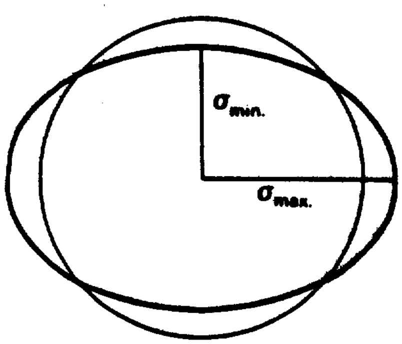
SUBSTITUTION OF THE CIRCULAR FORM FOR ELLIPTICAL ERROR DISTRIBUTIONS



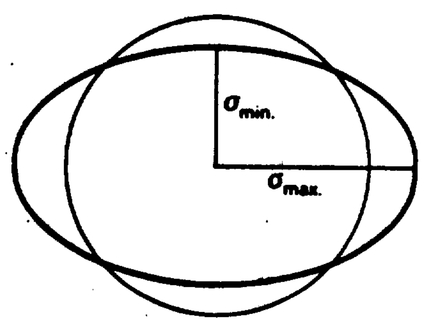
$\frac{\sigma_{min.}}{\sigma_{max.}} = 0.9$



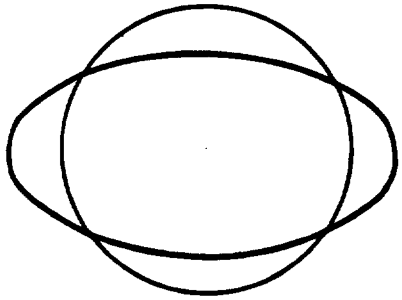
$\frac{\sigma_{min}}{\sigma_{max.}} = 0.8$



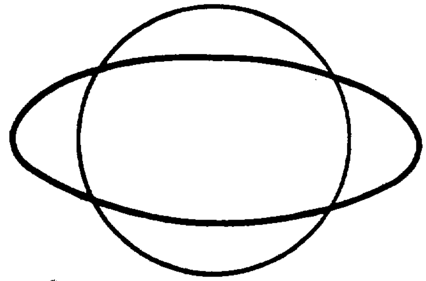
$\frac{\sigma_{min.}}{\sigma_{max.}} = 0.7$



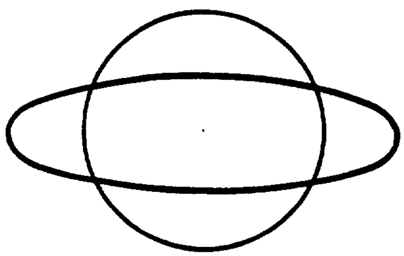
$\frac{\sigma_{min.}}{\sigma_{max.}} = 0.6$



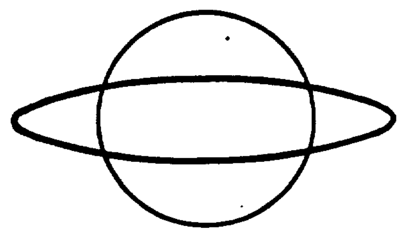
$\frac{\sigma_{min.}}{\sigma_{max.}} = 0.5$



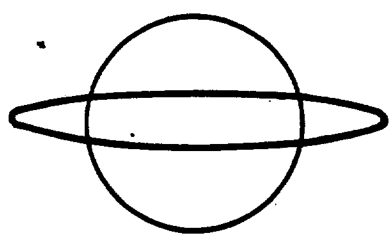
$\frac{\sigma_{min.}}{\sigma_{max.}} = 0.4$



$\frac{\sigma_{min.}}{\sigma_{max.}} = 0.3$



$\frac{\sigma_{min.}}{\sigma_{max.}} = 0.2$



$\frac{\sigma_{min.}}{\sigma_{max.}} = 0.1$

Appendix G
Error Probability of a Quantity Affected by a Bias

In error analysis it is assumed that all systematic errors have been removed from the observational data. This is seldom true for it is virtually impossible to eliminate all systematic errors. In the analysis it is hoped that the systematic errors of consequence have been eliminated leaving only numerous small systematic errors whose combinations cannot be distinguished from random errors. When a systematic error of consequence has not been removed, the value obtained is said to be biased. That is, the value deviated from its true or accepted value by some known or undetermined amount.

This appendix is concerned with the effect these undetected, biased quantities have on the probability interpretation applied in error analysis [16]. In the two-dimensional case, this would be the radius of circle which will include a certain portion of an error distribution affected by a biased quantity. As an example, consider a missile with a circular standard error $\sigma_c = r$, aimed at a point T. The missile is biased (or the target misidentified) by an amount d; therefore, the distribution of missile impacts is around the point A (a distance of d from T) not T. (see Figure G-1)

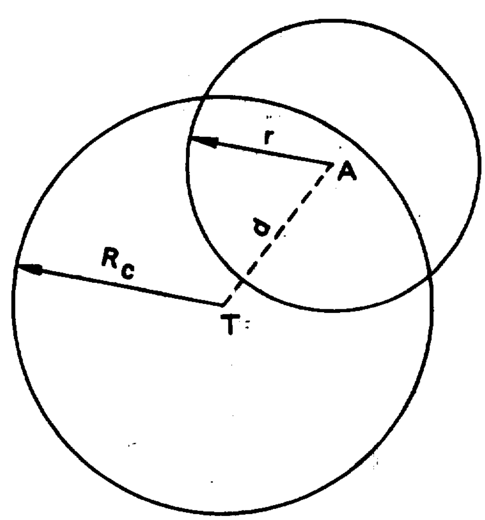


Figure G-1. Circular Error with a Bias

The probability error space containing the true value when the error source has both a random and a bias component can be expressed mathematically. The probability density function for an error, x, in one dimension, f(x) is:

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma_x} e^{-\frac{1}{2}\left[(x - \bar{x})^2\right]} \qquad \text{(G-1)}$$

where x = random variable

$\bar{x}$ = the bias in x.

The probability function for two dimensions is:

$$f(x,y) = \frac{1}{2\pi\,\sigma_x\,\sigma_y} e^{-\frac{1}{2}\left[\frac{(x - \bar{x})^2}{\sigma_x^2} + \frac{(y - \bar{y})^2}{\sigma_y^2}\right]} \qquad \text{(G-2)}$$

where x,y = random variables

$\bar{x}$ = bias in x

$\bar{y}$ = bias in y

In three-dimensional, the probability density function is:

$$f(x,y,z) = \frac{1}{(2\pi)^{\frac{3}{2}}\,\sigma_x\,\sigma_y\,\sigma_z} e^{-\frac{1}{2}\left[\frac{(x - \bar{x})^2}{\sigma_x^2} + \frac{(y - \bar{y})^2}{\sigma_y^2} + \frac{(z - \bar{z})^2}{\sigma_z^2}\right]} \qquad \text{(G-3)}$$

where x,y,z = random variables

x = bias in x

y = bias in y

z = bias in z

The one standard deviation probability level may be found from the integrals of the probability density functions, Equations (G-1), (G-2), and (G-3), using numerical integration. In one dimension, using equation (G-1), the linear standard deviation probability is:

$$\Pr\ (-\ \sigma\ell_x < x < \sigma\ell_x) = \int_{-\sigma\ell_x}^{\sigma\ell_x} f(x)dx \qquad \text{(G-4)}$$

$$\Pr\ (-\sigma\ell_x < x < \sigma\ell_x) = .6827 \qquad \text{(G-5)}$$

where $\sigma\ell_x$ is the linear one-sigma error in x.

The probability density function is two dimensions, Equation (G-2), may be numerically integrated to find the circular one standard deviation probability level:

$$\Pr\ [(x^2 + y^2) < \sigma_c^2] = \int \int_R f(x,y)\ dxdy \qquad \text{(G-6)}$$

$$\Pr\ [(x + y^2) < \sigma_c^2] = .3935 \qquad \text{(G-7)}$$

In Equation (G-6), R denotes a circular disc of radius σ_c .

For the three-dimensional case, Equation (G-3) may be numerically integrated to find the spherical one standard deviation probability level:

$$\Pr\ [(x^2 + y^2 + z^2) < \sigma_s^2] = \int \int \int_S f(x,y,z)\ dxdydz \qquad \text{(G-8)}$$

$$\Pr\ [(x^2 + y^2 + z^2) < \sigma_s^2] = .1990 \qquad \text{(G-9)}$$

In Equation (G-8), S denotes a sphere of radius σ_s .

For the linear case, Equation (G-4), the numerical integration may be performed using Simpson's Rule. For the two and three-dimensional cases, Equations (G-6) and (G-8), numerical integration may use the 15th degree Gauss Product Formulas [17].

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LINEAR ERROR

From \ To	50%	68.27%	90%	99.73%
50%	1.0000	1.4826	2.4387	4.4475
68.27	0.6745	1.0000	1.6449	3.0000
90	0.4101	0.6080	1.0000	1.8239
99.73	0.2248	0.3333	0.5483	1.0000

CIRCULAR ERROR

From \ To	39.35%	50%	90%	99.78%
39.35%	1.0000	1.1774	2.1460	3.5000
50	0.8493	1.0000	1.8227	2.9726
90	0.4660	0.5486	1.0000	1.6309
99.78	0.2857	0.3364	0.6131	1.0000

SPHERICAL ERROR

From \ To	19.9%	50%	90%	99.89%
19.9%	1.000	1.538	2.500	4.000
50	0.650	1.000	1.625	2.600
90	0.400	0.615	1.000	1.600
99.89	0.250	0.385	0.625	1.000

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