Control Systems

控制系統

- 清雲科技大學電機工程系
- Ya-Fu Peng

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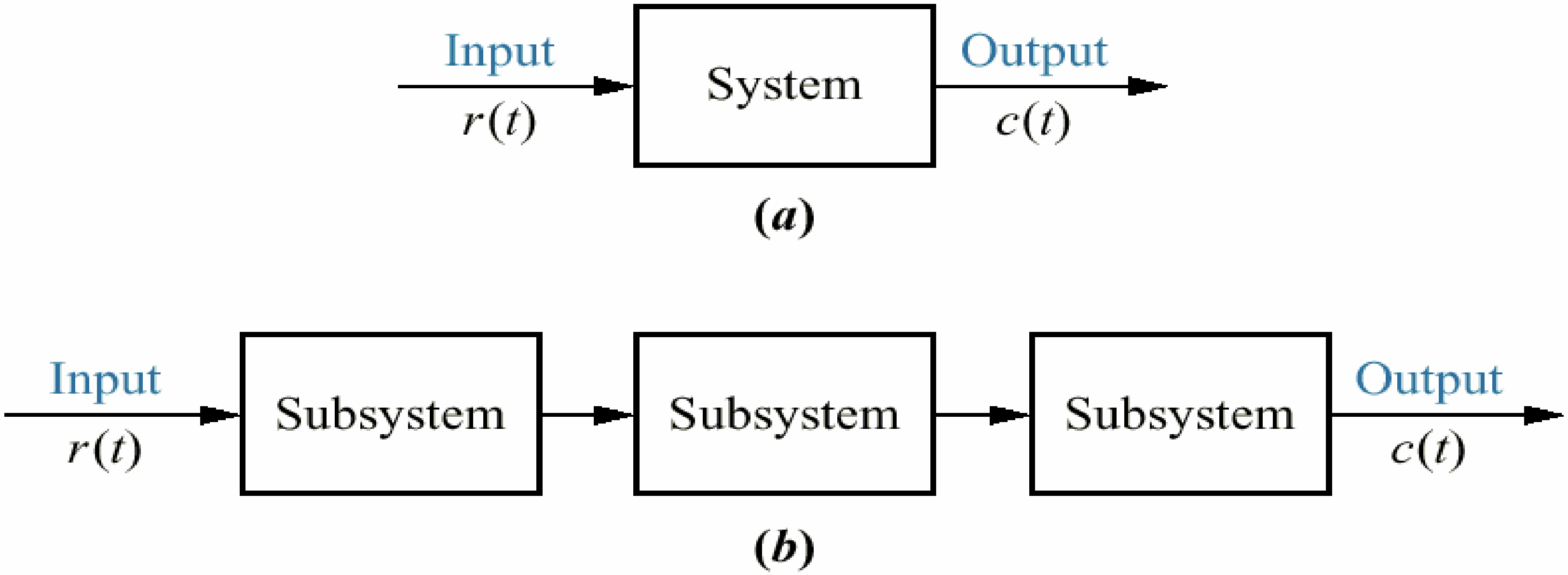
內容

- 簡介
- 頻域模型
- 時域模型
- 時間響應
- 互聯子系統之簡化
- 穩定度
- 穩態誤差

Chapter 2 Modeling in the Frequency Domain

- 從物理系統的示意圖推導數學模型
  - 頻域:轉移函數 Ch2
  - 時域:狀態方程式 Ch3

為何採用頻域分析-便利性



Note: The input, $r(t)$, stands for *reference input*.
The output, $c(t)$, stands for *controlled variable*.

拉氏轉換(Laplace Transform)

- 定義

$$\mathscr{L}[f(t)] = F(s) = \int_{0^-}^{\infty} f(t)e^{-st}\,dt$$

其中 $s = \sigma + j\omega$

拉氏轉換表

Item no.	*f(t)*	*F(s)*
1.	$\delta(t)$	1
2.	$u(t)$	$\frac{1}{s}$
3.	$tu(t)$	$\frac{1}{s^2}$
4.	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
5.	$e^{-at}u(t)$	$\frac{1}{s+a}$
6.	$\sin \omega t u(t)$	$\frac{\omega}{s^2+\omega^2}$
7.	$\cos \omega t u(t)$	$\frac{s}{s^2+\omega^2}$

拉氏轉換表

Item no.	Theorem	Name
1.	$\mathscr{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$	Definition
2.	$\mathscr{L}[kf(t)] = kF(s)$	Linearity theorem
3.	$\mathscr{L}[f_1(t) + f_2(t)] = F_1(s) + F_2(s)$	Linearity theorem
4.	$\mathscr{L}[e^{-at}f(t)] = F(s+a)$	Frequency shift theorem
5.	$\mathscr{L}[f(t-T)] = e^{-sT}F(s)$	Time shift theorem
6.	$\mathscr{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$	Scaling theorem

拉氏轉換表

7.	$\mathcal{L}\left[\frac{df}{dt}\right]$	$=$	$sF(s) - f(0-)$	Differentiation theorem
8.	$\mathcal{L}\left[\frac{d^2f}{dt^2}\right]$	$=$	$s^2F(s) - sf(0-) - \dot{f}(0-)$	Differentiation theorem
9.	$\mathcal{L}\left[\frac{d^nf}{dt^n}\right]$	$=$	$s^nF(s) - \sum_{k=1}^{n} s^{n-k}f^{k-1}(0-)$	Differentiation theorem
10.	$\mathcal{L}\left[\int_{0-}^{t} f(\tau)\,d\tau\right]$	$=$	$\frac{F(s)}{s}$	Integration theorem
11.	$f(\infty)$	$=$	$\lim_{s\to 0} sF(s)$	Final value theorem[1]
12.	$f(0+)$	$=$	$\lim_{s\to\infty} sF(s)$	Initial value theorem[2]

Ex:

- 試求 $f(t) = Ae^{-at}u(t)$ 的拉氏轉換

$$F(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty Ae^{-at}e^{-st}dt$$

$$= A\int_0^\infty e^{-(s+a)t}dt = -\frac{A}{s+a}e^{-(s+a)t}\Big|_{t=0}^\infty$$

$$= 0 - (-\frac{A}{s+a})$$

$$= \frac{A}{s+a}$$

Ex:

- 試求 $F(s)=\dfrac{1}{(s+3)^2}$ 的反拉氏轉換

$$\frac{1}{s^2} \to tu(t)$$

$$F(s+a) \to e^{-at} f(t)$$

$$\mathscr{L}^{-1}\left[\frac{1}{(s+3)^2}\right] = e^{-3t}tu(t)$$

部分分式展開

- 狀況1： $F(s)$分母含有不同的實數根

$$F(s)=\frac{N(s)}{D(s)}$$

$$=\frac{N(s)}{(s+p_1)(s+p_2)\cdots(s+p_m)\cdots(s+p_n)}$$

$$=\frac{k_1}{(s+p_1)}+\frac{k_2}{(s+p_2)}+\cdots+\frac{k_m}{(s+p_m)}+\cdots\frac{k_n}{(s+p_n)}$$

部分分式展開

- 狀況2：$F(s)$分母的根為實根且重根

$$F(s)=\frac{N(s)}{D(s)}$$

$$=\frac{N(s)}{(s+p_1)^r(s+p_2)\cdots(s+p_n)}$$

$$=\frac{k_1}{(s+p_1)^r}+\frac{k_2}{(s+p_1)^{r-1}}+\cdots+\frac{k_r}{(s+p_1)}+\cdots\frac{k_{r+n-1}}{(s+p_n)}$$

部分分式展開

- 狀況3：$F(s)$分母的根為複數根或虛根

$$F(s)=\frac{N(s)}{D(s)}=\frac{N(s)}{(s+p_1)(s^2+qs+r)\cdots}$$

$$=\frac{k_1}{(s+p_1)}+\frac{k_2 s+k_3}{(s^2+qs+r)}+\cdots$$

Ex:

- $y''(t)+12y'(t)+32y(t)=32u(t)$
  ，其中 $y'(0^-)=y(0^-)=0$

$$y''(t)\to s^2Y(s)-sy(0^-)-y'(0^-)$$

$$y'(t)\to sY(s)-y(0^-)$$

$$u(t)\to \frac{1}{s}$$

$$s^2Y(s)+12sY(s)+32Y(s)=\frac{32}{s}$$

$$Y(s)=\frac{32}{s(s^2+12s+32)}$$

$$Y(s)=\frac{32}{s(s^2+12s+32)}=\frac{32}{s(s+4)(s+8)}$$

$$=\frac{k_1}{s}+\frac{k_2}{(s+4)}+\frac{k_3}{(s+8)}=\frac{1}{s}-\frac{2}{(s+4)}+\frac{1}{(s+8)}$$

$$Y(s)=\frac{1}{s}-\frac{2}{(s+4)}+\frac{1}{(s+8)}$$

$\mathscr{L}^{-1}$

$$y(t)=u(t)-2e^{-4t}u(t)+e^{-8t}u(t)$$

$$y(t)=(1-2e^{-4t}+e^{-8t})u(t)$$

練習題

- $2y''(t)+9y'(t)+7y(t)=6$
  ，其中 $y'(0^-)=y(0^-)=0$

EX:

- $$F(s)=\frac{2}{(s+1)(s+2)^2}=\frac{k_1}{(s+1)}+\frac{k_2}{(s+2)^2}+\frac{k_3}{(s+2)}$$

$$k_1=\left.\frac{2}{(s+2)^2}\right|_{s\to -1}=2,\qquad k_2=\left.\frac{2}{(s+1)}\right|_{s\to -2}=-2$$

$$k_3=\left.\frac{1}{(2-1)!}\frac{dF_1(s)}{ds}\right|_{s\to -2}=\left.\frac{-2}{(s+1)^2}\right|_{s\to -2}=-2$$

$$Y(s)=\frac{2}{s+1}-\frac{2}{(s+2)^2}-\frac{2}{(s+2)}$$

$\mathcal{L}^{-1}$

$$y(t)=(2e^{-t}-2te^{-2t}-2e^{-2t})u(t)$$

Ex: $F(s)=\dfrac{3}{s(s^2+2s+5)}$

$$=\frac{k_1}{s}+\frac{k_2 s+k_3}{(s^2+2s+5)}$$

$$k_1=\left.\frac{3}{(s^2+2s+5)}\right|_{s\to 0}=\frac{3}{5}$$

接著兩邊同乘分母可得：

$$3=k_1(s^2+2s+5)+k_2 s^2+k_3 s$$

$$3=\frac{3}{5}(s^2+2s+5)+k_2 s^2+k_3 s$$

$$\Rightarrow k_2=-\frac{3}{5},\qquad k_3=-\frac{6}{5}$$

$$F(s)=\frac{\frac{3}{5}}{s}+\frac{-\frac{3}{5}s-\frac{6}{5}}{s^2+2s+5}$$

$$=\frac{\frac{3}{5}}{s}+\frac{-\frac{3}{5}(s+1)-\frac{3}{5}}{(s+1)^2+2^2}$$

$$=\frac{\frac{3}{5}}{s}+\frac{-\frac{3}{5}(s+1)}{(s+1)^2+2^2}+\frac{-\frac{3}{5}\cdot\frac{1}{2}\cdot 2}{(s+1)^2+2^2}$$

$$\frac{k_2 s+k_3}{s^2+qs+r}\Rightarrow\frac{A(s+a)+B\omega}{(s+a)^2+\omega^2}$$

$$\frac{A(s+a)}{(s+a)^2+\omega^2}\Rightarrow Ae^{-at}\cos\omega t$$

$$\frac{B\omega}{(s+a)^2+\omega^2}\Rightarrow Be^{-at}\sin\omega t$$

$\mathscr{L}^{-1}$

$$y(t)=\frac{3}{5}-\frac{3}{5}e^{-t}\cos 2t-\frac{3}{10}e^{-t}\sin 2t$$

EX: $$F(s) = \frac{10}{s(s+2)(s+3)^2}$$

$$= \frac{k_1}{s} + \frac{k_2}{(s+2)} + \frac{k_3}{(s+3)} + \frac{k_4}{(s+3)^2}$$

$$= \frac{\frac{5}{9}}{s} + \frac{-5}{(s+2)} + \frac{\frac{40}{9}}{(s+3)} + \frac{\frac{10}{3}}{(s+3)^2}$$

$\mathscr{L}^{-1}$

$$y(t) = \frac{5}{9} - 5e^{-2t} + \frac{40}{9}e^{-3t} + \frac{10}{3}te^{-3t}$$

轉移函數(Transfer Function): $T(s)$

$$T(s) \equiv \left.\frac{Y(s)}{R(s)}\right|_{\text{IC}=0}$$

- n 階線性非時變微分方程

$$a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_0 y(t)$$

$$= b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + \cdots + b_0 r(t)$$

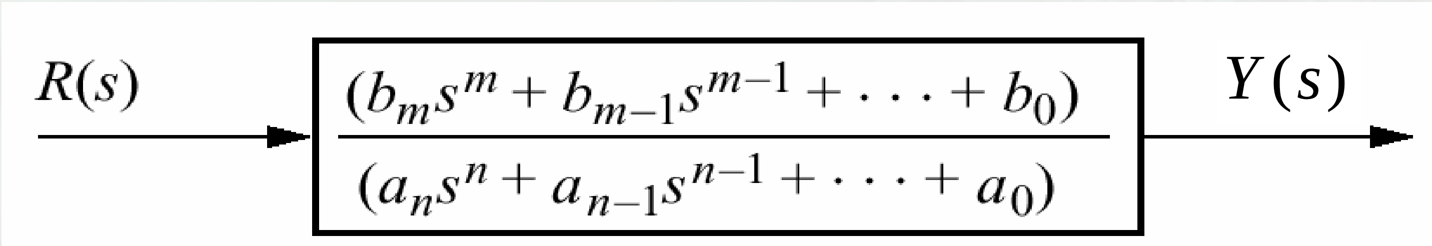
$\mathscr{L}$

$$a_n s^n Y(s) + a_{n-1} s^{n-1} Y(s) + \cdots + a_0 Y(s)$$

$$= b_m s^m R(s) + b_{m-1} s^{m-1} R(s) + \cdots + b_0 R(s)$$

$$(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0) Y(s) = (b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0) R(s)$$

$$\frac{Y(s)}{R(s)} = T(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 + a_0}$$



Ex:試求轉移函數$T(s)$，若$r(t)$=1試求$y(t)$

Sol:

$$\frac{dy(t)}{dt}+2y(t)=r(t)$$

$\mathscr{L}$

$$sY(s)+2Y(s)=R(s)\Rightarrow T(s)=\frac{Y(s)}{R(s)}=\frac{1}{s+2}$$

$$Y(s)=T(s)\cdot R(s)=\frac{1}{s+2}\cdot\frac{1}{s}=\frac{0.5}{s}+\frac{-0.5}{s+2}$$

$\mathscr{L}^{-1}$

$$y(t)=\frac{1}{2}-\frac{1}{2}e^{-2t}$$

Ex:試求轉移函數$T(s)$

$$\frac{d^3y(t)}{dt^3}+3\frac{d^2y(t)}{dt^2}+7\frac{dy(t)}{dt}+5y(t)=\frac{d^2r(t)}{dt^2}+4\frac{dr(t)}{dt}+3r(t)$$

Sol:

$$s^3Y(s)+3s^2Y(s)+7sY(s)+5Y(s)=s^2R(s)+4sR(s)+3R(s)$$

$$\Rightarrow (s^3+3s^2+7s+5)Y(s)=(s^2+4s+3)R(s)$$

$$\Rightarrow T(s)=\frac{Y(s)}{R(s)}=\frac{s^2+4s+3}{s^3+3s^2+7s+5}$$

Ex:試由轉移函數$T(s)$求出對應的微分方程式

$$T(s)=\frac{Y(s)}{R(s)}=\frac{2s+1}{s^2+6s+2}$$

Sol:

$$(s^2+6s+2)Y(s)=(2s+1)R(s)$$

$$\Rightarrow s^2Y(s)+6sY(s)+2Y(s)=2sR(s)+R(s)$$

$$\Rightarrow \frac{d^2y(t)}{dt^2}+6\frac{dy(t)}{dt}+2y(t)=2\frac{dr(t)}{dt}+r(t)$$

電路之轉移函數

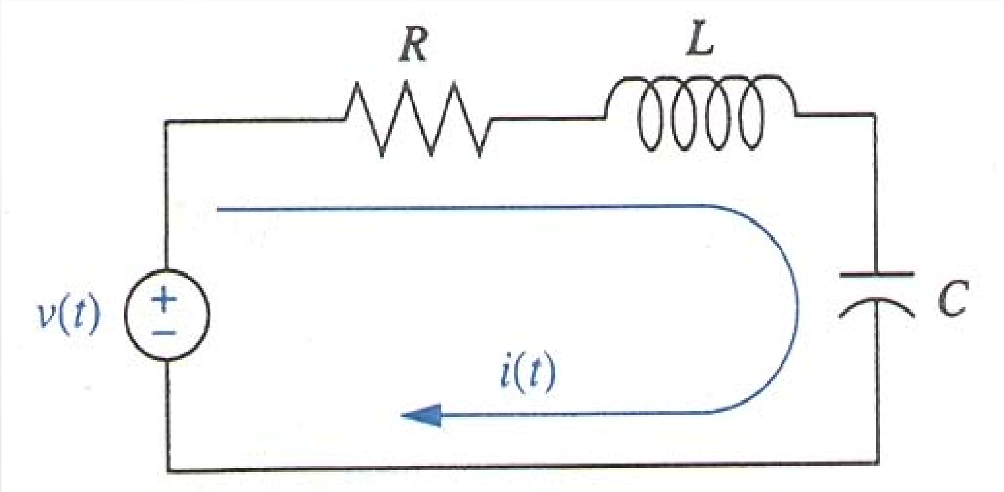
電阻 ⇒ $V_R(s) = RI(s)$　　電感 ⇒ $V_L(s) = sLI(s)$

電容 ⇒ $V_C(s) = \dfrac{1}{Cs} I(s)$　　阻抗 ⇒ $Z(s) = \dfrac{V(s)}{I(s)}$

Capacitor	$v(t) = \frac{1}{C}\int_0^t i(\tau)\,d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\frac{1}{Cs}$	Cs
Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^t v(\tau)d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

Note: The following set of symbols and units is used throughout this book: $v(t)$ = V (volts), $i(t)$ = A (amps), $q(t)$ = Q (coulombs), C = F (farads), R = Ω (ohms), G = ℧ (mhos), L = H (henries).

Ex:試求輸入電壓與電容電壓之轉移函數



$$L\frac{di(t)}{dt}+Ri(t)+\frac{1}{C}\int_0^t i(\tau)d\tau=v(t)$$

$\mathscr{L}$

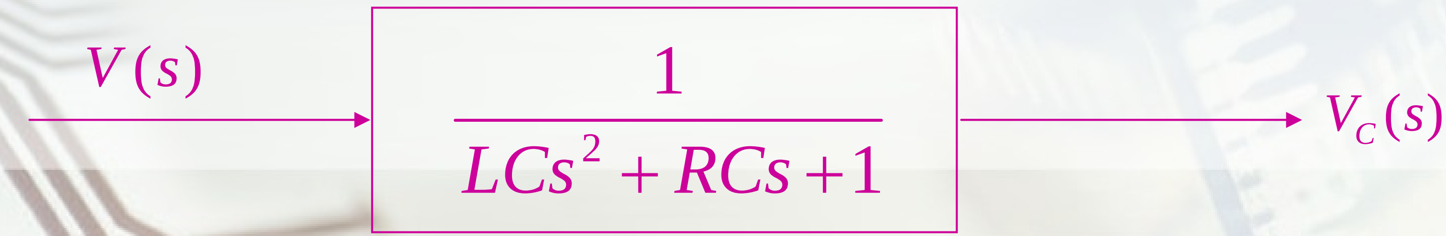
$$sLI(s)+RI(s)+\frac{1}{sC}I(s)=V(s)$$

$$(sL + R + \frac{1}{sC})I(s) = V(s)$$

$$\Rightarrow I(s) = \frac{1}{sL + R + \frac{1}{sC}}V(s) = \frac{sC}{LCs^2 + RCs + 1}V(s)$$

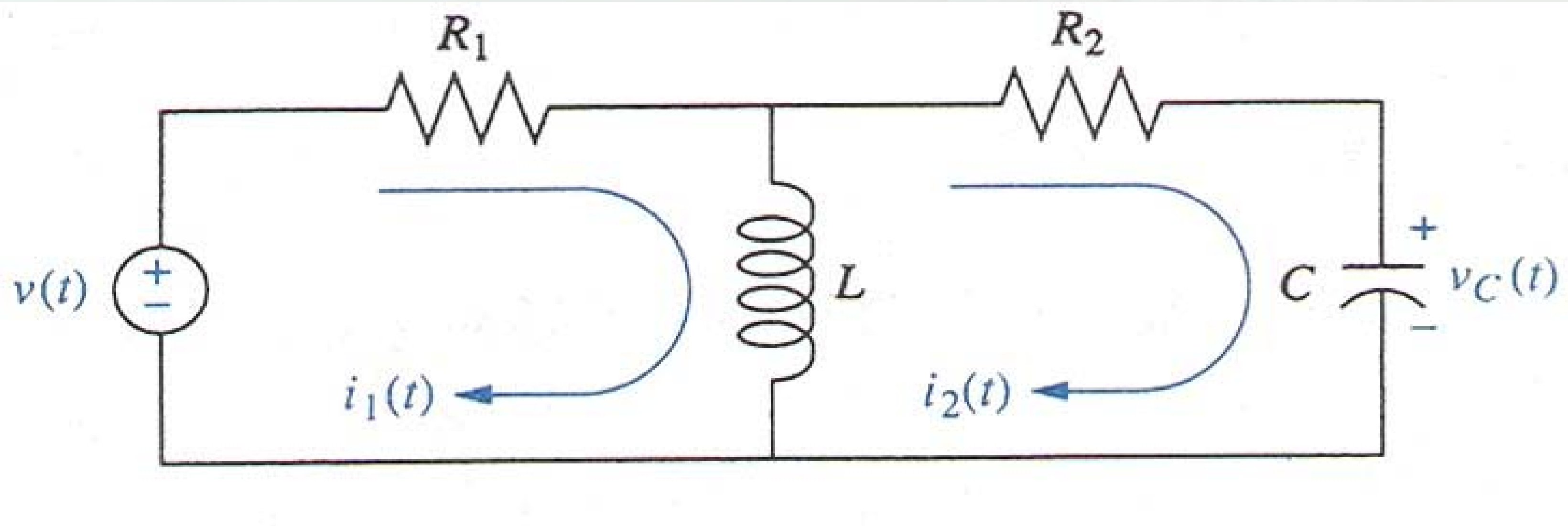
$$\Rightarrow V_C(s) = \frac{1}{sC}I(s) = \frac{1}{LCs^2 + RCs + 1}V(s)$$

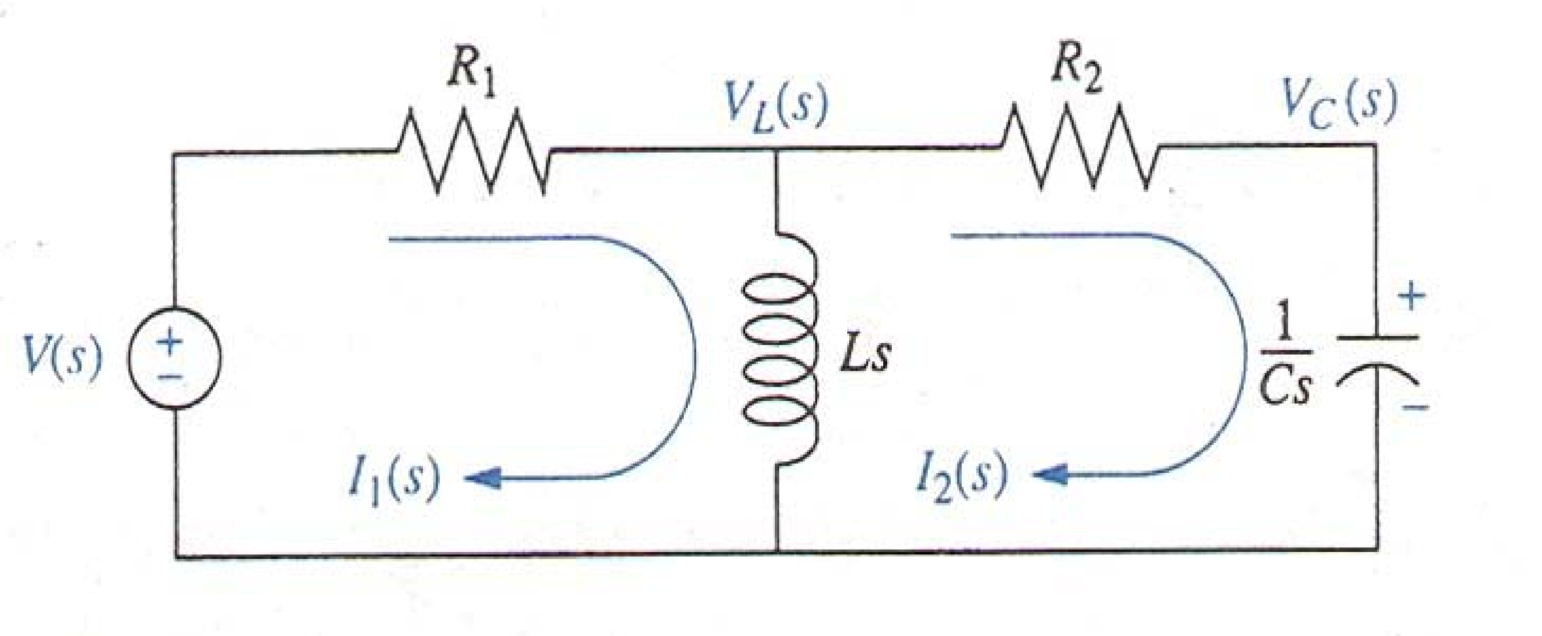
$$\Rightarrow \frac{V_C(s)}{V(s)} = \frac{1}{LCs^2 + RCs + 1}$$



Ex:試求輸入電壓與電容上電流之轉移函數

$$T(s) = \frac{I_2(s)}{V(s)}$$





$$\Rightarrow \begin{cases} R_1 I_1(s) + sL[I_1(s) - I_2(s)] = V(s) \\ sL[I_2(s) - I_1(s)] + R_2 I_2(s) + \dfrac{1}{sC} I_2(s) = 0 \end{cases}$$

$$\Rightarrow \begin{bmatrix} R_1 + sL & -sL \\ -sL & R_2 + sL + \dfrac{1}{sC} \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} V(s) \\ 0 \end{bmatrix}$$

- 利用Cramer 法則

$$I_2(s) = \frac{\begin{vmatrix} R_1 + sL & V(s) \\ -sL & 0 \end{vmatrix}}{\begin{vmatrix} R_1 + sL & -sL \\ -sL & R_2 + sL + \dfrac{1}{sC} \end{vmatrix}} = \frac{sL}{R_1 R_2 + L(R_1 + R_2)s + \dfrac{R_1}{sC} + \dfrac{L}{C}} V(s)$$

$$\Rightarrow T(s) = \frac{I_2(s)}{V(s)} = \frac{LCs^2}{LC(R_1 + R_2)s^2 + (R_1 R_2 C + L)s + R_1} V(s)$$