# Control Systems

# 控制系統


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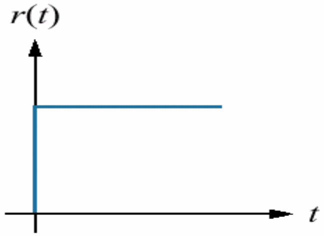
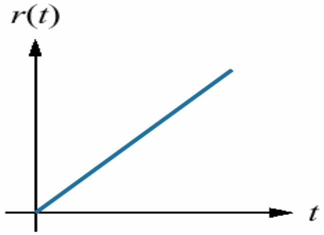
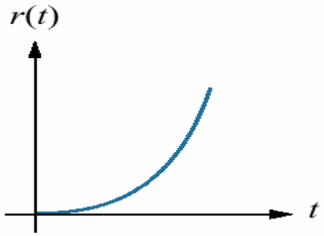
# 內容

- 簡介
- 頻域模型
- 時域模型
- 時間響應
- 互聯子系統之簡化
- 穩定度
- 穩態誤差

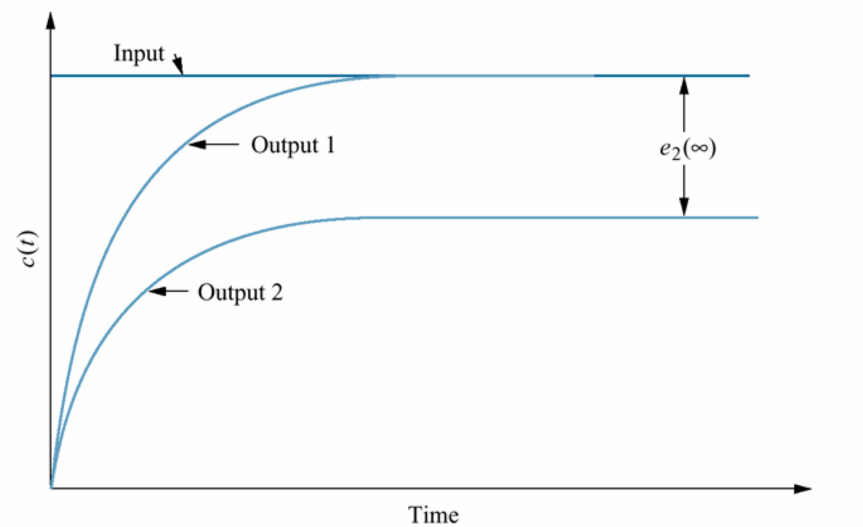
# Chapter 7　Steady-state error

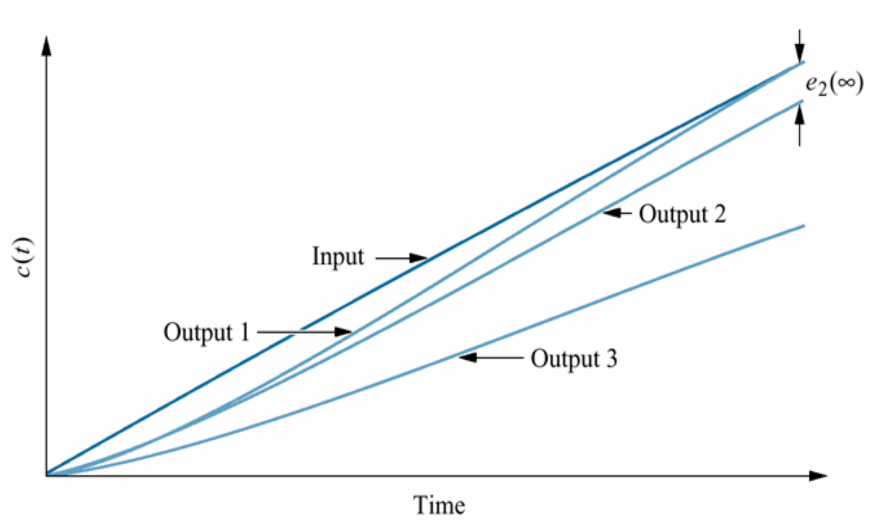
- 穩態誤差(steady-state error)：用一規定的測試輸入，當 $t \to \infty$ 時輸入與輸出之間的差。
- 穩態誤差的來源
  - 非線性源產生—齒隙，馬達的驅動電源…
  - 本身結構與應用的輸入型態而來

# ✵測試輸入信號：

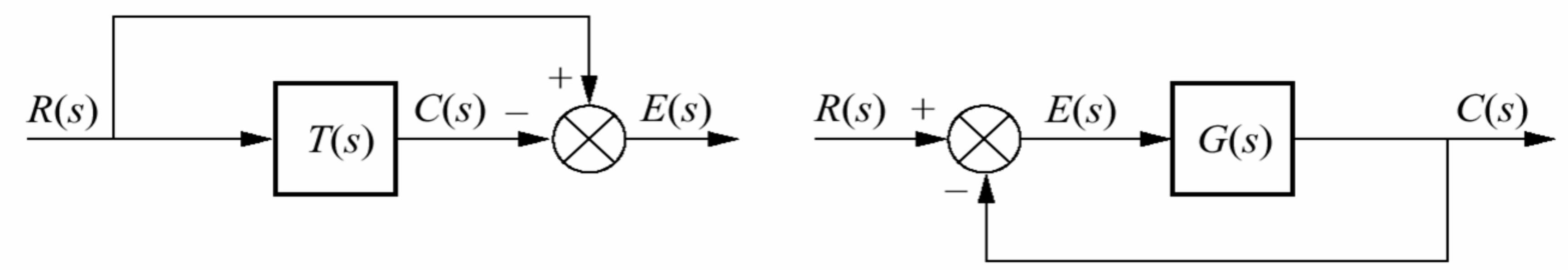
| Waveform | Name | Physical interpretation | Time function | Laplace transform |
|---|---|---|---|---|
|   | Step | Constant position | $1$ | $\frac{1}{s}$ |
|   | Ramp | Constant velocity | $t$ | $\frac{1}{s^2}$ |
|   | Parabola | Constant acceleration | $\frac{1}{2}t^2$ | $\frac{1}{s^3}$ |

## 應用於穩定系統—“穩態”誤差的計算




## 方塊圖表示



## 單位負回授系統的穩態誤差

$$C(s) = T(s)R(s) = \frac{G(s)}{1+G(s)}R(s)$$

$$E(s) = R(s) - C(s) = R(s) - T(s)R(s)$$

$$= [1 - T(s)]R(s) = \frac{1}{1+G(s)}R(s)$$

- 利用終值定理

$$e_{ss} = e(\infty) = \lim_{t \to \infty} e(t) = \lim_{s \to 0} s \cdot E(s)$$

$$= \lim_{s \to 0} s[1 - T(s)]R(s) = \lim_{s \to 0} s \frac{1}{1+G(s)} R(s)$$

若輸入為單位步階之系統 $T(s)=\dfrac{5}{s^2+7s+10}$ 試求穩態誤差。

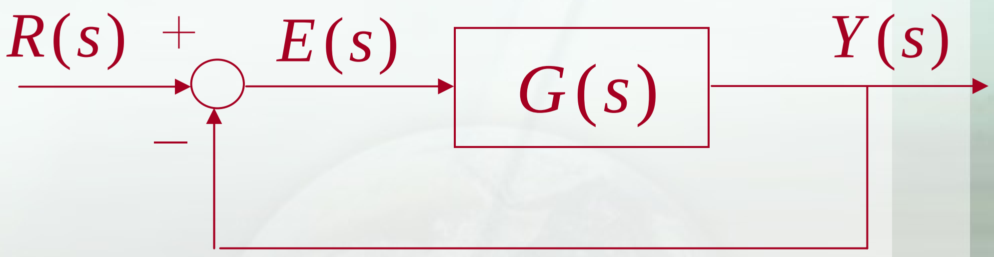
$$E(s)=[1-T(s)]\cdot R(s)=\frac{1}{s}\cdot\frac{s^2+7s+5}{s^2+7s+10}$$

$$e_{ss}=\lim_{t\to\infty}e(t)=\lim_{s\to 0}s\cdot E(s)$$

$$=\lim_{s\to 0}s\cdot\frac{s^2+7s+5}{s^2+7s+10}\cdot\frac{1}{s}$$

$$=\frac{1}{2}$$

## 以G(s)表穩態誤差




$$Y(s) = \frac{G(s)}{1+G(s)} R(s)$$

$$E(s) = R(s) - Y(s)$$

$$= R(s) - \frac{G(s)}{1+G(s)} R(s)$$

$$= \frac{1}{1+G(s)} R(s)$$

$$e_{ss} = \lim_{s \to 0} s \cdot E(s) = \lim_{s \to 0} \frac{s \cdot R(s)}{1+G(s)}$$

步階輸入：$R(s)=\dfrac{A}{s}$

$$e_{ss}=\lim_{s\to 0}\frac{A}{1+G(s)}=\frac{A}{1+\lim_{s\to 0}G(s)}$$

$$if\ :G(s)=\frac{(s+z_1)(s+z_2)\cdots\cdots}{s^n(s+p_1)(s+p_2)\cdots\cdots}$$

$$(a)\ n=0\ \Rightarrow \lim_{s\to 0}G(s)=\frac{z_1z_2\cdots}{p_1p_2\cdots}=k\quad\Rightarrow e(\infty)=\frac{A}{\infty}=0$$

$$(b)\ n\geq 1\ \Rightarrow \lim_{s\to 0}G(s)=\infty\quad\Rightarrow e(\infty)=\frac{A}{1+k}$$

## 斜坡輸入：$R(s)=\dfrac{A}{s^2}$

$$e_{ss}=\lim_{s\to 0}\frac{s}{1+G(s)}\frac{A}{s^2}=\lim_{s\to 0}\frac{A}{s+sG(s)}=\frac{A}{\lim\limits_{s\to 0}sG(s)}$$

$$n=0 \Rightarrow \lim_{s\to 0} sG(s)=0 \qquad \Rightarrow e(\infty)=\frac{A}{0}=\infty$$

$$n=1 \Rightarrow \lim_{s\to 0} sG(s)=\frac{z_1 z_2\cdots}{p_1 p_2\cdots}=k \qquad \Rightarrow e(\infty)=\frac{A}{k}$$

$$n\geq 2 \Rightarrow \lim_{s\to 0} sG(s)=\infty \qquad \Rightarrow e(\infty)=\frac{A}{\infty}=0$$

## 拋物線輸入：$R(s)=\dfrac{2A}{s^3}$

$$e_{ss}=\lim_{s\to 0}\frac{s}{1+G(s)}\frac{2A}{s^3}=\lim_{s\to 0}\frac{2A}{s^2+s^2G(s)}=\frac{2A}{\lim\limits_{s\to 0}s^2G(s)}$$

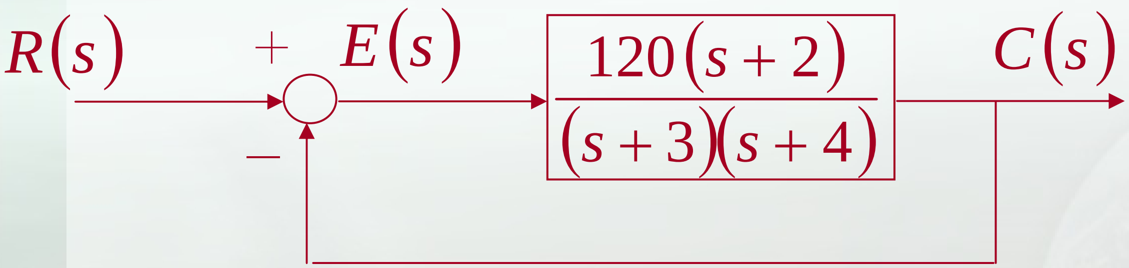
$$n<2\ \Rightarrow \lim_{s\to 0}s^2G(s)=0 \qquad \Rightarrow e(\infty)=\frac{2A}{\infty}=0$$

$$n=2\ \Rightarrow \lim_{s\to 0}s^2G(s)=\frac{z_1z_2\cdots}{p_1p_2\cdots}=k \qquad \Rightarrow e(\infty)=\frac{2A}{k}$$

$$n\geq 3\ \Rightarrow \lim_{s\to 0}s^2G(s)=\lim_{s\to 0}s^2\cdot\frac{(s+z_1)\cdots}{s^n(s+p_1)\cdots}=\infty$$

$$\Rightarrow e(\infty)=\frac{2A}{0}=\infty$$

# Ex：系統無任何積分，求各自之穩態誤差




$r(t)$分別為 $5u(t)$

$5tu(t)$

$5t^2u(t)$

$$E(s)=R(s)-C(s)$$

$$=R(s)-E(s)\cdot G(s)$$

$$E(s)=\frac{R(s)}{1+G(s)}$$

$$\text{(a) } r(t) = 5 \;\Rightarrow\; R(s) = \frac{5}{s}$$

$$e_{ss} = \lim_{s\to 0} s \cdot E(s) = \lim_{s\to 0} s \cdot \frac{R(s)}{1+G(s)} = \lim_{s\to 0} \frac{5}{1+G(s)}$$

$$= \frac{5}{1+\lim\limits_{s\to 0} G(s)} = \frac{5}{1+20} = \frac{5}{21}$$

$$\text{(b) } r(t) = 5t \;\Rightarrow\; R(s) = \frac{5}{s^2}$$

$$e_{ss} = \lim_{s\to 0} s \cdot \frac{\frac{5}{s^2}}{1+G(s)} = \lim_{s\to 0} \frac{5}{s+sG(s)}$$

$$= \frac{5}{\lim\limits_{s\to 0} sG(s)} = \frac{5}{\lim\limits_{s\to 0} s \cdot \frac{120(s+2)}{(s+3)(s+4)}} = \frac{5}{0} = \infty$$

$$\text{(c) } r(t) = 5t^2 u(t) \;\Rightarrow R(s) = \frac{10}{s^3}$$

$$e_{ss} = \lim_{s\to 0} \frac{\frac{10}{s^3}}{1+G(s)} = \frac{10}{\lim\limits_{s\to 0} s^2 \cdot G(s)} = \frac{10}{0} = \infty$$

# 靜態誤差常數(單位回授系統)

- 步階輸入 $(A) \Rightarrow$ 位置常數(position constant) $k_p$

$$k_p = \lim_{s \to 0} G(s) \qquad \Rightarrow e_{ss} = \frac{A}{1+k_p}$$

- 斜坡輸入 $(At) \Rightarrow$ 速度常數(velocity constant) $k_v$

$$k_v = \lim_{s \to 0} sG(s) \qquad \Rightarrow e_{ss} = \frac{A}{k_v}$$

- 拋物線輸入 $(At^2) \Rightarrow$ 加速度常數(Acceleration constant) $k_a$

$$k_a = \lim_{s \to 0} s^2 G(s) \qquad \Rightarrow e_{ss} = \frac{2A}{k_a}$$

例題：對下圖求其穩態誤差常數. 並在標準的步階、斜坡、拋物線輸入下求穩態誤差。

$$(a)\ G(s)=\frac{540(s+2)(s+5)}{(s+8)(s+10)(s+12)}$$

$$k_p=\lim_{s\to 0}G(s)=\frac{540\times 2\times 5}{8\times 10\times 12}=15\quad\Rightarrow e_{ss}=\frac{1}{1+k_p}=\frac{1}{16}$$

$$k_v=\lim_{s\to 0}sG(s)=0\quad\Rightarrow e_{ss}=\frac{1}{k_v}=\infty$$

$$k_a=\lim_{s\to 0}s^2G(s)=0\quad\Rightarrow e_{ss}=\frac{1}{k_a}=\infty$$

$$(b)\ G(s)=\frac{540(s+2)(s+5)}{s(s+8)(s+10)(s+12)}$$

$$k_p=\lim_{s\to 0} G(s)=\infty \quad \Rightarrow e_{ss}=\frac{1}{1+k_p}=\frac{1}{1+\infty}=0$$

$$k_v=\lim_{s\to 0} sG(s)=\frac{540\times 2\times 5}{8\times 10\times 12}=15 \quad \Rightarrow e_{ss}=\frac{1}{k_v}=\frac{1}{15}$$

$$k_a=\lim_{s\to 0} s^2G(s)=0 \quad \Rightarrow e_{ss}=\frac{1}{k_a}=\infty$$

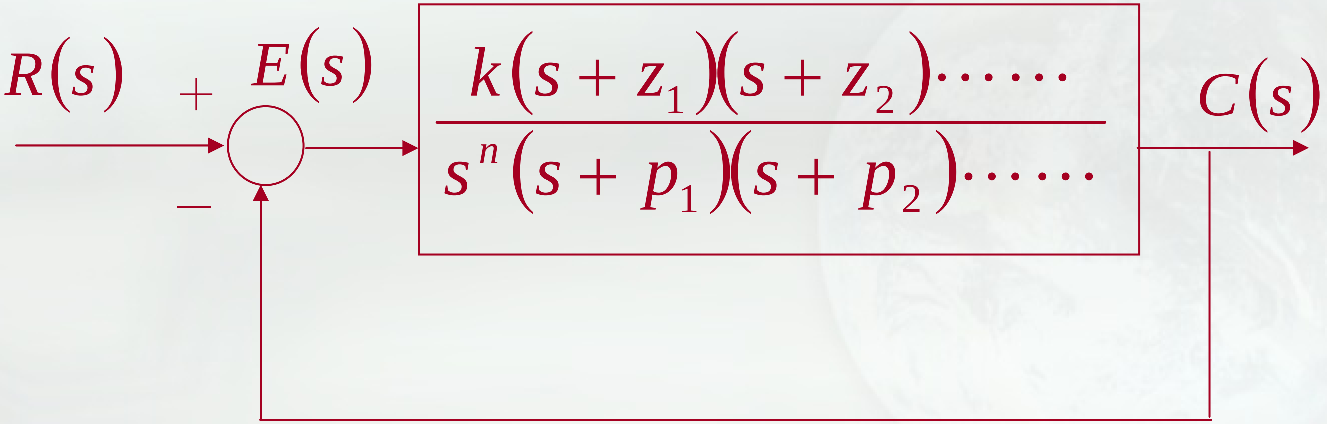
$$(c)\ G(s)=\frac{540(s+2)(s+5)}{s^2(s+8)(s+10)(s+12)}$$

$$k_p=\lim_{s\to 0}G(s)=\infty \quad \Rightarrow e_{ss}=\frac{1}{1+k_p}=\frac{1}{1+\infty}=0$$

$$k_v=\lim_{s\to 0}sG(s)=\infty \quad \Rightarrow e_{ss}=\frac{1}{k_v}=\frac{1}{\infty}=0$$

$$k_a=\lim_{s\to 0}s^2G(s)=\frac{540\times 2\times 5}{8\times 10\times 12}=15 \quad \Rightarrow e_{ss}=\frac{2}{k_a}=\frac{2}{15}$$

# 單位負回授系統之系統組態(System Type)：順向路徑的純積分次數，即為*n*值。




- $n$=0則type 0； $n$=1則type 1；依此類推⋯

| Input | Steady-state error formula | Type 0 | | Type 1 | | Type 2 | |
|---|---|---|---|---|---|---|---|
| | | Static error constant | Error | Static error constant | Error | Static error constant | Error |
| Step, $u(t)$ | $\frac{1}{1+K_p}$ | $K_p =$ Constant | $\frac{1}{1+K_p}$ | $K_p = \infty$ | 0 | $K_p = \infty$ | 0 |
| Ramp, $tu(t)$ | $\frac{1}{K_v}$ | $K_v = 0$ | $\infty$ | $K_v =$ Constant | $\frac{1}{K_v}$ | $K_v = \infty$ | 0 |
| Parabola, $\frac{1}{2}t^2u(t)$ | $\frac{1}{K_a}$ | $K_a = 0$ | $\infty$ | $K_a = 0$ | $\infty$ | $K_a =$ Constant | $\frac{1}{K_a}$ |

**EX:** 試求單位負回授系統 $G(s)=\dfrac{1000(s+8)}{(s+7)(s+9)}$ 之型態、靜態誤差常數及穩態誤差。

Sol:

type 0

$$k_p=\frac{1000\times 8}{7\times 9}=\frac{8000}{63}=126.98$$

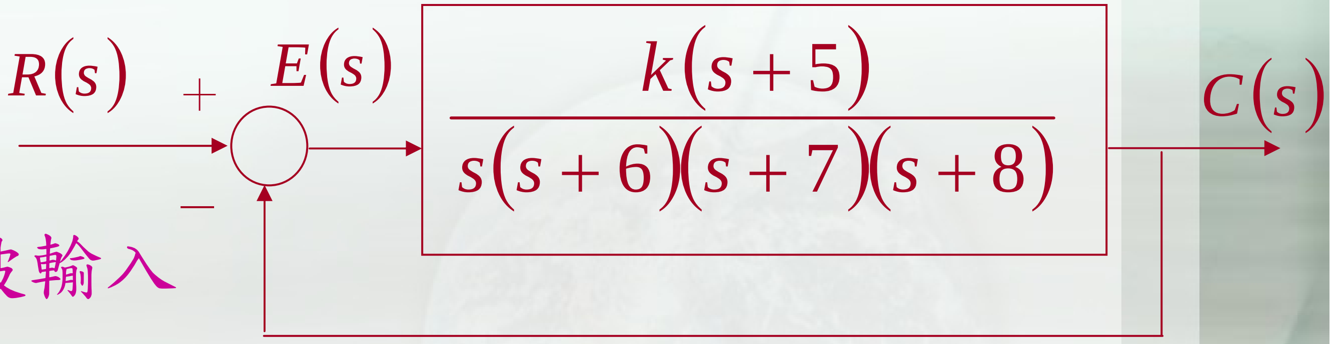
$$\Rightarrow e_{ss}=\frac{1}{1+k_p}=\frac{1}{128}\cong 0.0079$$

$$k_v=0\Rightarrow e_{ss}=\infty$$

$$k_a=0\Rightarrow e_{ss}=\infty$$

**EX:試求k值使系統在穩定狀態有10%穩態誤差。**

Sol:



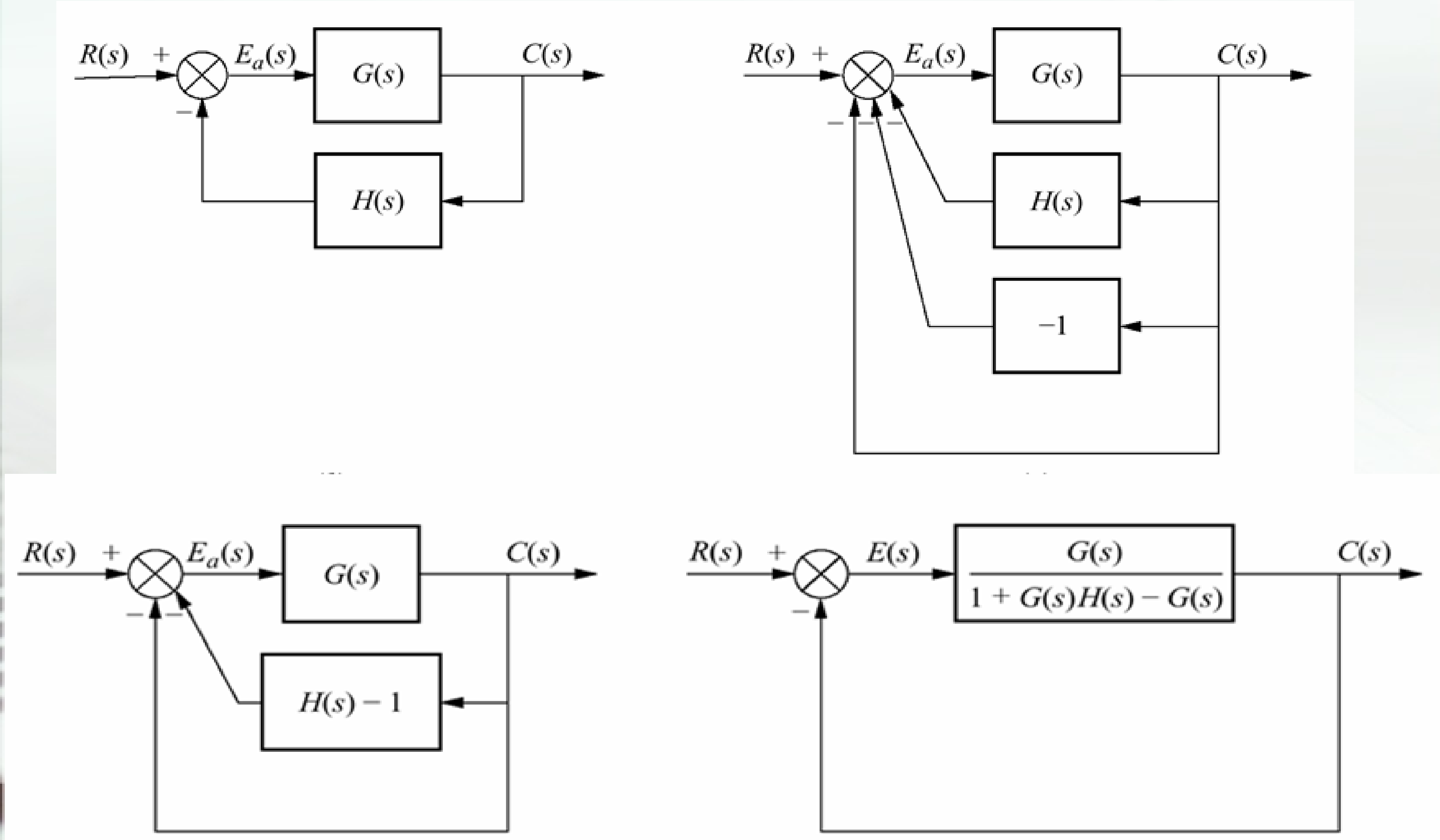

type 1 斜坡輸入

$$e_{ss}=\frac{1}{k_v}=\frac{1}{\lim_{s\to 0} sG(s)}$$

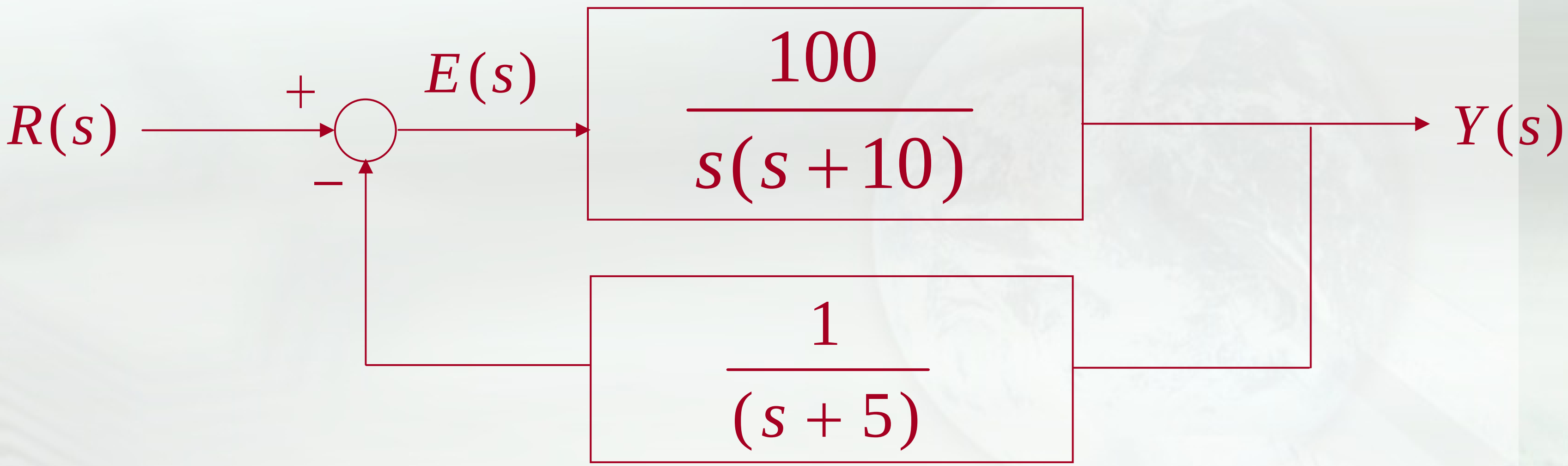
$$k_v=10=\lim_{s\to 0} s\cdot\frac{k(s+5)}{s(s+6)(s+7)(s+8)}=\frac{5\cdot k}{6\cdot 7\cdot 8}$$

$$\Rightarrow k=\frac{10\cdot 6\cdot 7\cdot 8}{5}=672$$

# 非單位回授的穩態誤差



**EX:試求型態及適當之誤差常數和對一單位步階輸入之穩態誤差。**



Sol:

$$G_e = \frac{G(s)}{1+G(s)H(s)-G(s)} = \frac{\frac{100}{s(s+10)}}{1+\frac{100}{s(s+10)}\cdot\frac{1}{(s+5)}-\frac{100}{s(s+10)}}$$

$$= \frac{100(s+5)}{s^3+15s^2+50s+100-100s-500}$$

$$= \frac{100(s+5)}{s^3+15s^2-50s-400} \Rightarrow \text{type 0}$$

$$\Rightarrow k_p = \lim_{s\to 0} G_e(s) = \frac{500}{-400} = -\frac{5}{4}$$

$$\Rightarrow e_{ss} = \frac{1}{1+k_p} = \frac{1}{1+(-\frac{5}{4})} = -4$$

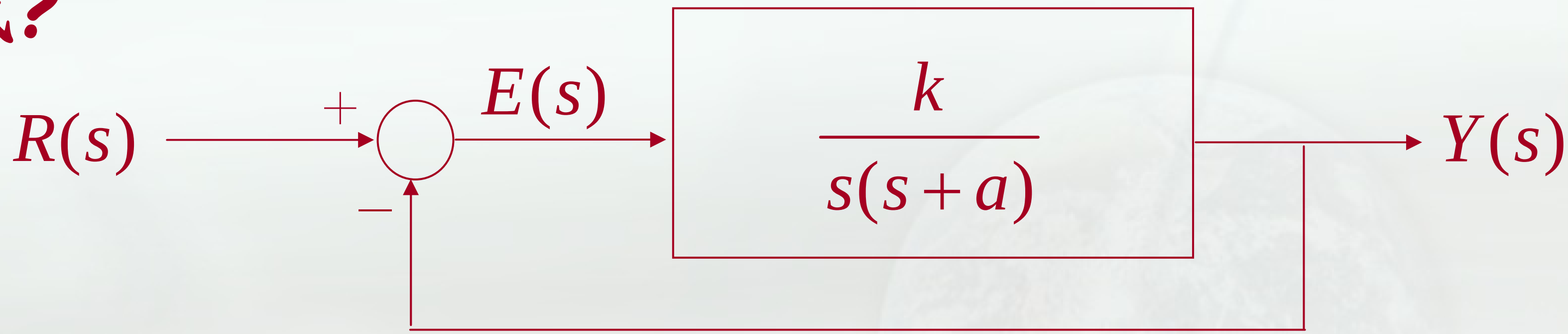
靈敏度：函數內的比例變化與參數比例變化之比值。

$$S_{F:P} = \lim_{\Delta P \to 0} \frac{\text{函數的比例變化}}{\text{參數的比例變化}} = \lim_{\Delta P \to 0} \frac{\Delta F / F}{\Delta P / P} = \lim_{\Delta P \to 0} \frac{P}{F} \cdot \frac{\Delta F}{\Delta P}$$

$$= \frac{P}{F} \lim_{\Delta P \to 0} \frac{\Delta F}{\Delta P}$$

$$S_{F:P} = \frac{P}{F} \cdot \frac{\partial F}{\partial P}$$

**EX:算出改變參數*a*對系統之靈敏度，又如何降低?**




Sol:

$$T(s)=\frac{k}{s^2+as+k}$$

$$S_{T:a}=\frac{a}{T}\cdot\frac{\partial T}{\partial a}=\frac{a}{\left(\frac{k}{s^2+as+k}\right)}\frac{-s\cdot k}{(s^2+as+k)^2}=\frac{-as}{s^2+as+k}$$

**EX:求上圖參數 *k* 和 *a* 對穩態誤差之靈敏度。**

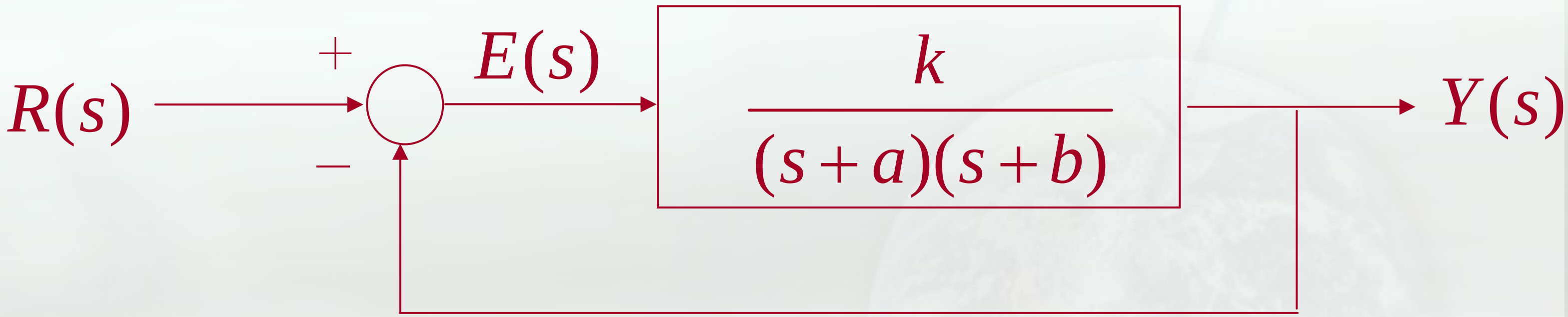
Sol:

$$e_{ss} = \frac{1}{k_v} = \frac{1}{\lim_{s\to 0} s\cdot\frac{k}{s(s+a)}} = \frac{a}{k}$$

$$S_{e:a} = \frac{a}{e}\cdot\frac{\partial e}{\partial a} = \frac{a}{\frac{a}{k}}\cdot\frac{1}{k} = 1$$

$$S_{e:k} = \frac{k}{e}\cdot\frac{\partial e}{\partial k} = \frac{k}{\frac{a}{k}}\cdot\frac{-a}{k^2} = -1$$

## EX:試求參數 $k$ 和 $a$ 對穩態誤差之靈敏度。




Sol:

$$\Rightarrow e_{ss} = \frac{1}{1+k_p} = \frac{1}{1+\lim_{s\to 0}\frac{k}{(s+a)(s+b)}} = \frac{1}{1+\frac{k}{ab}} = \frac{ab}{ab+k}$$

$$S_{e:a} = \frac{a}{e} \cdot \frac{\partial e}{\partial a} = \frac{a}{\dfrac{ab}{ab+k}} \cdot \frac{b(ab+k) - ab^2}{(ab+k)^2} = \frac{k}{ab+k}$$

$$S_{e:k} = \frac{k}{e} \cdot \frac{\partial e}{\partial k} = \frac{k}{\dfrac{ab}{ab+k}} \cdot \frac{-ab}{(ab+k)^2} = \frac{-k}{ab+k}$$

$\because \left|S_{e:a}\right|, \left|S_{e:k}\right|$ 小於 1 $\Rightarrow$ 比例回授系統會降低靈　敏度

## 狀態空間

$$\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{Bu}$$

$$\boldsymbol{y} = \boldsymbol{Cx} + \boldsymbol{Du}$$

$E(s) = R(s) - Y(s)$ 且 $Y(s) = T(s)R(s)$

$$\Rightarrow E(s) = [1 - T(s)]R(s) = [1 - \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}]R(s)$$

$$\Rightarrow e_{ss} = \lim_{s \to 0} s \cdot E(s) = \lim_{s \to 0} s \cdot [1 - \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}]R(s)$$

**EX:** $A = \begin{bmatrix} -5 & 1 & 0 \\ 0 & -2 & 1 \\ 20 & -10 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $C = \begin{bmatrix} -1 & 1 & 0 \end{bmatrix}$

Sol:

$$e_{ss} = \lim_{s \to 0} s \cdot \left(1 - \frac{s+4}{s^3 + 6s^2 + 13s + 20}\right) R(s)$$

$$= \lim_{s \to 0} s \cdot \left(\frac{s^3 + 6s^2 + 12s + 16}{s^3 + 6s^2 + 13s + 20}\right) R(s)$$

$$R(s) = \frac{1}{s} \quad \Rightarrow e_{ss} = \lim_{s \to 0} \frac{16}{20} = \frac{4}{5}$$

$$R(s) = \frac{1}{s^2} \quad \Rightarrow e_{ss} = \lim_{s \to 0} \frac{16}{0} = \infty$$